Description trees for some families of planar maps.

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Abstract

In this paper, we introduce *description trees*, to give a general framework for the recursive decompositions of several families of planar maps studied by W.T. Tutte. These trees reflect the combinatorial structure of the decompositions and carry out various combinatorial parameters. We also introduce *left regular trees* as canonical representants of some new *conjugacy classes* on planted plane trees. We give an enumeration formula for these trees. In several cases the combination of these two ingredients yields a purely combinatorial proof of some elegant formulae of W.T. Tutte and gives uniform random generation algorithms for the corresponding planar maps.

Rsum

Dans cet article, nous introduisons des *arbres de description* pour coder la dcomposition rcursive de plusieurs familles de cartes planaires tudies par W.T. Tutte. Ces arbres refltent la structure combinatoire de la dcomposition et portent diffrents paramtres combinatoires. Nous introduisons aussi les arbres rguliers gauches qui sont des représentants canoniques de nouvelles classes de conjugauisons d'arbre plan plantés. Nous donnons une formule d'numration pour ces arbres. Dans plusieurs cas la combinaison de ces deux ingrdients aboutit la preuve combinatoire de certaines formules de W.T. Tutte et donne des algorithmes de gnration alatoire uniforme pour les cartes planaires associes.

Introduction

In this paper, a *valuated tree* is a rooted plane tree with nonnegative integer labels on its vertices. The size of a tree T, denoted $|T|$, is its number of vertices. The label of a vertex v is denoted $\lambda(v)$ and its arity $a(v)$, is the number of its sons. The sequence of subtrees under v (the tree sons of v) is denoted by $\sigma(v) = (\sigma_1(v), \ldots, \sigma_k(v))$, where $k = a(v)$. If $a(v) = 0$, then $\sigma(v) = \emptyset$ and the vertex v is a *leaf*. Otherwise it is an *internal node*.

A *description tree* with rule $\mathcal{R}$ is a valuated tree such that the label of each vertex v belongs to a set of values that depends only on the labels of the tree sons of v according to the given rule $\mathcal{R}$.

A *map* is a partition of a compact oriented surface into three finite sets: a set of *vertices* (points), a set of *edges* (disjoint Jordan arcs) and a set of *faces* (disjoint simply connected domains). The *genus* of a map is the genus of the corresponding surface. Maps are considered up to orientation preserving homeomorphisms of the surface. Such maps can be defined combinatorially, without any reference to surfaces, either in terms of slicings or in terms of permutations. We shall be mainly interested in *planar* map, i.e. in the case of the sphere. Following W.T. Tutte, we shall eliminate symmetries by considering *rooted maps*, that is maps with an oriented edge, which therefore defines a root-vertex and a root-face.

Our aim is to illustrate how recursive decompositions of maps induced by root-edge deletion can be encoded into description trees. The technique of root-edge deletion was used by W.G. Brown and W.T. Tutte in the 60's to enumerate several families of planar maps together with several other combinatorial transformations. In the first part of this article we show that with only two main rules for description trees, one can encode several classical families of maps and that several combinatorial parameters of these maps are taken into account. In the second part, we give functional equations for the generating functions of our trees and we use the quadratic method to give parameterizations which contain those of W.T. Tutte as special cases.

In the last part, we introduce *conjugacy classes* of planted regular plane trees. Using Lukacievicz language we then define *left regular trees*. We obtain a combinatorial enumeration of these trees and in four special cases, we give bijections between left regular trees and description trees. This allows us to give purely combinatorial proofs of some nice formulae of W.T. Tutte. In particular these proofs do not rely any more on the quadratic method and yield uniform random generation algorithms of reasonable complexity (mean complexity in $n^{3/2}$, worst case in n^2) for planar maps, Eulerian maps, nonseparable maps and nonseparable cubic maps.

In [GW96], a bijection between nonseparable planar maps and two-stack-sortable permutations is obtained, via a family of trees called decomposition trees by the authors. This family of trees can easily be related to our description trees of type $\beta(1,0)$ (see further). Hence our purely combinatorial proof of W.G. Brown and W.T. Tutte's formulae for nonseparable planar maps extends to a combinatorial proof of the enumeration of two-stack-sortable permutations. Moreover, in this case, the bijection with left ternary trees can be refined to take in account more parameters and to give a combinatorial proof of W.G. Brown and W.T. Tutte formula for the number of nonseparable planar maps with i vertices and j faces. Once again this result extends to two-stack-sortable permutations (with i ascents and j descents).

In [Bón], our description trees of type $\beta(0,1)$ are used to derive the number of permutations of $[n]$ avoiding the pattern 1342.

1 Description trees and planar maps

1.1 α-description trees and planar maps

Let α-description trees of type (a,b) be description trees satisfying the following rules:

$\alpha(a,b)$-leaves: if v is a leaf, $\lambda(v) = b$.

$\alpha(a,b)$-nodes: if v is an internal node, and $\lambda_1, \dots, \lambda_k$ are the labels of its sons,

$$a + \sum_{i=1}^{k-1} \lambda_i \quad \leqslant \quad \lambda(v) \quad \leqslant \quad b + \sum_{i=1}^{k} \lambda_i.$$

Let $\mathcal{D}^{\alpha(a,b)}$ denote the set of α-description trees of type (a,b) and for all $n \geqslant 0$:

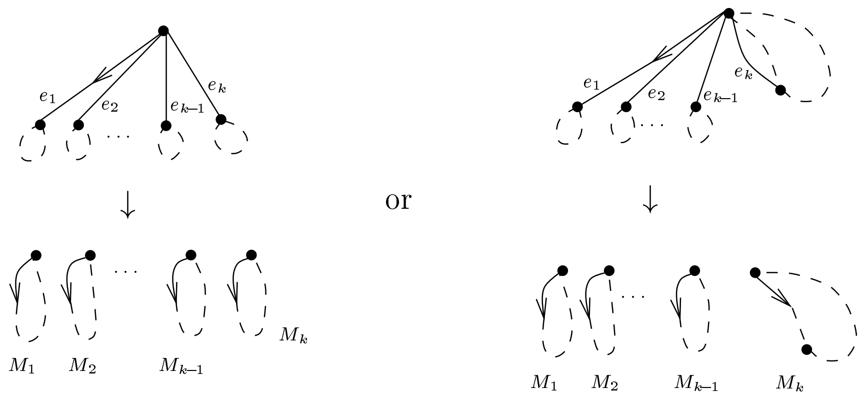
$$\mathcal{D}_n^{\alpha(a,b)} = \{T \in \mathcal{D}^{\alpha(a,b)} \mid |T| = n\}.$$

Theorem 1 *The following families of maps are encoded by α-description trees of size $n+1$ of the following types:*

$\mathcal{M}_n$: *planar maps with n edges, type* $(3,2)$.

$\mathcal{E}_n$: *Eulerian planar maps with n edges (or dually bipartite planar maps), type* $(2,1)$.

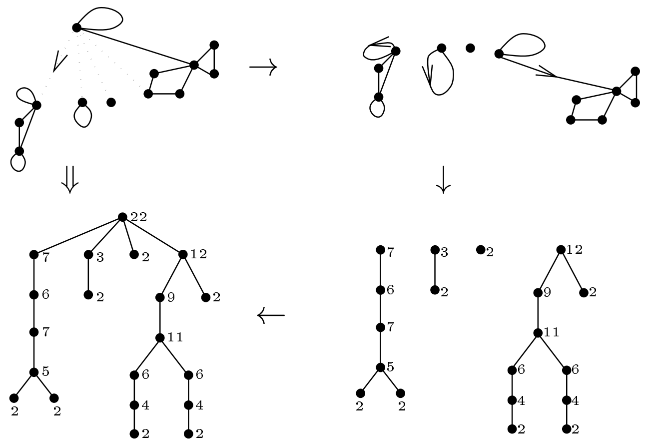
The underlying decomposition is derived from [Tut63]. Let M be a rooted planar map with $|M|$ edges and denote by $r(M)$ the degree of the root-face. The decomposition is recursive, hence we suppose that we know how to encode smaller maps. In particular, the map with no edge is encoded by the tree reduced to one vertex with label 0. The map M is then decomposed as follows:



where $e_1, \dots, e_k$ is the longest sequence of distinct edges around the root-vertex such that $e_1, \dots, e_{k-1}$ are bridges. Removing e_i for $i \leqslant k-1$ gives two components, one of which is M_i. The map M_k is obtained after the deletion of $e_1, \dots, e_k$ and $M_1, \dots, M_{k-1}$. The maps $M_1, \dots, M_k$ are smaller maps hence can be recursively encoded into description trees $T_1, \dots, T_k$ and $\lambda(T_i) = r(M_i)+2$. However to recover the original map, the end point of the edge e_k in M_k is needed. But the planarity condition forces this last edge to end on the exterior face of M_k. Hence the number of possible end points is $r(M_k)+2$. Suppose that the pth possible end point is chosen, then $r(M) = \sum_{i=1}^{k-1}(r(M_i)+2)+p$ with $1 \leqslant p \leqslant r(M_k)+2$.

The map M is then encoded by the tree T with root label $\lambda(T) = r(M)+2$, whose root-vertex has k tree sons, $T_1, \dots, T_k$.

We illustrate the recursive encoding:



A label is a *full label* if it is equal to $b + \sum_{i=1}^{k} \lambda_i$. When a label is not full, its *fall* is $(b + \sum \lambda_i) - \lambda$. The following parameters of the maps are coded in the corresponding description trees by:

maps	description trees
number of edges	number of vertices -1
number of vertices	number of vertices with full labels
number of faces	number of vertices with non-full labels $+1$
non-root-face degrees	falls of non-full labels
root-face degree	root-label -2

The second part of the theorem is obtained by restricting to even labels and dividing all labels by two. Type $(3,2)$ becomes type $(3/2,1)$ hence $(2,1)$ since all labels are integers.

1.2 β-description trees and non-separability

Let β-description trees of type (a,b) be description trees satisfying the following rules:

$\beta(a,b)$-leaves: if v is a leaf, $\lambda(v) = a$.

$\beta(a,b)$-non-root-nodes: if v is an internal node, and $\lambda_1, \ldots, \lambda_k$ are the labels of its sons, then

$$a \quad \leqslant \quad \lambda(v) \quad \leqslant \quad b + \sum_{i=1}^{k} \lambda_i.$$

$\beta(a,b)$-root-vertex: if v is the root-vertex and $\lambda_1, \ldots, \lambda_k$ are the labels of its sons, then $\lambda(v) = \sum_{i=1}^{k} \lambda_i$.

Let $\mathcal{D}^{\beta(a,b)}$ denote the set of β-description trees of type (a,b) and for all $n \geqslant 0$:

$$\mathcal{D}_n^{\beta(a,b)} = \{T \in \mathcal{D}^{\beta(a,b)} \mid |T| = n\}.$$

Theorem 2 *The following families of maps are encoded by β-description trees of size $n+1$ of the following types:*

$\mathcal{NS}_{n+1}$: *nonseparable planar maps with $n+1$ edges, type* $(1,0)$.

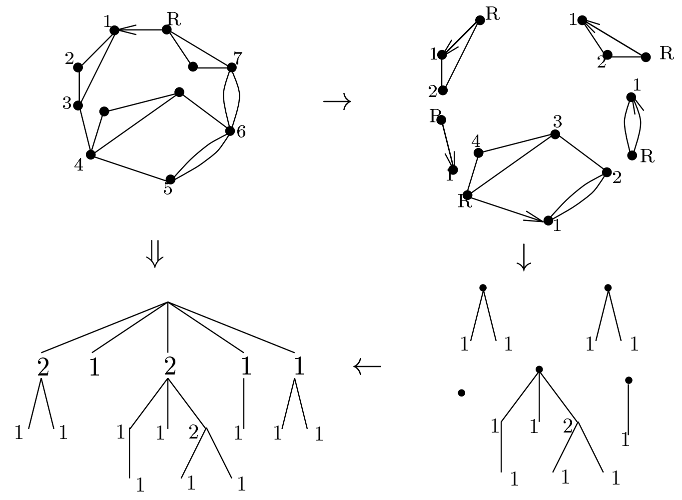
$\mathcal{NSC}_n$: *cubic nonseparable planar maps with $3n$ edges, type* $(2,2)$.

$\mathcal{BC}_n$: *2-colorable cubic planar maps with $3n$ edges, type* $(0,1)$.

$\mathcal{TC}_{n+1}$: *3-connected cubic planar maps with $3(n+1)$ edges, type* $(1,1)$.

Each of the four results involves a variation of W.G. Brown's decomposition. The common idea is that the deletion of the root edge merges two faces hence creating a sequence of separators (which can be vertices, edges or couples of edges, depending on the family of maps). This induces a sequence of nonseparable components on which the recursion can be applied after some auxiliary transformations.

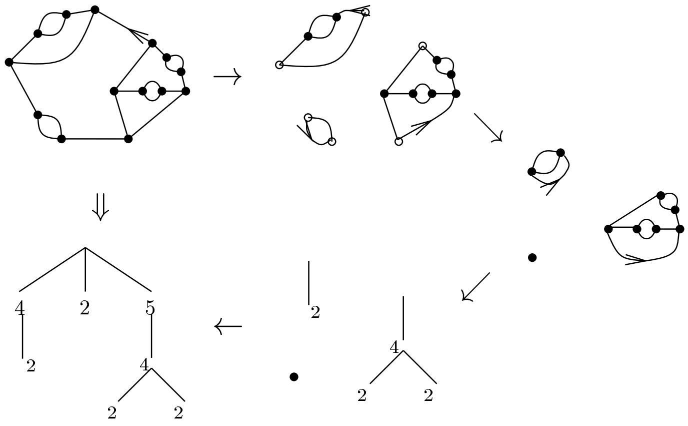
nonseparable planar maps: the underlying decomposition is exactly W.G. Brown's [Bro63]. See also [JS96] for more details. The separators are cut-vertices and the components are nonseparable maps with a marked vertex on the root-face. The position of these marked vertices can be directly encoded by the labels and the auxiliary transformation simply consists in forgetting the position of this vertex.



The following parameters of maps are encoded into the corresponding trees:

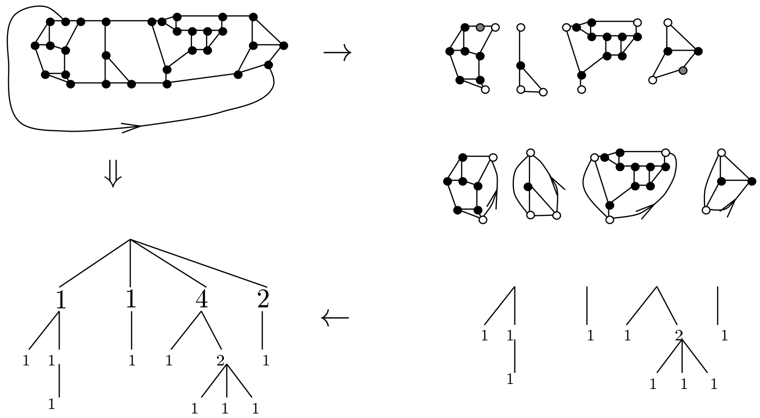
maps	description trees
number of edges	number of vertices
number of vertices	number leaves +1
number of faces	number of internal nodes +1
root-face degree	root-label +1

cubic nonseparable planar maps: this decomposition is a variation on the previous one. Indeed, decomposing nonseparable planar maps yields nonseparable components which are only near-cubic but can be converted into cubic maps by an auxiliary smoothing transformation. In this form, this decomposition is new: the number of these maps was derived in [Tut62] by introducing the notion of core of a cubic nonseparable map.



2-colorable cubic maps: Since 2-colorable cubic maps are also nonseparable, the same method can be carried out. It is however more difficult to describe because of the auxiliary transformation which is less round about.

3-connected cubic planar maps: as far as we know this decomposition is new. It is however based on the same principle, although details are more complicated. Here the separators are 2-separators, (i.e. pairs of edges) and the auxiliary transformation is similar to that of cubic nonseparable maps.



2 Generating functions and quadratic equations

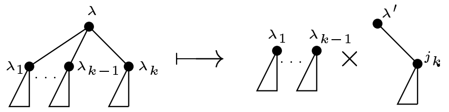
2.1 α-description trees

Let $\zeta(x,y)$ be the generating function of description trees of parameter $\alpha(a,b)$ where x and y respectively count the size of the tree and the label of the root-vertex:

$$\zeta(x,y) = \sum_{n\geqslant 1} \sum_{T\in\mathcal{D}_n^{\alpha(a,b)}} x^{|T|} y^{\lambda(T)}.$$

Consider the following decomposition of an α-description tree:

$$\mathcal{D}^{\alpha(a,b)} \setminus \{.\} \;\stackrel{\sim}{\longrightarrow}\; (\mathcal{D}^{\alpha(a,b)})^{\star} \times \mathcal{D}^{*\alpha(a,b)} :$$



where $\lambda' = \lambda - \sum_{i=1}^{k-1}\lambda_i$. Since λ' can take all values between a and $b+\lambda_k$, this decomposition yields the functional equation:

$$\zeta(x,y) = xy^b + \frac{1}{1-\zeta(x,y)}\frac{x}{1-y}\left(y^a\zeta(x,1) - y^{b+1}\zeta(x,y)\right) \quad .$$

Of course, more combinatorial parameters could have been taken into account in the generating functions thus resulting in more complicated equations. In particular, the number of internal nodes or the number of nodes with full labels can be coded into the equations.

In some restricted cases, the quadratic method yields a Lagrangian parameterization of the series $\zeta(x,1)$ which is of combinatorial interest (see [GJ83] for a good survey on this method). For instance $a = b+1$ leads to the following parameterization:

$$x = t\cdot\frac{(1-at)}{(1+t)^{2-a}(1+(2-a)t)}, \tag{1}$$

$$\zeta(x,1) = t\cdot\frac{(1+(1-2a)t+(1-a)^2t^2)}{(1-at)(1+(2-a)t)}. \tag{2}$$

In particular, for $a=3$ and $a=2$ the parameterizations are equivalent to the parameterizations of W.T. Tutte for the corresponding planar maps and Lagrange inversion yields summation free formulae:

$$\left|\mathcal{D}_{n+1}^{\alpha(2,1)}\right| = 3\frac{2^{n-1}(2n)!}{n!(n+2)!}, \qquad \left|\mathcal{D}_{n+1}^{\alpha(3,2)}\right| = 2\frac{3^{n}(2n)!}{n!(n+2)!}.$$

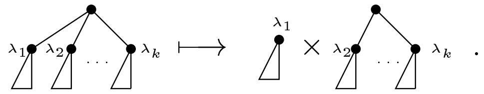
2.2 β-description trees

Let $\zeta(x,y)$ be the generating function of description trees of parameter $\beta(a,b)$ where x and y mark respectively the size of the tree and the label of the root-vertex:

$$\zeta(x,y) = \sum_{n\geqslant 1} \sum_{T\in\mathcal{D}_n^{\beta(a,b)}} x^{|T|} y^{\lambda(T)}.$$

The following decomposition of β-description trees:

$$\mathcal{D}^{\beta(a,b)} \setminus \{.\} \quad \xrightarrow{\sim} \quad \mathcal{D}^{*\beta(a,b)} \times \mathcal{D}^{\beta(a,b)} :$$



yields the functional equation:

$$\zeta(x,y) = x(1+\zeta(x,y))\left(y^a + \frac{1}{1-y}\left(y^a\zeta(x,1) - y^{b+1}\zeta(x,y)\right)\right).$$

Once again, more combinatorial parameters could have been taken into account in the generating functions.

In some restricted cases, the quadratic method yields a Lagrangian parameterization of the series $\zeta(x,1)$ which is of combinatorial interest. For instance $a = b$ leads to the following parameterization:

$$x = t \cdot \frac{(1-at)^2(1-t)^a}{(1+(1-a)t)}, \tag{3}$$

$$\zeta(x,1) = t \cdot \frac{(1+(1-2a)t+t^2)}{(1-at)^2(1-t)}. \tag{4}$$

In particular, for $a = 1$ and $a = 2$ the parameterizations are equivalent to the parameterizations of W.T. Tutte and Lagrange inversion yields summation free formulas:

$$\left|\mathcal{D}^{\beta(1,1)}_{n+1}\right| = 6\frac{(4n+1)!}{n!(3n+3)!}, \qquad \left|\mathcal{D}^{\beta(2,2)}_{n+1}\right| = \frac{2^{n+1}(3n)!}{n!(2n+2)!}.$$

For $a = 0$, $b = 1$ and $a = 1$, $b = 0$ one also gets W.T. Tutte's parameterization and formulae:

$$\left|\mathcal{D}^{\beta(1,0)}_{n+1}\right| = 4\frac{(3n)!}{n!(2n+2)!}, \qquad \left|\mathcal{D}^{\beta(0,1)}_{n+1}\right| = 3\frac{2^{n-1}(2n)!}{n!(n+2)!}.$$

These results suggest considering other values of the parameters. However we tried many combinations of the parameters a and b for α- or β-description trees and we observed the astonishing fact that no other values ($a \leqslant 6$, $b \leqslant 6$) give such simple summation free closed formulae as the quoted ones.

3 Conjugation of plane trees

A *p-ary planted tree* is a planted plane tree whose vertices have arity either p or 0 (i.e. $a(v) = p$ or $a(v) = 0$). Hence, for any internal node v, $\sigma(v) = (\sigma_1(v), \ldots, \sigma_p(v))$. From the graph theoretical point of view, vertices have degree either $p+1$ or 1.

The number of p-ary planted trees with n nodes can easily be computed using for instance the encoding into Lukacievicz words and the theorem of Dvorezki-Motzkin stated below. It is equal to

$$\frac{1}{(p-1)n+1}\binom{pn}{n}.$$

A *C-oriented p-ary planted tree* is a p-ary planted tree whose internal edges can be oriented or not, depending on the value of C: $C_0 = \{n\}$, $C_1 = \{o\}$ and $C_2 = \{n, o\}$. Let T be a p-ary planted tree with k internal edges. By definition T is a $\{n\}$-oriented p-ary planted tree and there are 2^k $\{o\}$-oriented p-ary planted trees of underlying shape T and 3^k such $\{n, o\}$-oriented p-ary planted trees.

Let $\mathcal{R}_n^{C,p}$ be the set of C-oriented p-ary planted trees with n internal nodes. Then

$$\begin{aligned}
|\mathcal{R}_n^{\{n\},p}| &= \frac{1}{(p-1)n+1}\binom{pn}{n}, \\
|\mathcal{R}_n^{\{o\},p}| &= \frac{1}{(p-1)n+1}\binom{pn}{n}2^{n-1}, \\
|\mathcal{R}_n^{\{n,o\},p}| &= \frac{1}{(p-1)n+1}\binom{pn}{n}3^{n-1}.
\end{aligned}$$

The following values of the parameters are of special interest:

$(C = \{o\}, p = 2)$: complete binary trees with oriented internal edges.

$(C = \{n, o\}, p = 2)$: complete binary trees with oriented and non-oriented internal edges.

$(C = \{n\}, p = 3)$: complete ternary trees with non-oriented edges.

$(C = \{o\}, p = 3)$: complete ternary trees with oriented edges.

Definition 1 *Two C-oriented p-ary planted trees are conjugate if they share the same underlying unplanted C-oriented tree. Equivalently they are conjugate if they differ only by the choice of the planted leaf.*

On drawings, conjugation is can be viewed as a rotation of the tree.

Since the number of leaves of a p-ary tree is $(p-1)n+2$ (including the planted one), there are at most $(p-1)n+2$ different trees in a conjugacy class. However there might be fewer trees because of rotational symetries.

3.1 Labels of leaves and border words

Let $p > 1$ be an integer and $\varepsilon = (p \bmod 2) + 1$, $\delta = \varepsilon(p-1)/2$. Let T be a C-oriented p-ary tree with n internal nodes. The prefix order on leaves defines a cyclical order on leaves, which is invariant by conjugation of the tree. Let x be a leaf and y be its successor according to this order. There is a unique path $w(y)$ going from x to y. This path is made of 2 external edges, $r(y)$ internal edges, $s(y)$ of which are properly oriented edges and $t(y)$ are counter-oriented edges. Let then $e(y) = \delta r(y) + s(y) - t(y) - \varepsilon$ be the label of the leaf y. It satisfies $e(y) \leqslant \varepsilon$ since $\delta \geqslant 1$ and $t(y) \leqslant r(y)$. These labels of leaves depend only on the underlying unplanted tree, hence are invariant by conjugation.

The prefix order of leaves induces a word $w(T) = w(y_1)w(y_2)\dots w(y_{(n-1)p+2})$ on the alphabet $\{i \mid i \geqslant -\varepsilon\}$. This word $w(T)$ characterizes the tree T. Moreover, two trees T_1 and T_2 are conjugated if and only if $w(T_1)$ and $w(T_2)$ are conjugated words.

By definition of the labels, we have

$$\begin{aligned}\sum_{i=1}^{(p-1)n+2} e(y_i) &= \sum \delta r(y) + \sum (s(y) - r(y)) - ((p-1)n+2)\varepsilon \\ &= \delta 2(n-1) + 0 - ((p-1)n+2)\varepsilon = (p+1)\varepsilon.\end{aligned}$$

Definition 2 *The C-oriented p-ary planted tree T is* ***canonical*** *if $w(T)$ is canonical (i.e. is a product of extended Lukacievicz words).*

The following theorem is then a direct consequence of Dvorezkin-Motskin theorem for Lukacievicz words:

Theorem 3 *The number of canonical C-oriented p-ary trees is*

$$\frac{(p+1)\varepsilon}{(p-1)n+2}\frac{1}{(p-1)n+1}\binom{pn}{n}c^{n-1}$$

where $c = 1, 2, 3$ respectively if $C = \{n\}, \{o\}, \{n, o\}$.

In particular we have:

$$\left|\mathcal{CT}_n^{(C=\{o\},p=2)}\right| = 3\cdot 2^{n-1}\frac{(2n)!}{n!(n+2)!}, \qquad \left|\mathcal{CT}_n^{(C=\{n\},p=3)}\right| = 4\cdot\frac{(3n)!}{n!(2n+2)!},$$
$$\left|\mathcal{CT}_n^{(C=\{n,o\},p=2)}\right| = 6\cdot 3^{n-1}\frac{(2n)!}{n!(n+2)!}, \qquad \left|\mathcal{CT}_n^{(C=\{o\},p=3)}\right| = 4\cdot 2^{n-1}\frac{(3n)!}{n!(2n+2)!}.$$

3.2 Bijection between canonical trees and maps

Theorem 4 *There exist bijections between the following families of canonical regular trees and description trees:*

$$\begin{aligned}\mathcal{CT}_n^{(C=\{o\},p=2)} &\simeq \mathcal{D}_{n+1}^{\alpha(2,1)}, \\ \mathcal{CT}_n^{(C=\{n,o\},p=2)} &\simeq \mathcal{D}_{n+1}^{\alpha(3,2)}, \\ \mathcal{CT}_n^{(C=\{n\},p=3)} &\simeq \mathcal{D}_{n+1}^{\beta(1,0)}, \\ \mathcal{CT}_n^{(C=\{o\},p=3)} &\simeq \mathcal{D}_{n+1}^{\beta(2,2)}.\end{aligned}$$

These bijections, yield unified bijective proof od W.T. Tutte's formulae for planar maps, Eulerian planar maps, nonseparable planar maps and cubic nonseparable maps.

This result is based on a recursive decomposition of canonical regular trees into description trees. The case ($C = \{n\}, p = 3$) was first conjectured by A. Del Lungo et al. and in this case proofs were independently found in [JS96] and [DLDRP97].

The four bijection are based on the same principle. The case ($C = \{n\}, p = 3$) corresponds to nonseparable planar maps and is already described in [JS96] in a different but equivalent language. The case ($C = \{o\}, p = 2$) is completely similar to the previous one, whereas the case ($C = \{o\}, p = 3$) require more attention since the second parameter of the decomposition is less straightforward to describe. The case ($C = \{n, o\}, p = 2$) corresponds to planar maps and these maps are already treated by a direct method in [Sch97]. Therefore we give details only for the more third case.

3.3 charabia

Let T be a canonical $\{o\}$-oriented 3-ary tree with n internal vertices. We define first the parameter $l(T)$ which is going to play the role of labels in the description trees of type $\beta(2,2)$. The word $w(T)$ is canonical hence $w(T) = w_1w_2w_3w_4$ where all w_i are standard Lukacievicz words.

Lemma 1 *Let x be an internal edge of T. Since x is oriented, cutting x yields two planted trees: T_1 planted at the head of x and T_2 planted at the tail of x. If $w(T_1) = a_1w_1b_1$ and $w(T_2) = a_2w_2b_2$ where a_i and b_i are letters, $w(T)$ is conjugated to $(b_2+a_1+3)w_1(b_1+a_2+1)w_2$ as illustrated on figure* **??**.

Let X be the set of internal edges x such that the root-leaf of T belongs to T_2. Therefore $w(T) = w_{12}(b_1 + a_2 + 1)w_2(b_2 + a_1 + 3)w_{11}$. Let T_1' be the conjugate of T_1 which is planted on the same leaf as T. We have $w(T_1') = w_{12}b_1b_2w_{11}$.

Let then $X_1 \subset X$ be the set of edges x such that:

- $w(T_2)$ is canonical and the word $w_{12}b_1$ ends at level -3.

and $X_2 \subset X$ be the set of edges x such that

- $w(T_2)$ is almost canonical (ie $\forall uv = w(T_2),\ d(u) \geqslant 4$) and the word $w_{12}b_1$ ends at level -2 and none of its left factor reaches level -3.

Then the mapping $(T, x) \mapsto (T_1, T_2)$ defines a bijection from $\{(T, x) \mid T \in CT_n,\ x \in X_1 \cup X_2\}$ onto $\{T' \mid T' \in CT_{n+1},\ w(T') = w_0(-1)\}$.

Moreover the prefix order on edges defines an order on elements x of $X_1 \cup X_2$ and if $y \in X_1(T) \cup X_2(T)$ then ($y \in X_1(T') \cup X_2(T')$ iff $y < x$).

Therefore $r(T') \geqslant r(T)$.

3.4 Proof

Let $\mathcal{CT}_n = \mathcal{CT}_n^{\{o\},3}$. There is an immediate decomposition of $T \in \mathcal{CT}_n$ as follows: T is planted and the top internal vertex v has three (possibly empty) tree-sons T_1, T_2 and T_3. Since T is canonical, either T_3 is empty or the internal edge linking v to T_3 is oriented toward T_3 and T_3 is canonical. Let $\mathcal{CT}_n^*$ be the set of trees such that T_3 is empty. Now by iterating the previous decomposition, the following lemma is obvious.

Lemma 2 *There is a bijection between elements of $\mathcal{CT}$ and sequences of elements of $\mathcal{CT}^*$ such that their total number of internal vertices are the same.*

Proposition 1 *For any tree $T \in \mathcal{CT}_n$ we can define a sequence $X(T)$ of edges such that the mapping X satifies the following conditions:*

- *If $T = (T_1, \dots, T_k)$ is the decomposition of T into a sequence of elements of $\mathcal{CT}^*$ then*
$$X(T) = X(T_1), a_1, b_1, X(T_2)a_2, b_2, \dots, a_{k-1}, b_{k-1}, X(T_k).$$
- *For $n \geqslant 1$, there is a bijection*
$$\phi : \{(T, i) \mid T \in \mathcal{CT}_{n-1},\ 1 \leqslant i \leqslant |X(T)|\} \longrightarrow \mathcal{CT}_n^*$$
such that if $T' = \phi(T, i)$ then $X(T')$ is the restriction of $X(T)$ to its i first elements.
- *If T_0 is the unique tree with one internal vertex, $X(T_0)$ is empty.*

Corollary 1 *Then a tree T of $\mathcal{CT}_n$ can be recursively decomposed and encoded into a description tree D with n vertex with root label $\lambda(D) = |X(T)| + 2$.*

Proof of the corollary. Suppose that the bijection is known for smaller trees and let $T = (T'_1, \dots, T'_k)$ and $(T_i, x_i) = \phi^{-1}(T'_i)$. Then each tree T_i can be encoded by a description tree D_i according to the induction hypothesis and $\lambda(D_i) = |X(T_i)| + 2$. Let then D be the description tree whose root has k sons, $D_1, \dots, D_k$ with respective root label $\lambda_i = |X(T'_i)| + 2$ instead of $\lambda(D_i)$. Since by construction $|X(T_i)'| \leqslant |X(T_i)|$, we have $2 \leqslant \lambda_i \leqslant \lambda(D_i) + 2$, which is exactly the $\beta(2, 2)$ non-root node condition. Moreover let $\lambda(D) = |X(T)| + 2$. According to proposition **??** $|X(T)| = \sum(|X(T_i)| + 2) - 2$. Therefore $\lambda(D) = \sum \lambda_i$ which is the root-vertex condition. Finally $X(T_0)$ is empty, hence $|X(T_0)| + 2 = 2$ and the leaf-condition is also satisfied. □

Proof of the proposition.

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