# Murnaghan-Nakayama and Littlewood-Richardson type rules for Kazhdan-Lusztig representations of Coxeter groups.

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**Abstract.** We present recursive formulas for computing Kazhdan-Lusztig characters of Hecke algebras of Coxeter groups. These formulas hold for products of elements of parabolic subgroups in an arbitrary Coxeter group.

The first formula is obtained by direct calculations of the action on the Kazhdan-Lusztig basis. This formula yields a new combinatorial interpretation (of Murnaghan-Nakayama type) for the characters of the Iwahori-Hecke algebras. In particular, it suggests a new proof to the Murnaghan-Nakayama rule.

The second formula is obtained by generalizing theorems of Barbasch and Vogan. These theorems give beautiful rules for restricting and inducing Kazhdan-Lusztig representations of Weyl groups. We show that these results imply and generalize the Littlewood-Richardson rule for decomposing outer products of representations of the symmetric groups. A new recursive rule for computing characters of arbitrary Coxeter groups follows from this observation. Another application is a generalization of Garsia-Remmel's algorithm for decomposing tensor products of symmetric groups representations to other Weyl groups.

**Résumé.** On présente des formules de récurrence pour le calcul des caractères de Kazhdan-Lusztig des algèbres de Hecke pour les groupes de Coxeter. Ces formules sont valables pour les produits d'éléments de sous-groupes paraboliques dans un groupe de Coxeter quelconque.

La première formule s'obtient par un calcul direct de l'action sur la base de Kazhdan-Lusztig. Cette formule donne une nouvelle interprétation (de type Murnaghan-Nakayama) des caractères des algèbres d'Iwahori-Hecke. En particulier, elle suggère une nouvelle démonstration de la règle de Murnaghan-Nakayama.

La deuxième formule s'obtient en généralisant des théorèmes de Barbasch et Vogan. Ces théorèmes donnent de jolies règles pour la restriction et pour l'induction des représentations de Kazhdan-Lusztig pour les groupes de Weyl. Nous démontrons que ces règles impliquent et généralisent la règle de Littlewood-Richardson pour la décomposition des produits extérieurs des représentations les groupes symétriques. Une nouvelle règle récursive pour le calcul des caractères des groupes de Coxeter quelconques se déduit de cette observation. Une autre application est une généralisation de l'algorithme de Garsia-Remmel pour la décomposition des produits tensoriels des représentations des groupes symétriques à d'autres groupes de Weyl.

## 1. Introduction.

The Murnaghan-Nakayama rule is a most efficient recursive rule for computing characters of the symmetric groups. The classical proofs of this rule use Littlewood-Richardson rule, Jacobi-Trudi formula, and Schur polynomials. A new proof of this rule, using the Young orthogonal form appear in [Gr]. Recent papers generalize these proofs to Hecke algebras of type $A$, $B$ and $D$. See [CK], [HR], [KW], and [Ra1] and references therein. Unfortunately, the machinery used in these papers is not applicable for Hecke algebras of arbitrary Coxeter groups. We suggest a new approach to the Murnaghan-Nakayama rule. We use Kazhdan-Lusztig theory to prove this rule in a more general setting, i.e. arbitrary Coxeter groups and their Hecke algebras. The classical Murnaghan-Nakayama rule is obtained as a special case.

## 2. Background.

### 2.1. The Murnaghan-Nakayama rule.

Let $n$ be a positive integer. The partitions of $n$ are designated by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_k \leq \ldots \leq \lambda_2 \leq \lambda_1$ are positive integers with $n = \lambda_1 + \ldots + \lambda_k$. The set $\{(i,j) | i,j \in \mathbb{Z}$ , $0 < i \leq k$ , $0 < j \leq \lambda_i\}$ is called the diagram of $\lambda$, and denoted by $[\lambda]$. A $\lambda$-tableau is obtained by replacing each box in $[\lambda]$ by one of the integers $1, 2, \ldots, n$, allowing no repeats. A tableau is called standard if its entries increase along rows and columns.

Every partition $\lambda$ of $n$ has an associated irreducible representation of $S_n$ over $\mathbb{C}$. This representation is called the Specht module $S^\lambda$. Moreover, the set $\{S^\lambda | \lambda \text{ a partition of } n\}$ forms the complete list of irreducible representations of $S_n$ over $\mathbb{C}$.

The boundary of a diagram $[\lambda]$ consists of all boxes $(i,j) \in [\lambda]$, so that $(i+1, j+1) \notin [\lambda]$. A skew hook is a connected part of the boundary of a diagram which can be removed to leave a proper diagram. A $k$ skew hook is a skew hook having $k$ boxes.

The leg length of a skew hook $\tau$ is $ll(\tau) = (\text{the number of rows of } \tau) - 1$.

The following is a recursive method for computing the characters of $S_n$.

**Theorem 2.1. The Murnaghan-Nakayama rule**. Let $\sigma = \pi\gamma$ be the disjoint product of $\pi \in S_{n-k}$ and a cycle of length $k$, $\gamma$. Then

$$\chi^\lambda(\sigma) = \sum_\tau (-1)^{ll(\tau)} \chi^{\lambda \setminus \tau}(\pi)$$

where the sum runs over all $k$ skew hooks, $\tau$, of $[\lambda]$. An empty sum is interpreted as zero.

### 2.2. Schützenberger's version of the Littlewood-Richardson rule.

Let $\rho$ be a representation of $S_k$ and $\phi$ be a representation of $S_{n-k}$. The outer product of these representations is the representation of $S_n$ induced from the tensor product $\rho \otimes \phi$.

The Littlewood-Richardson rule gives a procedure to decompose the outer products of representations of the symmetric groups. Schützenberger's version of this rule interprets the multiplicities of the irreducible representations in terms of shuffles of Knuth equivalence classes.

Let $Q$ be a standard tableaux of shape $\lambda$. We denote $sh(Q) = \lambda$.

The Robinson-Schensted correspondence defines a bijection between permutations $w \in S_n$ and pairs of standard tableaux of same shape $(P(w), Q(w))$. We say that the permutations $w_1$ and $w_2$ are left Knuth equivalent if $P(w_1) = P(w_2)$ and denote it by $\sim_L$.

Denote $sh(P(w))$ by $sh(w)$.

For two permutations $\pi, \sigma$ let $A_{\pi,\sigma}$ be the collection of all shuffles of $\pi$ and $\sigma$, where $\pi$ and $\sigma$ are interpreted as permutations in disjoint and consequent sets of digits.

For example, $\pi = 231$ and $\sigma = 21$. Then we interpret $\sigma$ as 54 and

$$A_{231,21} = \{23154, 23514, 25314, 52314, 23541, 25341, 52341, 25431, 52431, 54231\}.$$

Let $C_{\pi,\sigma}$ be the set $\bigcup_{w \sim_L \pi, v \sim_L \sigma} A_{w,v}$.

The following is Schützenberger's version of the Littlewood-Richardson rule (see [Sc] and [GR]).

**Theorem 2.2. Schützenberger-Littlewood-Richardson rule.** Let $\mu$ be a partition of $n$, $\nu$ be a partition of $m$, and $\lambda$ be a partition of $n+m$. Let $S^\mu$, $S^\nu$ and $S^\lambda$ be the corresponding Specht modules. Denote the multiplicity of $S^\lambda$ in $S^\mu \otimes S^\nu \uparrow_{S_m \times S_n}^{S_{n+m}}$ by $c^\lambda_{\mu,\nu}$. Let $\pi$ be a permutation in $S_m$ with $sh(\pi) = \mu$, and let $\sigma$ be a permutation in $S_n$ with $sh(\sigma) = \nu$. Then $c^\lambda_{\mu,\nu}$ equals to the number of Knuth equivalence classes in $C_{\pi,\sigma}$ of shape $\lambda$.

**Corollary 2.3.** $C_{\pi,\sigma}$ is a union of Knuth equivalence classes.

It should be noted that Theorem 2.2 is equivalent to the classical Littlewood-Richardson rule [FG].

### 2.3. Hecke algebras and their Kazhdan-Lusztig representations.

Let $W$ be a Coxeter group whose set of simple reflections is $S$. A parabolic subgroup of $W$ is a subgroup generated by a subset of simple reflections $S_1 \subseteq S$. The resulting $q$ Hecke algebra of $W$, $\mathcal{H}_q(W)$, is the algebra obtained by the set of generators $\{T_s | s \in S\}$ with the following relations:

$$(T_sT_t)^{m_{s,t}} = (T_tT_s)^{m_{s,t}} \quad \text{if } s,t \in S, \text{ and } (st)^{m_{s,t}} = (ts)^{m_{s,t}}.$$

$$T_t(T_sT_t)^{m_{s,t}} = T_s(T_tT_s)^{m_{s,t}} \quad \text{if } s,t \in S, \text{ and } t(st)^{m_{s,t}} = s(ts)^{m_{s,t}}.$$

$$T_s^2 = (q-1)T_s + qT_1 \quad \text{for every } s \in S.$$

The specialization $q = 1$ gives back the group algebra of $W$.

We denote $T_v = T_{s_1}T_{s_2}\cdots T_{s_k}$ where $s_1s_2\cdots s_k$ is a reduced expression of $v$. This notation is independent of the choice of the reduced expression. The set $\{T_v | v \in W\}$ forms the natural basis of $\mathcal{H}_q(W)$.

In [KL] Kazhdan and Lusztig present a canonical basis to the the $q$ Hecke algebra $\mathcal{H}_q(W)$, which is compatible with a rich set of representations. This basis is also indexed by the elements of $W$. For any $w \in W$ denote the corresponding Kazhdan-Lusztig basis element by $C_w$.

Kazhdan and Lusztig define a transitive preorder on the Coxeter group $W$. We say that $u \leq_L v$ if $C_u$ appears in the expansion of $T_sC_v$ in the Kazhdan-Lusztig basis, for some simple reflection $s \in W$. The Kazhdan-Lusztig preorder, $\preccurlyeq_L$, is the transitive closure of $\leq_L$. The Kazhdan-Lusztig left cells are the equivalence classes under $\preccurlyeq_L$.

Let $\mathcal{C}$ be a union of left cells. Define $\mathcal{I}_\mathcal{C}$ to be the subspace of the regular representation of $\mathcal{H}$ spanned by the elements $C_x$, which satisfy $x \preccurlyeq_L w$ for some $w \in \mathcal{C}$. Let $\mathcal{J}_\mathcal{C}$ be the the subspace spanned by the elements $C_x$, which satisfy $x \preccurlyeq_L w$ for some $w \in \mathcal{C}$ but $x \notin \mathcal{C}$. Then the quotient $\mathcal{I}_\mathcal{C}/\mathcal{J}_\mathcal{C}$ is a left representation denoted by $\rho^\mathcal{C}$. We call this representation the Kazhdan-Lusztig cell representation associated with $\mathcal{C}$.

The Knuth relation plays an important role in Kazhdan-Lusztig theory. In the symmetric group case the Kazhdan-Lusztig left cells form all left irreducible representations [KL Prop. 4.1]. It turns out that the Kazhdan-Lusztig cells are the Knuth equivalence classes [Sh 1.7.2].

**Theorem 2.4.** A subset of the Kazhdan-Lusztig basis of the symmetric group indexed by an equivalence class under left Knuth relation constructs a left cell representation of $S_n$, which is isomorphic to the Specht module of the corresponding shape.

## 3. First approach: The action on the Kazhdan-Lusztig basis.

**3.1.** By direct calculations of actions on the Kazhdan-Lusztig basis we prove a recursive rule for computing characters of certain elements in Hecke algebras of arbitrary Coxeter groups (Theorems 1).

Our rule yields a new combinatorial interpretation (of Murnaghan-Nakayama type) to the characters of the Hecke algebras. These characters are presented as weighted sums of Kazhdan-Lusztig elements (Corollary 2). In the symmetric group case we obtain a new presentation of the characters as weighted sums of standard tableaux, where the weights are significantly easy to compute (no recursion is needed!) (Theorem 3). We show that this new interpretation implies the classical Murnaghan-Nakayama rule. Applications in different directions follow.

**3.2. Main results.** Let $x, v, w$ be elements in a Coxeter group $W$, $T_v$ be a corresponding element in the Hecke algebra $\mathcal{H}_q(W)$, and $C_w, C_x$ be the corresponding Kazhdan-

Lusztig elements. Denote by $T_v C_w(x)$ the coefficient of $C_x$ in $T_v C_w$, where $T_v C_w$ is expressed as a linear combination of Kazhdan-Lusztig elements.

In [Ro1] we prove the following recursive rule.

**Theorem 1.** [Ro1 Theorems 1-2] Let $W$ be a Coxeter group, and let $s_1, \ldots, s_k$ be a subset of simple reflections, so that for $1 \leq i < k$ $(s_i s_{i+1})^3 = 1$; and for $1 \leq i < k$ and $1 < j$ $s_i s_{i+j} = s_{i+j} s_i$. Let $I$ be a subset of $S$, so that for every $1 \leq i \leq k$ and every $s \in I$ $s_i s = s s_i$. Let $z$ be an element in the subgroup generated by $I$.

Then for every $w \in W$

$$T_{s_1 s_2 \cdots s_k} T_z C_w(w) = T_{s_1 s_2 \cdots s_k} C_w(w) \cdot T_z C_w(w),$$

where

$$T_{s_1 s_2 \cdots s_k} C_w(w) = \begin{cases} (-1)^{k-i_0} q^{i_0} & \exists i_0, s_i w \succ w \text{ for } 1 \leq i \leq i_0 \text{ and } s_i w \prec w \text{ for } i_0 < i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Here $\prec$ denotes the strong Bruhat order.

A Murnaghan-Nakayama type rule for characters follows.

**Corollary 2.** [Ro1 Cor. 3] Let $W$ be a Coxeter group and $\mathcal{H}_q(W)$ be its $q$ Hecke algebra. Let $\mathcal{C}$ be a finite left Kazhdan-Lusztig cell in $W$ and let $\chi_q^{\mathcal{C}}$ be the character of the representation of $\mathcal{H}_q(W)$ associated to $\mathcal{C}$.

Let $W_1, W_2, \ldots, W_t$ be parabolic subgroups of $W$ isomorphic to symmetric groups, so that $s_i s_j = s_j s_i$ for all pairs of simple reflections $\{(s_i, s_j) | s_i \in W_i, s_j \in W_j, i \neq j\}$. Let $\{s_{i1}, s_{i2}, \ldots, s_{ik}\} \subseteq S$ be the simple reflections of $W_i$, indexed so that $(s_{ij} s_{i,j+1})^3 = 1$, and let $d_i = \prod_{j=1}^{k} s_{ij}$.

Define the following functions $h_q^i$

$$h_q^i(w) = \begin{cases} (-1)^{k-j_0} q^{j_0} & \exists j_0, s_{ij} w \succ w \text{ for } 1 \leq j \leq j_0 \text{ and } s_{ij} w \prec w \text{ for } j_0 < j \leq k \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\chi_q^{\mathcal{C}}(T_{d_1 \cdot d_2 \cdots d_t}) = \sum_w \prod_{i=1}^{t} h_q^i(w)$$

where the sum runs over all $w \in \mathcal{C}$.

For Hecke algebras of type $A$ the sum in the last corollary may be interpreted as a weighted sum of standard tableaux.

**Theorem 3.** [Ro1 Theorem 4] Let $\lambda$ be a partition of $n$, and $\chi_q^\lambda$ be the corresponding character of the $q$ Hecke algebra of type $A_{n-1}$. Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be another partition of $n$, and $T_\mu$ be an element in the $q$ Hecke algebra indexed by a permutation in $A_{n-1}$ of cycle type $\mu$. Then

$$\chi_q^\lambda(T_\mu) = \sum_Q \mathrm{w}_q^\mu(Q),$$

where the sum is over all standard tableaux $Q$ of shape $\lambda$ and the $\mu$-weight of a standard tableau is given by

$$\mathrm{w}_q^\mu(Q) = \prod_{\substack{1\le i\le k\\ i\notin B(\mu)}} f_\mu(i,Q), \quad \text{where } B(\mu) = \{\mu_1 + \cdots + \mu_r \mid 1 \le r \le \ell\},$$

and

$$f_\mu(i,Q) = \begin{cases} -1 & i+1 \text{ is in the southwest of } i \\ 0 & i+1 \text{ is in the northeast of } i\ ,\ i+2 \text{ is in the southwest of } i+1 \text{ and } i+1 \notin B(\mu) \\ q & \text{otherwise} \end{cases}$$

It should be noted that Theorem 3 yields a new proof to the classical Murnaghan-Nakayama rule for the symmetric groups [Ro1 Theorem 6].

**3.3. Applications and generalizations.** The above character formulas are generalized and applied in different directions in the following papers of Ram et al.

(a) Halverson, Leduc and Ram [HLR] apply Theorem 3 to compute the bitrace of the regular representation of the Iwahori-Hecke algebra as a weighted sum of matrices with nonnegative integer entries.

(b) Ram [Ra2] proved that Theorem 3 (the type $A$ case) is equivalent to the $q$-analogue of the Frobenius formula presented in [Ra1]. This allows Ram to generalize Theorem 3 to get new character formulas for Brauer and Birman-Wenzl algebras.

## 4. Second approach: Generalizing the Littlewood-Richardson rule.

**4.1.** The Littlewood-Richardson rule gives a combinatorial procedure to decompose the outer products of representations of the symmetric groups. Schützenberger [Sc] suggested a new version to the Littlewood-Richardson rule, using the language of Knuth equivalence classes. The Knuth equivalence classes are the Kazhdan-Lusztig cells of the symmetric groups. This gives rise to the question whether it is possible to generalize Schützenberger's results to arbitrary Coxeter groups. A positive answer to this question is given in [BV], [Sri] and [Ro2].

**4.2.** Our goal is finding a general recursive rule for characters of arbitrary Coxeter groups. In the symmetric groups case the occurrence of a recursion may be observed as a result of the Littlewood-Richardson rule [J ch. 21]. In [Ro2] we show that a generalized

Littlewood-Richardson formula implies a recursion theorem for the Kazhdan-Lusztig characters arbitrary Coxeter groups. This result explains and generalizes the recursion, which was found by direct calculations (Theorem 1).

In [BV] Barbasch and Vogan prove a beautiful property of the Kazhdan-Lusztig basis of Weyl groups. They show that this basis is compatible with restriction and induction of representations in a certain sense. In [Ro2] we show that this property holds for arbitrary Coxeter groups.

**Proposition 4.** [Ro2 Thm. 5.2] Let $W$ be an arbitrary Coxeter group, and let $H$ be a parabolic subgroup of $W$. Let $R$ be the set of representatives of minimal length of the the right cosets of $H$ in $W$. Let $\mathcal{C}$ be a Kazhdan-Lusztig left cell of $W$, and $\rho^{\mathcal{C}}$ the representation of $W$ corresponding to $\mathcal{C}$.

Then $\mathcal{C}$ is a union of sets of the form $\mathcal{D}r$, where $r \in R$ and $\mathcal{D}$ is a Kazhdan-Lusztig left cell of $H$. This decomposition is associated (in Kazhdan-Lusztig sense) with the decomposition of the restricted representation $\rho^{\mathcal{C}} \downarrow_H^W$.

The following Theorem is complementary.

**Theorem 5.** [BV Prop. 3.15] Let $H$ be a parabolic subgroup of a Weyl group $W$. Let $\mathcal{C}$ a Kazhdan-Lusztig left cell of $H$ and let $\rho^{\mathcal{C}}$ be its associated Kazhdan-Lusztig representation of $H$.

Let $L$ be the set of all representatives of minimal length of the left cosets of $H$ in $W$.

Then $L\mathcal{C}$ is a union of Kazhdan-Lusztig left cells. The associated Kazhdan-Lusztig representation $\rho^{L\mathcal{C}}$ is isomorphic to the induced representation $\rho^{\mathcal{C}} \uparrow_H^W$.

The Littlewood-Richardson rule may be obtained as a special case of Theorem 5.

Proposition 4 implies the following recursive rule for computing characters.

**Theorem 6**. [Ro2 Theorem 3] Let $W_1$ and $W_2$ be two parabolic subgroups of an arbitrary Coxeter group $W$, so that every pair $z_1 \in W_1$ and $z_2 \in W_2$ commutes.

Let $w_1 \in W_1$, $w_2 \in W_2$, and $v \in W$, so that $v = v_1 v_2 r$ where $v_1 \in W_1$, $v_2 \in W_2$ and $r$ is a representative of minimal length of a right coset of $W_1 \times W_2$ in $W$. Then with the notations of Theorem 1

$$T_{w_1 w_2} C_v(v) = T_{w_1} C_v(v) \cdot T_{w_2} C_v(v) = T_{w_1} C_{v_1}(v_1) \cdot T_{w_2} C_{v_2}(v_2).$$

This Theorem generalizes the recursive part of Theorem 1. The following corollary is immediate.

**Corollary 7.** [Ro2 Cor. 4] With the above notations let $\mathcal{C}$ be a finite left Kazhdan-Lusztig cell of $W$, and let $\chi^{\mathcal{C}}$ be its associated character. Then

$$\chi^{\mathcal{C}}(T_{w_1 w_2}) = \sum_{v \in \mathcal{C}} T_{w_1} C_v(v) \cdot T_{w_2} C_v(v).$$

**4.3.** Garsia and Remmel [GR] apply Schützenberger's version of the Littlewood-Richardson formula to obtain a combinatorial procedure for decomposing tensor products of symmetric groups representations. An immediate consequent of Barbasch-Vogan rules is a generalization of this work to other Weyl groups.

Let $W$ be a Weyl group with $S$ the set of simple reflections. Let $T$ be a subset of $S$, and let $\langle T\rangle$ be the parabolic subgroup generated by $T$. Let $\bar{R}(T)$ the set of all elements in $W$ whose set of descents is contained in $S\backslash T$. Namely, $\bar{R}(T) = \{w \in W | \nexists s \in T, \ell(ws) < \ell(w)\}$, where $\ell$ is the length function. Let $1_T$ be the trivial representation of $\langle T\rangle$.

Let $\mathcal{C}$ be a Kazhdan-Lusztig cell in $W$, and let $\rho^{\mathcal{C}}$ be the cell representation corresponding to $\mathcal{C}$. By Barbasch-Vogan rule $\mathcal{C}$ is a union of subsets of the form $\mathcal{D}r$, where $\mathcal{D}$ is a Kazhdan-Lusztig left cell of $\langle T\rangle$, and $r$ is a shortest representative of a right coset of $\langle T\rangle$. Denote by $\mathcal{C}_T$ the union of these $\mathcal{D}$ cells.

**Proposition 8.** [Ro2 Prop. 5] The set $\bar{R}(T)\mathcal{C}_T$ is a union of Kazhdan-Lusztig cells in $W$. This union is associated (in the Kazhdan-Lusztig sense) with a representation isomorphic to $1_T \uparrow^W \otimes \rho^{\mathcal{C}}$.

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