A Schensted insertion for an infinite dimensional representation, v. 2.1

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Abstract

We give the first known example of a Robinson-Schensted type insertion for a class of infinite dimensional representations. The tensor powers of the Weil representation W of $\mathfrak{sp}(2n, \mathbb{C})$ decompose as a direct sum of certain highest weight $\mathfrak{sp}(2n, \mathbb{C})$-modules $L_k(\lambda)$ tensored with some corresponding finite dimensional irreducible $O(k, \mathbb{C})$-module V_λ. Concentrating only on the $\mathfrak{sp}(2n, \mathbb{C})$-module structure, we can see this as iterating the following decomposition:

$$L_k(\lambda) \otimes W \simeq \bigoplus_{\mu} L_{k+1}(\mu)\,. \tag{2}$$

If k is sufficently large relative to n and λ, then the module $L_k(\lambda)$ and all the terms on the right hand side belong to the holomorphic discrete series and our insertion algorithm allows us to give a weighted bijection proving the formal character identity corresponding to the decomposition (2). It seems likely that a correspondence along these lines can be given to combinatorially explain similar identities of formal characters for small k as well, but this is still under investigation. On donne le premier

exemple d'une insertion de type Robinson-Schensted pour une classe representations de dimension infinies. La puissance tensorielle de la representation de Weil decomposee en somme directe de certains $\mathfrak{sp}(2n, \mathbb{C})$-modules de highest weight est tensoree avec des $O(k, \mathbb{C})$-modules V_λ irreducibles de dimensions finies. En etudiant seulement les structures des $\mathfrak{sp}(2n, \mathbb{C})$-module on peut obtenir ces resultats par iteration de la decomposition suivante

$$L_k(\lambda) \otimes W \simeq \bigoplus_{\mu} L_{k+1}(\mu)\,. \tag{2}$$

Si k est suffisament grand par rapport a n et λ alors le module $L_k(\lambda)$ et tous les termes de droite appartiennent a la serie discrete holomorphe et notre algorithm d'insertion permet de donner une weighted bijection demontrant ainsi l'indentite du caractere formel correspondant a la decomposition (2). Il semblerait probable qu'une correspondence similaire pourait etre donnee de facon a donner une expliquation combinatoire sur les identites des caracteres formels pour petit k. Ceci est a l'heure actuelle un projet.

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1 Introduction

Our goal is to extend the notion of tableaux, insertion and the Robinson-Schensted correspondence (or the R-S correspondence for short) to a class of infinite-dimensional representations of the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. (Here n is fixed; see §2 for definitions of this and terms to follow.) The Weil representation of $\mathfrak{g}$ is an infinite-dimensional representation which can be realized on the space of polynomials $W = \mathbb{C}[x_1, x_2, \ldots, x_n]$ using multiplication operators and certain simple differential operators. We can reformulate results of M. Kashiwara and M. Vergne ([KV]) (see also the work of R. Howe [How]) to show that the centralizer of the action of $\mathfrak{g}$ on the k-th tensor power $W^{\otimes k}$ of W is given by an action of the orthogonal group $O(k, \mathbb{C})$, and that they constitute a dual pair, namely the space $W^{\otimes k}$ decomposes into a multiplicity-free sum:

$$W^{\otimes k} \simeq \bigoplus_{\lambda} L_k(\lambda) \otimes V_\lambda, \tag{1}$$

where each $L_k(\lambda)$ is an irreducible $\mathfrak{g}$-module, each V_λ is an irreducible $O(k, \mathbb{C})$-module (using the notation in [KV]), and the summation is for $\lambda \in \mathcal{L}_k$ where $\mathcal{L}_k = \{\, \lambda \in \mathbb{Y} \mid l(\lambda) \leq n,\ \lambda'_1 + \lambda'_2 \leq k \,\}$. Here $\mathbb{Y}$ denotes Young's lattice, the lattice of all partitions of nonnegative integers (ordered by inclusion). These $L_k(\lambda)$ all turn out to be irreducible highest weight modules (for a fixed choice of Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$). The highest weight of $L_k(\lambda)$ is

$$\Lambda(\lambda, k) = -(\frac{k}{2} + \lambda_n)\epsilon_1 - (\frac{k}{2} + \lambda_{n-1})\epsilon_2 - \ldots - (\frac{k}{2} + \lambda_1)\epsilon_n\,.$$

The highest weights $\Lambda(\lambda, k)$ obtained in this manner from (1) for all $k \in \mathbb{Z}_{\geq 0}$ lie in (but do not fill) a cone $C_0 = \{\, \sum_{i=1}^n a_i\epsilon_i \in \widetilde{Q} \mid 0 > a_1 \geq a_2 \geq \cdots \geq a_n \,\}$ in the lattice $\widetilde{Q} = \{\, \sum_{i=1}^n a_i\epsilon_i \mid a_i \in \mathbb{Z}/2\ (1 \leq i \leq n),\ \text{and}\ a_i - a_{i+1} \in \mathbb{Z}\ (1 \leq i \leq n-1) \,\}$. Since the V_λ are finite-dimensional $O(k, \mathbb{C})$-modules, it is possible to describe their behavior in combinatorial terms. R. King and B. Wybourne exploited this connection to obtain some combinatorial results for the infinite-dimensional representations $L_k(\lambda)$ ([KW]).

Our model is the following version of the classical R-S correspondence which depicts the decomposition of the k-th tensor power of the natural representation of $\mathfrak{gl}(n, \mathbb{C})$:

$$[1, n]^k \xrightarrow{\sim} \coprod_{\substack{\lambda \vdash k \\ l(\lambda) \leq n}} \mathrm{CST}^{(n)}(\lambda) \times \mathrm{SYT}(\lambda).$$

Here $\mathrm{CST}^{(n)}(\lambda)$ (resp. $\mathrm{SYT}(\lambda)$) denotes the set of column strict or semistandard tableaux (resp. standard tableaux) of shape λ with entries from $[1, n]$ and the interval $[1, n]$ is taken inside $\mathbb{Z}_{\geq 0}$. This bijection is constructed as a repetition of a procedure called *row insertion*, $\mathrm{CST}^{(n)}(\lambda) \times [1, n] \xrightarrow{\sim} \coprod_{\substack{\mu \supset \lambda \\ l(\mu) \leq n}} \mathrm{CST}^{(n)}(\mu)$, which depicts the decomposition of the tensor product $V_\lambda \otimes V_\square \simeq \bigoplus_\mu V_\mu$ (where V_λ and V_μ are irreducible finite-dimensional $\mathfrak{gl}(n, \mathbb{C})$-modules now, $V_\square$ is the space of natural representation of $\mathfrak{gl}(n, \mathbb{C})$, and μ runs over partitions such that μ/λ is one box and $l(\mu) \leq n$). Following this line, our correspondence should consist of a repetition of a bijection reflecting the decomposition of the tensor product

$$L_k(\lambda) \otimes W \simeq \bigoplus_{\mu} L_{k+1}(\mu), \tag{2}$$

where μ runs over partitions in $\mathcal{L}_{k+1}$ such that μ/λ is a horizontal strip.

The first ingredient we need is a set $\mathrm{SIST}(\lambda)$ of certain tableaux which, with a notion of weight depending on k, gives the weight generating function equal to the formal character of $L_{\Lambda(\lambda,k)}$. Here, for any $\Lambda \in \mathfrak{h}^*$, L_Λ denotes the (unique) irreducible highest weight $\mathfrak{g}$-module with highest weight Λ.

This has been done for a subclass of these $\mathfrak{g}$-modules in [TY], namely for those having highest weights in a smaller cone $C_1 = \{\, \sum_{i=1}^{n} a_i\epsilon_i \in C_0 \mid -n > a_1 \,\}$, in other words those in the holomorphic discrete series, in which case L_Λ is actually a generalized Verma module: If we put $\mathfrak{k} = \mathfrak{gl}(n, \mathbb{C})$ embedded in $\mathfrak{g}$ in a suitable fashion, then all weights in C_0 are $\mathfrak{k}$-dominant (with respect to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{k}$ of $\mathfrak{k}$). Therefore, if $\Lambda \in C_0$, we have a finite-dimensional irreducible $\mathfrak{k}$-module F_Λ with highest weight Λ. Let $\mathfrak{p}$ denote the parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and with Levi part $\mathfrak{k}$. Then we can form the generalized Verma module $N_\Lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\Lambda$, where F_Λ is regarded as a $\mathfrak{p}$-module by defining the action of the nilpotent radical of $\mathfrak{p}$ to be trivial. If $\Lambda \in C_0 - C_1$, L_Λ is the irreducible quotient of N_Λ (still possibly isomorphic to N_Λ for some such Λ).

It can be easily seen that, in (2), if $L_k(\lambda)$ has highest weight in C_1, then so does every summand on the right hand side. Our result here presents a weight-preserving bijection

$$\mathrm{SIST}(\lambda) \times \mathbb{Z}^n_{\geq 0} \xrightarrow{\sim} \coprod_{\mu} \mathrm{SIST}(\mu)\,, \tag{3}$$

where μ runs through partitions in $\mathcal{L}_{k+1}$ such that μ/λ is a horizontal strip. Here, the generating function of $\mathbb{Z}^n_{\geq 0}$, with a proper definition of weights, gives the formal character of W. Unlike the case of finite-dimensional representations, an equality of formal characters is in general far from sufficient to draw out any conclusion on the decomposition of a tensor product. In our case, however, it is known that the tensor product decomposes as a direct sum of irreducible highest weight modules. (See [Kob].)

W itself is a direct sum of two irreducible highest weight modules, but their highest weights belong to $C_0 - C_1$ (out of the holomorphic discrete series region). Because of this, our present work does not give the initial stages of a (hypothesized) full-blown R-S correspondence depicting the decomposition (1).

If we start from an $L_k(\lambda)$ which is already in the holomorphic discrete series, instead of starting from scratch, we can iterate our procedure describing (2) from there and depict the decomposition of $L_k(\lambda) \otimes W^{\otimes k'}$.

We conjecture the existence of a nice subset $\mathrm{SIST}^{(k)}(\lambda)$ of $\mathrm{SIST}(\lambda)$, for k violating the condition $\lambda_n + \frac{k}{2} > n$, such that the weight generating function of $\mathrm{SIST}^{(k)}(\lambda)$ equals the formal character of $L_k(\lambda)$. We are also hoping to find some modified correspondence that works for these "singular" cases, so that combining these with the ones we presently obtained would define the whole correspondence.

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2 The Weil representation and its modules

Definition 2.1 The *symplectic group* $Sp(2n;\mathbb{C})$ is the subgroup of $GL(2n,\mathbb{C})$ which preserves a fixed nondegenerate skew-symmetric form $\langle\ ,\ \rangle$. More precisely, we may take our form to be given as $\langle v,w\rangle = v^tJw$, where J is the matrix given as follows:

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The *symplectic Lie algebra* $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ is then given by

$$\mathfrak{sp}(2n,\mathbb{C}) = \{X \in \mathfrak{gl}(2n,\mathbb{C}) | X^tJ + JX = 0\}.$$

We may consider $\mathfrak{gl}(n,\mathbb{C})$ as living inside $\mathfrak{sp}(2n,\mathbb{C})$ via the following embedding

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}.$$

We fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$ in the manner read from Fig. 1 and its footnote. Let $\epsilon_i \in \mathfrak{h}^*$ be such that $\epsilon_i(E_{jj} - E_{n+j,n+j}) = \delta_{ij}$.

Note that $\mathfrak{gl}(n,\mathbb{C})$ shares a Cartan subalgebra $\mathfrak{h}$ with $\mathfrak{sp}(2n,\mathbb{C})$. Also, $\mathfrak{gl}(n,\mathbb{C}) \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{gl}(n,\mathbb{C})$.

Definition 2.2 The *Weil representation* ρ of $\mathfrak{sp}(2n,\mathbb{C})$ on $W = \mathbb{C}[x_1, x_2, \ldots, x_n]$ is given explicitely in Figure 1 by defining how various basis elements act on the polynomial ring. It extends, up to a twist by an automorphism of $\mathfrak{sp}(2n,\mathbb{C})$, the simpler action of $\mathfrak{gl}(n,\mathbb{C})$ given by sending $E_{ij} \mapsto x_i\partial_j$, where $\partial_j := \frac{\partial}{\partial x_j}$.

W has two independent highest weight vectors, namely 1 (weight $-\frac{1}{2}\omega_n$, where $\omega_n = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$) and x_n (weight $-\frac{1}{2}\omega_n - \epsilon_n$). $U(\mathfrak{g}) \cdot 1$ (resp. $U(\mathfrak{g}) \cdot x_n$) is the space of all polynomials of even total degree (resp. odd total degree). We have $W = U(\mathfrak{g}) \cdot 1 \oplus U(\mathfrak{g}) \cdot x_n$, and that these two submodules are irreducible. In the notation defined below, they are $L_{\Lambda((0),1)}$ and $L_{\Lambda((1),1)}$. These two highest weights belong to $C_0 - C_1$. (See Definition 2.6).

Definition 2.3 Let Λ be a weight for $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$, i.e, $\Lambda \in \mathfrak{h}^*$. Let $\mathfrak{p}$ be the parabolic subalgebra which contains $\mathfrak{b}$ and whose Levi part is $\mathfrak{k} = \mathfrak{gl}(n,\mathbb{C})$ (thought of as embedded in $\mathfrak{sp}(2n,\mathbb{C})$ by the map described immediately after Definition 2.1). Λ is called $\mathfrak{k}$-*dominant* if $\Lambda = \sum_{i=1}^n \alpha_i\epsilon_i$, $\alpha_i - \alpha_{i+1} \in \mathbb{Z}_{\geq 0}$ $(1 \leq i \leq n-1)$.

We define the following four modules.

- L_Λ = the unique irreducible highest weight $\mathfrak{g}$-module with highest weight Λ (for all $\Lambda \in \mathfrak{h}^*$)
- M_Λ := the Verma module for $\mathfrak{g}$ of highest weight Λ (for all $\Lambda \in \mathfrak{h}^*$).
- F_Λ =the finite dimensional irreducible $\mathfrak{k}$-module with highest weight Λ (for $\mathfrak{k}$-dominant Λ).

Figure 1: The Weil representation for $\mathfrak{sp}(2n, \mathbb{C})$.

Root	Element A	$n = 2$ Example	Operator $\rho(A)$
Cartan SA	$E_{ii} - E_{n+i,n+i}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$-x_i\partial_i - \frac{1}{2}$
$\epsilon_i - \epsilon_j$ for $i < j$	$E_{ij} - E_{n+j,n+i}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$x_j\partial_i$
$-\epsilon_i + \epsilon_j$ for $i < j$	$E_{ji} - E_{n+i,n+j}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$x_i\partial_j$
$\epsilon_i + \epsilon_j$ for $i \neq j$	$E_{i,n+j} + E_{j,n+i}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\sqrt{-1}\partial_i\partial_j$
$-\epsilon_i - \epsilon_j$ for $i \neq j$	$E_{n+i,j} + E_{n+j,i}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\sqrt{-1}x_ix_j$
$2\epsilon_i$	$E_{i,n+i}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\sqrt{-1}\partial_i^2$
$-2\epsilon_i$	$E_{n+i,i}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\sqrt{-1}x_i^2$

In this table, the positive and negative root vectors alternate $+, -, +, -, +, -$ after the Cartan subalgebra entry. There is a Lie algebra automorphism of $\mathfrak{sp}(2n, \mathbb{C})$ which exchanges E_β and $E_{-\beta}$ for all positive roots β, and which act as -1 on $\mathfrak{h}$. This means that there is another version of the Weil representation given by composing this automorphism with ρ. It is this version which extends the representation $E_{ij} \mapsto x_i\partial_j$ of $\mathfrak{gl}(n, \mathbb{C})$.

- N_Λ =the generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\Lambda$ for $\mathfrak{g}$, where F_Λ is regarded as a $U(\mathfrak{p})$-module by defining the action of the nilpotent radical of $\mathfrak{p}$ to be trivial.

Definition 2.4 For $\lambda \in \mathbb{Y}$ with $l(\lambda) \leq n$ and $k \in \mathbb{Z}_{\geq 0}$, we put:

$$\Lambda(\lambda, k) = -\frac{k}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) - \lambda_n \epsilon_1 - \lambda_{n-1}\epsilon_2 - \cdots - \lambda_1 \epsilon_n.$$

$$= -\sum_{i=1}^{n}(\lambda_{n-i+1} + \frac{k}{2})\epsilon_i$$

Definition 2.5 We define the following lattice $\tilde{Q}$ in $\mathfrak{h}^*$, which will contain all weights we consider:

$$\widetilde{Q} = \{ \sum_{i=1}^{n} a_i \epsilon_i \mid a_i \in \mathbb{Z}/2 \ (1 \leq i \leq n), \text{ and } a_i - a_{i+1} \in \mathbb{Z} \ (1 \leq i \leq n-1) \}$$

Unlike the usual weight lattice for $\mathfrak{sp}(2n, \mathbb{C})$ (see, e.g., [Hum]), which is just the one freely generated by the ϵ_i's, here we also need to include those half-integral linear combinations for which every coefficient is purely half-integral.

Definition 2.6 We define two cones in the lattice $\widetilde{Q}$ as follows

$$C_0 = \{ a_1\epsilon_1 + a_2\epsilon_2 + \ldots + a_n\epsilon_n \in \widetilde{Q} \mid 0 > a_1 \geq a_2 \geq \cdots \geq a_n \}.$$

$$C_1 = \{ a_1\epsilon_1 + a_2\epsilon_2 + \ldots + a_n\epsilon_n \in \widetilde{Q} \mid -n > a_1 \geq a_2 \geq \cdots \geq a_n \}.$$

Facts 2.7 *We have the following facts concerning the modules L_Λ.*

1. *([KV] and [How]) For $k \in \mathbb{Z}_{>0}$, we put $\mathcal{L}_k = \{ \lambda \in \mathbb{Y} \mid l(\lambda) \leq n \text{ and } \lambda'_1 + \lambda'_2 \leq k \}$, where λ'_j denotes the length of the jth column of the Young diagram of λ. Then we have*
$$W^{\otimes k} \simeq \bigoplus_{\lambda \in \mathcal{L}_k} L_k(\lambda) \otimes V_\lambda$$
where $L_k(\lambda) \simeq L_{\Lambda(\lambda,k)}$ and V_λ is a finite-dimensional irreducible $O(k, \mathbb{C})$-module.

2. *$C_1 \subset \{ \Lambda(\lambda, k) \mid \lambda \in \mathcal{L}_k, \ k \in \mathbb{Z}_{>0} \} \subset C_0$. (This is immediate.)*

3. *If $\Lambda \in \tilde{Q}$, L_Λ is in holomorphic discrete series if and only if $\Lambda \in C_1$. Moreover, in this case L_Λ is isomorphic to N_Λ.*

These conditions make it clear that, for k sufficiently large ($\geq 2n$), all the irreducible $\mathfrak{sp}(2n, \mathbb{C})$-modules appearing in the decomposition of $W^{\otimes k}$ are holomorphic discrete series modules. On the other hand, the two irreducible constituents of W (see above) are not in the holomorphic discrete series.

Lemma 2.8 *The formal character of the Weil representation W is given as the sum of all monomials in x_1^{-1}, x_2^{-1}, ..., x_n^{-1} with an additional factor*

$$\operatorname{ch} W = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} x^{-\alpha - (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})} = (x_1 x_2 \cdots x_n)^{-\frac{1}{2}} \prod_{i=1}^{n} (1 - x_i^{-1})^{-1}.$$

Namely, if we define the $\frac{1}{2}$-*adjusted weight* of a monomial x^α or $\alpha \in \mathbb{Z}_{\geq 0}^n$ to be $\alpha - \frac{1}{2}\omega_n$, then the weight generating function of $\mathbb{Z}_{\geq 0}^n$ for the $\frac{1}{2}$-adjusted weight equals the formal character of W.

Lemma 2.9

$$\operatorname{ch} N_{\Lambda(\lambda,k)} = \frac{\operatorname{ch} F_{\Lambda(\lambda,k)}}{\prod_{1 \leq i < j \leq n} (1 - x_i^{-1} x_j^{-1}) \prod_{1 \leq i \leq n} (1 - x_i^{-2})}.$$

It is easy to compute

$$\operatorname{ch} F_{\Lambda(\lambda,k)} = (x_1 x_2 \cdots x_n)^{-\frac{1}{2}} s_\lambda(x_1^{-1}, \ldots, x_n^{-1}),$$

where s_λ is the Schur function; hence we can compute the character of $L_k(\lambda) = L_{\Lambda(\lambda,k)} = N_{\Lambda(\lambda,k)}$ if $\Lambda(\lambda, k) \in C_1$.

The following definitions of semi-infinite symplectic tableaux and weight are from [TY].

Definition 2.10 Fix a positive integer n, and let $\lambda \in \mathbb{Y}$ have length $\leq n$. Let Γ_n denote the totally ordered set $\{1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}\}$. The *semi-infinite complementary shape* to λ is the following convex subset of $\mathbb{Z} \times \mathbb{Z}$:

$$S_\lambda = \{(i,j) | 1 \leq i \leq n, \; -\infty < j_i < -\lambda_{n+1-i}\}$$

A *semi-infinite* $\mathfrak{sp}(2n, \mathbb{C})$ *tableau* of *complementary shape* λ is a map $T : S_\lambda \to \Gamma_n$, satisfying the following conditions:

1. For each $i \in [n]$, there exists $N_i \in \mathbb{Z}$ such that $T(i,j) = i$ whenever $j < N_i$.

2. Each row is weakly increasing: $T(i,j) \leq T(i, j+1)$ whenever both are defined.

3. Each column is strictly increasing: $T(i,j) < T(i+1, j)$ whenever both are defined.

Note that the first two conditions imply that $T(i,j) \geq i$, which is part of the usual definition of finite symplectic tableaux.

We write $\operatorname{sch}(T) = \lambda$ to indicate that the semi-infinite tableau T has semi-infinite complementary shape λ, and denote the set of all SIST's of that shape by $\mathrm{SIST}(\lambda)$.

Definition 2.11 Define the *crude weight* of $T \in \mathrm{SIST}(\lambda)$ to be $\sum_{i=1}^n \alpha_i \epsilon_i$ $(\in \mathfrak{h}^*)$, which we abbreviate as $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$ where

$$\alpha_i := -(\text{number of } \bar{i}\text{'s in } T) - (\text{ number of columns of } T \text{ not containing } i) - \lambda_n.$$

Figure 2: A Semi-infinite symplectic tableau of complementary shape (9,9,4,1)

$\cdots$	1	1	1	1	1	1	1	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	2	3
$\cdots$	2	2	2	2	$\bar{2}$	$\bar{2}$	$\bar{2}$	3	3	3	3	3	$\bar{3}$			
$\cdots$	3	3	3	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	4								
$\cdots$	4	4	4	4	4	$\bar{4}$	$\bar{4}$	$\bar{4}$								

The idea is that, as for the usual symplectic tableaux, the letter i (resp. $\bar{i}$) has weight ϵ_i (resp. $-\epsilon_i$), and that the reference tableau (whose crude weight is defined to be zero) is the one of complementary shape empty with all entries in the ith row equal to i. For example, the crude weight of the tableau in Figure 2 is $(-13, -15, -13, -14)$. Also, for each $k \in \mathbb{Z}_{>0}$ we define the $\frac{k}{2}$*-adjusted weight* of T to be the crude weight of $T - \frac{k}{2}\epsilon_1 - \frac{k}{2}\epsilon_2 - \cdots - \frac{k}{2}\epsilon_n$. For example, the $\frac{7}{2}$-adjusted weight of the tableau in Figure 2 is $(-\frac{33}{2}, -\frac{37}{2}, -\frac{33}{2}, -\frac{35}{2})$.

Lemma 2.12 ([TY]) *If $\lambda \in \mathbb{Y}$ with $l(\lambda) \leq n$ and $k \in \mathbb{Z}_{>0}$, then the weight generating function of* $\mathrm{SIST}(\lambda)$ *for the $\frac{k}{2}$-adjusted weight equals the formal character of $N_{\Lambda(\lambda,k)}$. In particular, if $\Lambda(\lambda, k) \in C_1$, this also equals the formal character of $L_{\Lambda(\lambda,k)}$.*

3 Insertion

Using the duality (2.7 (1)) and some knowledge of $O(k, \mathbb{C})$-modules, it is not difficult to see the decomposition (2) holds. The goal of this section is to give an insertion scheme which shows that the formal characters of both sides of (2) are equal.

Definition 3.1 We define an insertion operator

$$I_i : \mathrm{SIST}(\lambda) \longrightarrow \coprod_{\mu} \mathrm{SIST}(\mu)$$

where μ will turn out to be a shape such that μ/λ is a single box as follows. Remove the rightmost entry in row i equal to i, making it a hole. Start sliding by jeu de taquin moving letters in the northwest direction, until a hole is left on the boundary of the SIST. (Or equivalently one could regard the slide starting from infinite left in the ith row.) The resulting tableau is a SIST because the sliding process preserves semistandardness. μ/λ is one box since the tableau loses one box of its shape at its outer corner.

Lemma 3.2 *Fix $\lambda \in \mathbb{Y}$ with $l(\lambda) \leq n$. Then the map*

$$\mathrm{SIST}(\lambda) \times [1, n] \to \coprod_{\mu} \mathrm{SIST}(\mu)$$

defined by $(T, i) \mapsto I_i(T)$ is a weight-preserving bijection, where μ runs over partitions $\mu \in \mathbb{Y}$ with $l(\mu) \leq n$ and such that μ/λ is one box, and the SIST's are assigned their crude weights, and the weight of $i \in [1, n]$ is defined to be $-\epsilon_i$.

Proof: By a note in Definition 3.1, the map is well-defined. The candidate for the inverse map is the slide, which moves letters in the southeast direction, starting from the hole at μ/λ. It is not difficult to see that this sliding process eventually converges in some single row i, and the output of this inverse candidate will be the pair consisting of the resulting tableau and the letter i. From known properties of sliding, it is again not difficult to see that this indeed gives the inverse map. The forward map forces one i to be lost in the tableaux, so by the definition of crude weight the map is weight-preserving. $\square$

Example 3.3 Consider $I_1(T)$ where $T =$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & 1 & \bar{1} & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \; .$$

We show the sliding step by step:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & & \bar{1} & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & & \bar{2} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & \bar{2} & & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-5} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & 1 & \bar{1} & 2 & 2 & \bar{2} & 3 \\ \hline \cdots & 2 & 2 & \bar{2} & \bar{3} & \bar{3} \\ \cline{1-6} \cdots & 3 & 3 & \bar{3} & \\ \cline{1-5} \end{array} \; .$$

$I_1(T)$ is the final tableau in the sequence above, where we forget about the empty square.

Definition 3.4 Let I_i^m denote the operator which repeats I_i m times, and for $\alpha \in \mathbb{Z}^n_{\geq 0}$, set $I^\alpha(T) = I_n^{\alpha_n} \circ I_{n-1}^{\alpha_{n-1}} \circ \cdots I_1^{\alpha_1}(T)$. In other words, $I^\alpha(T)$ is the map which successively slides α_1 1's to the left in T, α_2 2's to the left in T, and so on.

Our main theorem is as follows.

Theorem 3.5 *Let $\lambda \in \mathbb{Y}$ with $l(\lambda) \leq n$. Then the map*

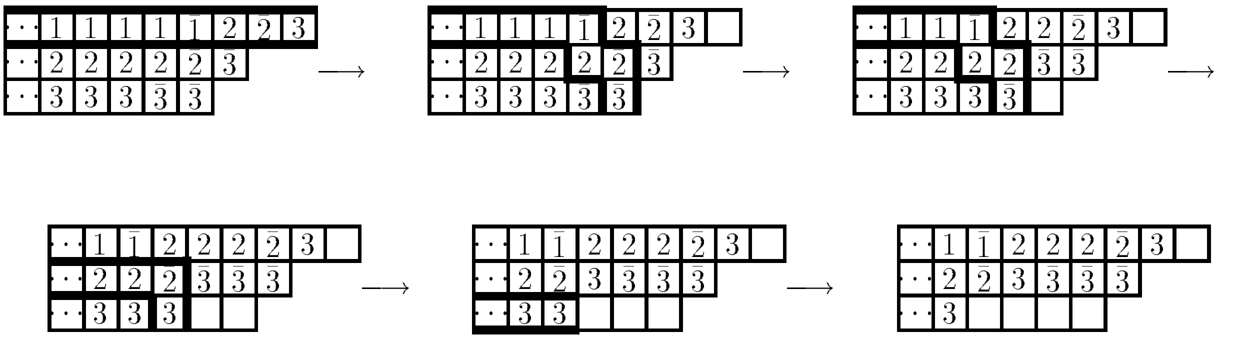
$$\mathrm{SIST}(\lambda) \times \mathbb{Z}^n_{\geq 0} \xrightarrow{\sim} \coprod_{\mu} \mathrm{SIST}(\mu)$$

defined by $(T, \alpha) \mapsto I^\alpha(T)$ is a weight-preserving bijection, where μ runs over partitions $\mu \in \mathbb{Y}$ with $l(\mu) \leq n$ and such that μ/λ is a horizontal strip, and the SIST's and the n-tuples are assigned their crude weights.

Example 3.6 Consider $I^{(3,1,1)}(T)$ where $T =$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \cdot\cdot & 1 & 1 & 1 & 1 & \bar{1} & 2 & \bar{2} & 3 \\ \hline \cdot\cdot & 2 & 2 & 2 & 2 & \bar{2} & \bar{3} \\ \cline{1-7} \cdot\cdot & 3 & 3 & 3 & \bar{3} & \bar{3} \\ \cline{1-6} \end{array} \quad .$$

We show the effect of successively inserting $1, 1, 1, 2, 3$, and indicate the sliding path by highlighting the sides of the squares involved. The sliding between the second and third tableaux is the same as that shown in Example 3.3 above.



Proof: [of Theorem 3.5] First note the following

Lemma 3.7 *Let P be a skew tableau, and let C_1, C_2, ..., C_l be a sequence of squares such that C_i is an inner (resp. outer) cocorner of the skew shape* $\mathrm{shape}(P) \cup C_1 \cup \cdots \cup C_{i-1}$. *Recursively define P_i by $P_0 = P$ and P_i to be the result of the sliding which moves letters northwest (resp. southeast) performed on P_{i-1} starting from the hole at C_i. Let D_i be the square lost at the end of the sliding from P_{i-1} to P_i. If the C_i form a horizontal strip from right to left (resp. left to right), then the D_i also form a horizontal strip from right to left (resp. right to left).*

Proof: By a result of M. Haiman ([Hai], Lemma 2.7), we can exchange the role of the skew tableau which the sliding process actually moves and the skew tableau which designates the order of the starting positions of the sliding. Therefore it suffices to show that, if a sliding is performed on a horizontal strip tableau numbered from left to right, then the result is also a horizontal strip tableau numbered from left to right. This is a well known fact. $\square$

(Continuation of Proof of Theorem 3.5) First we show that the result of $I^{\alpha}(T)$ has shape complementary to μ with μ/λ being a horizontal strip. We can cut off the infinite stable part of T (leaving some of its stable part) to obtain a finite skew tableau T^0 whose inner shape is such that we can put a horizontal strip, adjacent to the inner shape of T^0, with α_i boxes in row i. Performing slides (moving letters northwest) along this horizontal strip, right to left, has the same effect as I^{α}, and by Lemma 3.7 we know that μ/λ is a horitontal strip (actually created from right to left).

Next we define a candidate for the inverse map. Let U be a SIST which belongs to the left-hand side. We can cut off the stable part, similarly to the above, in such a way that the lengths of adjacent rows of the inner shape differ by at least $|\mu/\lambda|$. We perform slides (moving letters southeast) along the horizontal strip μ/λ from right to left. By Lemma 3.7, the inner boundary of U moves southeast by a horizontal strip (vacation being created from right to left). Denote by α_i the number of boxes in this horizontal strip lying in row i. Then we see that if we iterate the inverse map of Lemma 3.2 to U in such a way that the outer shape loses μ/λ from right to left, then the letters obtained are α_n n's, α_{n-1} $(n-1)$'s, ..., α_1 1's, in this order. The output of our inverse candidate to be defined by the pair formed by the SIST obtained at the end of this iteration of the inverse to Lemma 3.2, and the n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Now from the fact that map described as the inverse candidate in the proof of Lemma 3.2 is really the inverse, our map (in Theorem 3.5) and our inverse candidate (in this proof) are also inverses to each other. It preserves weights because it is an iteration of the weight-preserving map in Lemma 3.2 (and because of the definition of crude weights of n-tuples). □

Corollary 3.8 *Let $k \in \mathbb{Z}_{\geq 2}$, and suppose that $\lambda \in \mathcal{L}_k$ and $\Lambda(\lambda, k) \in C_1$. Then all μ appearing on the right-hand side of Theorem 3.4 belong to $\mathcal{L}_{k+1}$, and for those μ we have $\Lambda(\mu, k+1) \in C_1$. Moreover, the map in Theorem 3.4 is also weight-preserving if we assign $\frac{k}{2}$-adjusted weights to* $\mathrm{SIST}(\lambda)$, *$\frac{1}{2}$-adjusted weights to $\mathbb{Z}^n_{\geq 0}$, and $\frac{k+1}{2}$-adjusted weights to* $\mathrm{SIST}(\mu)$. *Thus interpreted, this bijection gives the formal character identity for* (2).

Proof: If $\Lambda(\lambda, k) \in C_1$, then $\Lambda(\mu, k+1) \in C_1$ for any $\mu \supset \lambda$. The condition $\mu \in \mathcal{L}_{k+1}$ is more subtle. $l(\mu) \leq n$ is assured from the beginning (Theorem 3.5), so we only have to check if $\mu'_1 + \mu'_2 \leq k+1$ holds. If $k+1 \geq 2n$, this is satisfied automatically. So suppose $k+1 \leq 2n-1$, or equivalently $\frac{k}{2} \leq n-1$. Now the condition $\Lambda(\lambda, k) \in C_1$ implies $\lambda_n + \frac{k}{2} > n$, so that we have $\lambda_n \geq 2$. This implies $\lambda'_1 + \lambda'_2 = 2n$, so that if $\lambda \in \mathcal{L}_k$, we must have $2n \leq k$, which is a contradiction. Therefore the case $k+1 \geq 2n$ never occurs, and we have $\mu \in \mathcal{L}_{k+1}$.

Since the map of Theorem 3.4 is weight-preserving for crude weights, it is also weight-preserving for the adjusted weights because the adjustments match. Using Lemma 2.7 and 2.9, and comparing with (2), we see that the weight generating function identity (for the adjusted weights) resulting from this bijection coincides with the formal character identity for (2). □

Definition 3.9 Following the idea of what R. Proctor calls N-orthogonal tableaux [Pro], we propose a skew semistandard tableau of shape μ/λ with entries from $[l+1, k] = \{l+1, l+2, \ldots, k\}$ to be called a *skew $O(l)/O(k)$-tableau* if it satisfies what he calls the q-orthogonal condition for $q = k, k+1, \ldots, l$ in the following sense. For such q, let $\lambda^{(q)}$ denote the shape containing λ and all letter $\leq q$. Thus $\lambda = \lambda^{(k)} \subset \lambda^{(k+1)} \subset \cdots \subset \lambda^{(l)} = \mu$, and the differences of adjacent terms are horizontal strips. The q-orthogonal condition requires that the sum of the first two columns of $\lambda^{(q)}$ is at most q. Let $O(l)/O(k)(\mu/\lambda)$ denote the set of all $O(l)/O(k)$-tableaux of shape μ/λ.

Corollary 3.10 *Let $k \in \mathbb{Z}_{\geq 2}$, and let $\lambda \in \mathcal{L}_k$. Also assume that $\Lambda(\lambda, k) \in C_1$. Iterating the map in Theorem 3.4, we obtain a weight-preserving bijection*

$$\mathrm{SIST}(\lambda) \times \left(\mathbb{Z}^n_{\geq 0}\right)^{k'} \xrightarrow{\sim} \coprod_{\mu} \mathrm{SIST}(\mu) \times O(k+k')/O(k)(\mu/\lambda)$$

where μ ranges over all partitions with $l(\mu) \leq n$. The SIST-component of the output is the result of inserting (by Lemma 3.2) k' n-tuples, and the skew-tableau component of the output is the one that records the complementary shapes of SIST's after each insertion of an n-tuple, namely the shape $\lambda^{(k+i)}$ is the shape complementary to the SIST after inserting the ith n-tuple.

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