From Motzkin to Catalan permutations: a "discrete continuity"

Elena Barcucci, Alberto Del Lungo, Elisa Pergola, Renzo Pinzani *

Abstract. A permutation π contains the subsequence of type τ iff π contains a subsequence having all the same pairwise comparisons as τ. In this paper, we consider the family of permutations $S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ where $j \geq 1$ is a parameter. $\pi \in S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ if π has length n, it does not contain a subsequence of type 321 and all its subsequences of type $(j+1)(j+2)1...j$ [i.e. $(j+2)(j+3)2...(j+1)$ restricted on $\{1, ..., j+2\}$] are of type $(j+2)1(j+3)2...(j+1)$. The family of Motzkin permutations: $M_n(321, 3\bar{1}42)$ is obtained by $S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ with $j=1$; the family of Catalan permutations $C_n(321)$ is obtained by $S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ with $j=\infty$. Any other case, $1 < j \neq \infty$, yields the inequalities: $M_n \leq |S_n(321, (j+2)\bar{1}(j+3)2...(j+1))| \leq C_n$ where M_n and C_n are the n-th Motzkin and Catalan number respectively. Let S^j be the set of all permutations with the forbidden subsequences 321 and $(j+2)\bar{1}(j+3)2...(j+1)$; we determine the functional equation satisfied by the generating function of S^j according to the length of the permutations, the number of their active sites, the number of their inversions and the number of their right minima. We give the generating function of S^j according to some of these parameters: i. e. the length of the permutations, the number of their right minima and the number of their inversions. Further we illustrate a bijection between a subclass of plane trees and the S^j permutations, for each $j \geq 1$. Finally we take the cases $j = 1, 2$ and $j = \infty$ into particular consideration as they involve the well-known Motzkin, left-Motzkin factors and Catalan numbers respectively.

Résumé. Une permutation π contient le motif de type τ si et seulement si elle contient une sousequence dont les elements sont dans la meme relation d'ordre que ceux de τ. Dans cet article nous considerons la famille des permutations $S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ où $j \geq 1$ est un paramètre. $\pi \in S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ si sa longueur est n et π ne contient aucune sousequence de type 321 et toutes ses sousequences de type $(j+1)(j+2)1...j$ [c'est à dire $(j+2)(j+3)2...(j+1)$ rapporté sur $\{1, ..., j+2\}$] sont de type $(j+2)1(j+3)2...(j+1)$. La famille des permutations de Motzkin $M_n(321, 3\bar{1}42)$ est obtenue de $S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ pour $j=1$; la famille des permutations de Catalan $C_n(321)$ est obtenue de $S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ pour $j=\infty$. Dans tous les autres cas, $1 < j \neq \infty$, les inégalites suivantes sont verifiées: $M_n \leq |S_n(321, (j+2)\bar{1}(j+3)2...(j+1))| \leq C_n$ où M_n et C_n sont respectivement le n-ième nombre de Motzkin et de Catalan. Soit S^j l'ensemble de toutes les permutations qui ne contiennent pas de sousequences de type 321 ou $(j+2)\bar{1}(j+3)2...(j+1)$; nous déterminons l'équation fonctionelle qui est satisfaite par la fonction génératrice selon la longueur des permutations, le nombre de leurs sites actifs,

*Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, 50134 Firenze, Italy, e-mail:pire@ingfi1.ing.unifi.it

le nombre d'inversions des permutations et le nombre de leurs minima à droite. Nous donnons la fonction génératrice de S^j selon quelques-uns d'entre de ces paramètres: c'est à dire la longueur des permutations, le nombre de leurs minima à droite et le nombre d'inversions des permutations. En plus nous introduisons une bijection entre les permutations S^j et des sousclasses des arbres planaires. Enfin nous illustrons, en particulier, les trois cas $j = 1, 2$ et $j = \infty$ parce-qu' ils concernent les bien connus nombres de Motzkin, des prefixes de Motzkin et de Catalan respectivement.

1 Introduction

Permutations with forbidden subsequences have been widely studied in the last few years. They are of interest in both Computer Science and Combinatorics as the permutations with forbidden subsequences are related to the characterization of words that do not present any regularity or to the analysis of some regularities in words [6, 17]; they are present in problems of sorting [19, 21, 22] and pattern matching [7]; they codify a great number of nontrivial combinatorial objects [9, 10, 11, 13, 14, 15]. Several classical sequences of numbers in Combinatorics appear in the problem of permutations with forbidden subsequences enumeration. We refer, for example, to the Pell numbers, the binomial coefficients, the Fibonacci, Motzkin and Schröder numbers. Some of these results have been obtained by J. West [20, 21, 23, 24] and S. Gire [13]. Knuth studied permutations sorting through a stack [16] showing that they are exactly the permutations with the forbidden subsequence 231. All the permutations with a forbidden subsequence of length three are enumerated by the Catalan numbers. West [22] proved that permutations sorting twice through a stack avoid the subsequences 2341 and $3\bar{5}241$. The conjectured formula for their number has been proved by Zeilberger [25]. Even if lot of work has been carried out on permutations with forbidden subsequences (see [14], for a survey), many interesting problems remain still open. The enumeration of both the involutions and the alternate permutations of well-known families of permutations or the enumeration of some permutations with given forbidden subsequences keep on to be subject of studies. Many efforts have been made to enumerate the permutations with the forbidden subsequence $12..(k+1)$ that are related to pairs of standard Young tableaux having the same shape and length at most k. The obtained results concern the enumeration of $S(123)$ [20]; $S(1234)$ [12] and an estimate of the asymptotic value of $S_n(1...(k+1))$ [18] i. e. the number of permutations with length n and avoiding the subsequence $1...(k+1)$. In this paper we solve a particular instance of the problem of enumerating families of permutations with a certain forbidden subsequence of arbitrary length. We completely characterize the permutations avoiding the subpatterns 321 and $(j+2)\bar{1}(j+3)2...(j+1)$; their enumeration give sequences of numbers providing a kind of "discrete continuity" among the well-known Motzkin and Catalan numbers sequences. Section 2 of this paper contains some definitions on permutations with forbidden subsequences. In Section 3, we describe the basic idea of ECO method. ECO is a method for Enumerating Combinatorial Objects; it allows to obtain each combinatorial object Q, belonging to a given class of combinatorial objects $\mathcal{S}$, by performing a "local expansion" on $P \in \mathcal{S}$. It is an useful tool for the enumeration of many classes of combinatorial objects [1, 2, 3, 4]; in this paper we apply it to both the S^j permutations and subclasses of plane trees, thus establishing a constructive bijection between these two classes of combinatorial objects and obtaining their enumeration. In Section 4, we determine a recursive construction for the S^j permutations, $j \geq 1$. By translating the construction into formulae we obtain a set of functional equations. Its solution gives the generating function of S^j according to the length of the permutations, the number of their active sites, the number of their inversions and the number of their right minima. We are able to determine the generating function of S^j according to the length, the number of right minima and the number of inversions of the permutations. In Section 5, we illustrate the bijections between each family $S_n(321,(j+2)\bar{1}(j+3)2...(j+1))$, $j \geq 1$, and some particular plane trees with $(n+1)$ internal nodes. In Section 6, we examine the particular cases:

- S^1 related to the Motzkin numbers. We show the bijections between the Motzkin permutations and the 1-2 trees.
- S^2 related to the left-Motzkin factors numbers. These permutations are in bijection with the forests of 1-2 trees.
- S^∞ related to the Catalan numbers. The bijections involve the Catalan permutations and the plane trees.

In Section 7, we present the research future perspectives on this subject.

2 Notations and Definitions

A permutation $\pi = \pi(1)\pi(2)\ldots.\pi(n)$ on $[n] = \{1, 2, \ldots, n\}$ is a bijection between $[n]$ and $[n]$. Let S_n be the set of permutations on $[n]$.

Definition 2.1 *A permutation $\pi \in S_n$ contains a subsequence of type $\tau \in S_k$ iff a sequence of indexes $1 \leq i_{\tau(1)} < i_{\tau(2)} < \ldots < i_{\tau(k)} \leq n$ exists such that $\pi(i_1) < \pi(i_2) < \ldots < \pi(i_k)$. We denote the set of permutations of S_n not containing subsequences of type τ by $S_n(\tau)$.*

Example 2.1 The permutation 6145732 belongs to $S_7(2413)$ because all its subsequences of length 4 are not of type 2413. This permutation does not belong to $S_7(3142)$ because there exist subsequences of type 3142:
$\pi(1)\pi(2)\pi(5)\pi(6) = 6173$, $\pi(1)\pi(2)\pi(5)\pi(7) = 6172$.

Definition 2.2 *A* barred *permutation $\bar{\tau}$ of $[k]$ is a permutation of S_k having a bar over one of its elements. Let τ be a permutation on $[k]$ identical to $\bar{\tau}$ but unbarred, and $\hat{\tau}$ the permutation of $[k-1]$ made up of the $k-1$ unbarred elements of $\bar{\tau}$, rearranged to be a permutation on $[k-1]$.*

Definition 2.3 *A permutation $\pi \in S_n$ contains a type $\bar{\tau}$ subsequence iff π contains a type $\hat{\tau}$ subsequence that, in turn, is not a type τ subsequence. We denote the set of permutations of S_n not containing type $\bar{\tau}$ subsequences by $S_n(\bar{\tau})$.*

Example 2.2 If $\bar{\tau} = 41\bar{3}52$, we have $\tau = 41352$ and $\hat{\tau} = 3142$. The permutation $\pi = 6145732$ belongs to $S_7(\bar{\tau})$ because all its subsequences of type $\hat{\tau}$: $\pi(1)\pi(2)\pi(5)\pi(6) = 6173$, and $\pi(1)\pi(2)\pi(5)\pi(7) = 6172$ are subsequences of $\pi(1)\pi(2)\pi(3)\pi(5)\pi(6) = 61473$ and $\pi(1)\pi(2)\pi(3)\pi(5)\pi(7) = 61472$, which are of type τ.

If we have the set $\tau_1 \in S_{k_1}, \ldots, \tau_p \in S_{k_p}$ of barred or unbarred permutations, we denote the set $S_n(\tau_1) \cap \ldots \cap S_n(\tau_p)$ by $S_n(\tau_1, .., \tau_p)$. We call the family $F = \{\tau_1, .., \tau_p\}$ a *family of forbidden subsequences*, the set $S_n(F)$ a *family of permutations with forbidden subsequences.*

Example 2.3 The permutation $\pi = 6145732$ belongs to $S_7(2413, 41\bar{3}52)$.

Let $\pi \in S_n$, we denote the position lying on the left of $\pi(1)$ by s_0, the position lying between $\pi(i)$,$\pi(i+1)$, $1 \leq i \leq n-1$, by s_i and the position lying on the right of $\pi(n)$ by s_n. $s_0, s_1, ..., s_{n-1}, s_n$ are the *site* of π.

Definition 2.4 *Let $F = \{\tau_1, .., \tau_p\}$, a site s_i, $0 \leq i \leq n$, of a permutation $\pi \in S_n(F)$ is* active *if the insertion of $(n+1)$ into s_i gives a permutation belonging to the set $S_{n+1}(F)$; otherwise it is said to be* inactive.

Definition 2.5 *Let $\pi \in S_n$. The pair (i,j) is an* inversion *if $\pi(i) > \pi(j)$. An element $\pi(i)$ is a* right minimum *if $\pi(i) < \pi(j)$, $\forall j \in [i+1, n]$.*

Example 2.4 The permutation $\pi = 6145732$ has 12 inversions: $(1,2)(1,3)(1,4)(1,6)(1,7)(3,6)$ $(3,7)(4,6)(4,7)(5,6)(5,7)(6,7)$ and 2 right minima: $\pi(2) = 1$ and $\pi(7) = 2$.

3 The ECO method

In this section we recall the basic ideas of ECO method. We refer to [1, 2, 3] for further details, applications and examples. Let $\mathcal{S}$ be a class of combinatorial objects; p a parameter of enumeration on $\mathcal{S}$ such that $p : \mathcal{S} \to \mathbb{N}$. We denote $\mathcal{S}_n = \{s \in \mathcal{S} : p(s) = n\}$; an operator θ on $\mathcal{S}$ is a function from $\mathcal{S}_n$ in $2^{\mathcal{S}_{n+1}}$, where $2^{\mathcal{S}_{n+1}}$ is the power set of $\mathcal{S}_{n+1}$.

Proposition 3.1 *Let θ be an operator on $\mathcal{S}$; if θ satisfies the following conditions:*

1. $\forall\ Y \in \mathcal{S}_{n+1} \quad \exists X \ \in \mathcal{S}_n$ *such that* $Y \in \theta(X)$,
2. *if* $X_1, X_2 \in \mathcal{S}_n$ *and* $X_1 \neq X_2$ *then* $\theta(X_1) \cap \theta(X_2) = \emptyset$,

then the following family of sets: $\mathcal{F}_{n+1} = \{\theta(X) : \forall X \in \mathcal{S}_n\}$ is a partition of $\mathcal{S}_{n+1}$.

Given a class $\mathcal{S}$ of combinatorial objects, if we are able to define an operator θ satisfying conditions 1 and 2, then Proposition 3.1 allows us to construct each object $Y \in \mathcal{S}_{n+1}$ from another object $X \in \mathcal{S}_n$ and every $Y \in \mathcal{S}_{n+1}$ is given by only one $X \in \mathcal{S}_n$. Therefore, if we have an operator on $\mathcal{S}$ which satisfies conditions 1 and 2, then we obtain a recursive description of the elements of $\mathcal{S}$ from which we can deduce a functional equation verified by the generating function of $\mathcal{S}$.

Example 3.1 Let $\mathcal{S}$ be the class of 1-2 trees and p the number of internal nodes. Let $P \in \mathcal{S}$ and $\mathcal{F}(P)$ be the set of external nodes that follow the last internal node in the preorder traversal. The operator θ replaces each external node in $\mathcal{F}(P)$ with an internal one (see fig. 1). It is easy to prove that θ satisfies Proposition 3.1. For further detailed definitions and proofs see [4].

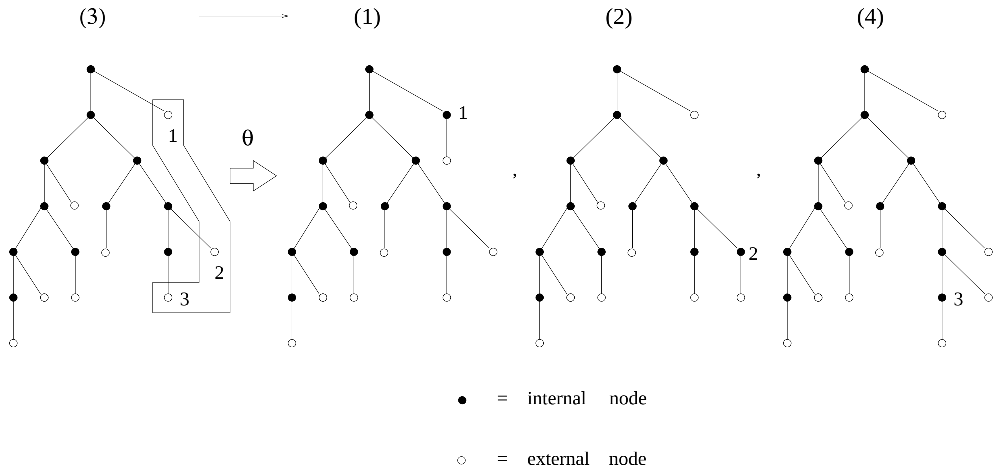

Figure 1: The operator θ on a 1-2 tree.

4 Permutations with one forbidden subsequence of increasing length

Let $S^j = \bigcup_{n\geq 1} S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$, we define an operator θ on the S^j permutations, $j \geq 1$, which satisfies Proposition 3.1; we obtain a recursive description of these combinatorial structures from which we deduce the set of functional equations verified by the generating function of S^j. Some computations allow us to determine their generating function according to the length, the number of right minima and the number of inversions of the permutations. Let $\pi = \pi(1)\pi(2).....\pi(n-1)\pi(n) \in S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$, $j \geq 1$, we denote the set of active sites of π by $\mathcal{F}(\pi)$. It is easy to prove the following Proposition:

Proposition 4.1 *Let* $\pi = \pi(1)\pi(2).....\pi(n-1)\pi(n) \in S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$, $j \geq 1$, *if* $|\mathcal{F}(\pi)| = k$ *then* $\mathcal{F}(\pi) = \left\{s_{n-(k-1)}, ..., s_n\right\}$.

$$\Pi = \Pi(1)\ldots\ldots \Pi(\text{n-k+1}) \underset{s_{n-(k-1)}}{\overset{k}{-}} \Pi(\text{n-k+2}) \underset{s_{n-(k-2)}}{\overset{.}{-}} \ldots\ldots \underset{s_{n-2}}{\overset{3}{-}} \Pi(\text{n-1}) \underset{s_{n-1}}{\overset{2}{-}} \Pi(\text{n}) \underset{s_n}{\overset{1}{-}}$$

Figure 2: A permutation $\pi \in S^j$ with $|\mathcal{F}(\pi)| = k$.

If $|\mathcal{F}(\pi)| = k$ the *operator* θ puts $(n+1)$ into s_i, $n-(k-1) \leq i \leq n$; it satisfies Proposition 3.1. Let $\pi \in S^j$; we denote: the length of π by $n(\pi)$, the number of its right minima by $m(\pi)$, the number of its active sites by $a(\pi)$ and the number of its inversions by $i(\pi)$. The generating function of S^j according to the above mentioned parameters is the following:

$$S^j(x, y, s, q) = \sum_{\pi \in S^j} x^{n(\pi)} y^{m(\pi)} s^{a(\pi)} q^{i(\pi)}.$$

We number the sites in $\mathcal{F}(\pi)$ from the right to the left in an increasing order (see fig. 2).
Let $\pi \in S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$ and π' be the permutation obtained from π by putting $(n+1)$ into the active site $s_{n(\pi)-(k-1)}$, $1 \leq k \leq a(\pi)$. The parameters change as follows (see fig. 3):

a) if $k = 1$ then $\mathcal{F}(\pi') = \left\{s_{n(\pi)-(a(\pi)-1)}, ..., s_{n(\pi)}, s_{n(\pi)+1}\right\}$ and:
$n(\pi') = n(\pi) + 1$, $m(\pi') = m(\pi) + 1$, $a(\pi') = a(\pi) + 1$, $i(\pi') = i(\pi)$.

b) if $2 \leq k \leq min\{a(\pi), j\}$ then $\mathcal{F}(\pi') = \left\{s_{n(\pi)-(k-2)}, ..., s_{n(\pi)}, s_{n(\pi)+1}\right\}$ and:
$n(\pi') = n(\pi) + 1$, $m(\pi') = m(\pi)$, $a(\pi') = k$, $i(\pi') = i(\pi) + k - 1$.

c) if $min\{a(\pi), j\} + 1 \leq k \leq a(\pi)$ then $\mathcal{F}(\pi') = \left\{s_{n(\pi)-(k-3)}, ..., s_{n(\pi)}, s_{n(\pi)+1}\right\}$ and:
$n(\pi') = n(\pi) + 1$, $m(\pi') = m(\pi)$, $a(\pi') = k - 1$, $i(\pi') = i(\pi) + k - 1$.

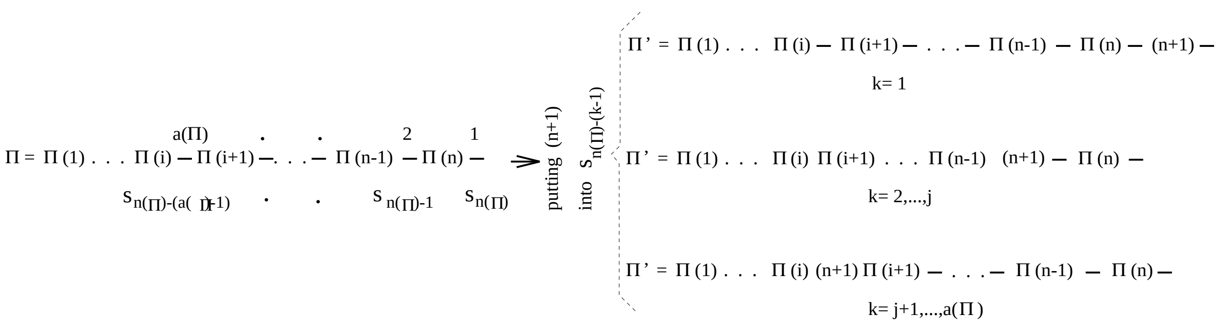

Figure 3: The operator θ on $\pi \in S^j$.

Remark 4.1 Let us note that the permutation of length one has 2 active sites and a permutation $\pi : |\mathcal{F}(\pi)| = k$ gives $(k+1)$ permutations with $(k)...(j)(j)...(2)(k+1)$ active sites respectively so the rule that describe the active sites changes has the form (see fig. 10):

$$\begin{cases} (2) \\ (k) \rightarrow (k)...(j)(j)..(2)(k+1) \end{cases}$$

Remark 4.2

- If $\boldsymbol{j = 1}$ then point **b)** is not considered. The permutation of length one, has 2 active sites and is represented by xys^2; the functional equation we obtain is:

$$S^1(x, y, s, q) = \frac{xys^2}{(1-xys)} + \frac{xsq\ S^1(x, y, 1, q)}{(1-xys)(1-sq)} - \frac{x\ S^1(x, y, sq, q)}{(1-xys)(1-sq)}. \tag{1}$$

- If $\boldsymbol{j = \infty}$ then point **c)** is not considered. The functional equation we obtain is:

$$S^\infty(x, y, s, q) = \frac{xys^2}{(1-xys)} + \frac{xs^2q\ S^\infty(x, y, 1, q)}{(1-xys)(1-sq)} - \frac{xs\ S^\infty(x, y, sq, q)}{(1-xys)(1-sq)}. \tag{2}$$

We refer to [1] for the solution of these equations and further considerations on Motzkin and Catalan permutations.

If $j \geq 2$, the translation of the construction into formulae gives a set of functional equations. We partition the set S^j into j subsets: $S^{j,(2)}, \dots, S^{j,(j)}, S^{j,(k\geq j+1)}$ such that $S^{j,(i)} = \{\pi \in S^j : a(\pi) = i\}$. This means that:

$$S^j(x,y,s,q) = \sum_{i=2}^{j} S^{j,(i)}(x,y,1,q)s^i + S^{j,(k\geq j+1)}(x,y,s,q).$$

Let $\pi \in S_n(321,(j+2)\bar{1}(j+3)2...(j+1))$ and $\mathcal{F}(\pi) = \left\{s_{n(\pi)-(a(\pi)-1)}, \dots, s_{n(\pi)-1}, s_{n(\pi)}\right\}$, then we have:

- $\pi' \in S^{j,(2)}$, $j > 2$, is obtained by $\pi \in S^{j,(2)} \bigcup \dots \bigcup S^{j,(j)} \bigcup S^{j,(k\geq j+1)}$ with an insertion into $s_{n(\pi)-1}$.
- $\pi' \in S^{j,(i)}$, $3 \leq i \leq j-1$, is obtained by both $\pi \in S^{j,(i-1)}$ with an insertion into $s_{n(\pi)}$ and $\pi \in S^{j,(i)} \bigcup \dots \bigcup S^{j,(j)} \bigcup S^{j,(k\geq j+1)}$ with an insertion into $s_{n(\pi)-(i-1)}$.
- $\pi' \in S^{j,(j)}$ is obtained by both $\pi \in S^{j,(j-1)}$ with an insertion into $s_{n(\pi)}$ and $\pi \in S^{j,(j)}$ with an insertion into $s_{n(\pi)-(j-1)}$ and by $\pi \in S^{j,(k\geq j+1)}$ with an insertion into both $s_{n(\pi)-(j-1)}$ and $s_{n(\pi)-j}$.
- $\pi' \in S^{j,(k\geq j+1)}$ is obtained by $\pi \in S^{j,(j)} \bigcup S^{j,(k\geq j+1)}$ with an insertion into $s_{n(\pi)}$ and by $\pi \in S^{j,(k\geq j+1)}$ with an insertion into $s_{n(\pi)-(i-1)}$, $j+2 \leq i \leq a(\pi)$.

The permutation of length one belongs to $S^{j,(2)}$ and is represented by xys^2, it follows the set of equations:

$$\begin{cases}
S^{j,(2)}(x,y,s,q) & = \; s^2\left[xy + xqS^j(x,y,1,q)\right], \; j > 2 \\[2ex]
S^{j,(i)}(x,y,s,q) & = \; s^i\Big[xyS^{j,(i-1)}(x,y,1,q) + xq^{i-1}\Big(S^{j,(i)}(x,y,1,q) + \dots \\
 & \quad \dots + S^{j,(k\geq j+1)}(x,y,1,q)\Big)\Big], \; 3 \leq i \leq j-1 \\[2ex]
S^{j,(j)}(x,y,s,q) & = \; s^j\Big[xyS^{j,(j-1)}(x,y,1,q) + xq^{j-1}S^{j,(j)}(x,y,1,q) + \\
 & \quad + xq^{j-1}(1+q)S^{j,(k\geq j+1)}(x,y,1,q)\Big] \\[2ex]
S^{j,(k\geq j+1)}(x,y,s,q) & = \; xysS^{j,(j)}(x,y,s,q) + \sum_{\pi\in S^{j,(k\geq j+1)}} x^{n(\pi)+1}y^{m(\pi)+1}s^{a(\pi)+1}q^{i(\pi)} + \\
 & \quad + \sum_{\pi\in S^{j,(k\geq j+1)}} \sum_{i=j+2}^{a(\pi)} x^{n(\pi)+1}y^{m(\pi)}s^{i-1}q^{i(\pi)+i-1}
\end{cases}$$

Proposition 4.2 *The generating function of S^j: $S^j(x,y,s,q)$, $j \geq 2$, is:*

$$S^j(x,y,s,q) = \sum_{i\geq 2} S^{j,(i)}(x,y,s,q)$$

where the $S^{j,(i)}(x,y,s,q)$ *satisfy:*

$$\begin{cases} S^{j,(2)}(x,y,s,q) & = xs^2\left[y + qS^j(x,y,1,q)\right],\ j>2 \\ S^{j,(i)}(x,y,s,q) & = \frac{xys^i}{1-xq^{i-1}} S^{j,(i-1)}(x,y,1,q) + \frac{xq^{i-1}s^i}{1-xq^{i-1}}\Big(S^{j,(i+1)}(x,y,1,q) + \ldots \\ & \ldots + S^{j,(k\geq j+1)}(x,y,1,q)\Big),\ 3\leq i\leq j-1 \\ S^{j,(j)}(x,y,s,q) & = \frac{xys^j}{1-xq^{j-1}} S^{j,(j-1)}(x,y,1,q) + \frac{xs^jq^{j-1}}{1-xq^{j-1}}(1+q)S^{j,(k\geq j+1)}(x,y,1,q) \\ S^{j,(k\geq j+1)}(x,y,s,q) & = \frac{xys}{1-xys} S^{j,(j)}(x,y,s,q) + \frac{x(sq)^{j+1}}{(1-xys)(1-sq)} S^{j,(k\geq j+1)}(x,y,1,q) - \\ & -\frac{x}{(1-xys)(1-sq)} S^{j,(k\geq j+1)}(x,y,sq,q) \end{cases}$$

We suppose $S^{j,(1)}(x,y,1,q)=1$ in the case of $j=2$.

The fourth equation of Proposition 4.2 can be solved by using the Lemma of Bousquet-Mélou [8]:

Lemma 4.3 *Let* $\mathcal{R}=\mathbb{R}[[x,y,s,q]]$ *be the algebra of the formal power series in variables* x,y,s *and* q *with real coefficients and let* $\mathcal{A}$ *be a sub-algebra of* $\mathcal{R}$ *such that the series converge for* $s=1$*. Let* $A(s)=A(x,y,s,q)$ *be a formal power series in* $\mathcal{A}$*. Let assume that:*

$$A(s) = xe(s) + xf(s)A(1) + xg(s)A(sq),$$

where $e(s)$*,* $f(s)$ *and* $g(s)$ *are some given power series in* $\mathcal{A}$*. Then:*

$$A(s) = \frac{J_1(s)+J_1(s)J_0(s)-J_1(s)J_0(s)}{1-J_0(1)},$$

where: $J_0(s) = \sum\limits_{n\geq 0} x^{n+1} g(s)g(sq)\ldots g(sq^{n-1})e(sq^n)$
and $J_1(s) = \sum\limits_{n\geq 0} x^{n+1} g(s)g(sq)\ldots g(sq^{n-1})f(sq^n).$

By means of Lemma 4.3 and the fourth equation of Proposition 4.2 we get the following:

Proposition 4.4 *The generating function* $S^{j,(k\geq j+1)}(x,y,s,q)$ *is given by:*

$$S^{j,(k\geq j+1)}(x,y,s,q) = \frac{J_1(s)J_0(1)-J_1(1)J_0(s)+J_1(1)}{J_0(1)}$$

where:

$$J_1(x,y,s,q) = \sum_{n\geq 0} \frac{(-1)^n x^{n+1} y s^{j+1} q^{n(j+1)}}{(xys,q)_{n+1}(sq,q)_n}\ S^{j,(j)}(x,y,1,q),$$
$$J_0(x,y,s,q) = 1 + \sum_{n\geq 0} \frac{(-1)^{n+1} x^{n+1} s^{j+1} q^{(n+1)(j+1)}}{(xys,q)_{n+1}(sq,q)_{n+1}};$$

and $(a,q)_n = \prod\limits_{k=0}^{n-1}\left(1-aq^k\right).$

Let us now take the generating function $S^{j,(k\geq j+1)}(x,y,1,q)$ into consideration. We denote the function $S^{j,(k\geq j+1)}(x,y,1,q)$, $S^{j,(j)}(x,y,1,q)$, $J_0(x,y,1,q)$ and $J_1(x,y,1,q)$ by $S^{j,(k\geq j+1)}(x,y,q)$, $S^{j,(j)}(x,y,q)$, $J_0(x,y,q)$ and $J_1(x,y,q)$ respectively, for the sake of brevity. From Proposition 4.4 it follows that:

$$S^{j,(k\geq j+1)}(x,y,q) = f(x,y,q)S^{j,(j)}(x,y,q), \tag{3}$$

with:

$$f(x,y,q) = y\frac{\sum_{n\geq 0} \frac{(-1)^n x^{n+1} q^{n(j+1)}}{(xy,q)_{n+1}(q,q)_n}}{\sum_{n\geq 0} \frac{(-1)^n x^n q^{n(j+1)}}{(xy,q)_n(q,q)_n}}.$$

Theorem 4.5 *The generating function of S^j: $S^j(x,y,q)$ is such that:*

- $S^2(x,y,q) = \frac{xy(1+f(x,y,q))}{1-xq-xq(1+q)f(x,y,q)}$;
- $S^j(x,y,q) = \frac{xy(1-xq^2)\Delta_j(x,y,q)}{(1-xq)(1-xq^2)\Delta_j(x,y,q)+xy\Delta_{j-1}(x,y,q)}, \qquad j\geq 3,$

  with $\Delta_j(x,y,q) = c_1(x,y,q)\lambda_1^j(x,y,q) + c_2(x,y,q)\lambda_2^j(x,y,q)$

  $$where: \quad \lambda_1(x,y,q) = \frac{1}{2}\left[-\left(1+\frac{xy}{1-xq^2}\right) + \sqrt{\left(1+\frac{xy}{1-xq^2}\right)^2 - \frac{4xy}{(1-xq^2)(1-xq^3)}}\right]$$

  $$and \qquad \lambda_2(x,y,q) = \frac{1}{2}\left[-\left(1+\frac{xy}{1-xq^2}\right) - \sqrt{\left(1+\frac{xy}{1-xq^2}\right)^2 - \frac{4xy}{(1-xq^2)(1-xq^3)}}\right].$$

  $c_1(x,y,q),\ c_2(x,y,q)$ *satisfy:*

  $$\begin{cases} c_1(x,y,q)\lambda_1^2(x,y,q) + c_2(x,y,q)\lambda_2^2(x,y,q) &= 1+f(x,y,q) \\ c_1(x,y,q)\lambda_1^3(x,y,q) + c_2(x,y,q)\lambda_2^3(x,y,q) &= f(x,y,q)\frac{xq^{j-1}(1+q)-xy}{1-xq^{j-1}} - \frac{1+xy-xq^{j-1}}{1-xq^{j-1}} \end{cases}$$

Proof.
• The generating function $S^2(x,y,s,q)$ satisfies the following equations:

$$\begin{cases} S^{2,(2)}(x,y,s,q) &= \frac{xys^2}{1-xq} + \frac{xs^2q}{1-xq}(1+q)S^{2,(k\geq 3)}(x,y,1,q) \\ S^{2,(k\geq 3)}(x,y,s,q) &= \frac{xys}{1-xys}S^{2,(2)}(x,y,s,q) + \frac{x(sq)^3}{(1-xys)(1-sq)}S^{2,(k\geq 3)}(x,y,1,q)- \\ & -\frac{x}{(1-xys)(1-sq)}S^{2,(k\geq 3)}(x,y,sq,q) \\ S^2(x,y,s,q) &= S^{2,(2)}(x,y,s,q) + S^{2,(k\geq 3)}(x,y,s,q) \end{cases} \tag{4}$$

The thesis is obtained by easy substitutions from this set of equations and Equation (3).

• Proposition 4.2 and Equation (3) give the set of equations: $A_j(x,y,q)X_j(x,y,q) = Y_j(x,y,q)$ where:

$A_{j-i}(x,y,q) =$

$$= \begin{pmatrix} 1 & 1 & 1 & \ldots\ldots & 1 & 1 & 1 & 1 \\ \frac{xy}{1-xq^{i+2}} & -1 & \frac{xq^{i+2}}{1-xq^{i+2}} & \cdots & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} \\ & \ddots & & & \vdots & \vdots & \vdots & \vdots \\ & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ & & & \ddots & \frac{xy}{1-xq^{j-2}} & -1 & \frac{xq^{j-2}}{1-xq^{j-2}} & \frac{xq^{j-2}}{1-xq^{j-2}} \\ & & & & & \frac{xy}{1-xq^{j-1}} & -1 & \frac{xq^{j-1}(1+q)}{1-xq^{j-1}} \\ & & & & & & -f(x,y,q) & 1 \end{pmatrix}, \quad i \geq 0$$

$$X_j(x,y,q) = \begin{pmatrix} S^{j,(2)}(x,y,q) \\ S^{j,(3)}(x,y,q) \\ \vdots \\ S^{j,(j)}(x,y,q) \\ S^{j,(k\geq j+1)}(x,y,q) \end{pmatrix}, \qquad Y_j(x,y,q) = \begin{pmatrix} S^j(x,y,q) \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Let:

$B_{j-i}(x,y,q) =$

$$= \begin{pmatrix} -1 & \frac{xq^{i+1}}{1-xq^{i+1}} & \frac{xq^{i+1}}{1-xq^{i+1}} & \ldots\ldots & \frac{xq^{i+1}}{1-xq^{i+1}} & \frac{xq^{i+1}}{1-xq^{i+1}} & \frac{xq^{i+1}}{1-xq^{i+1}} & \frac{xq^{i+1}}{1-xq^{i+1}} \\ \frac{xy}{1-xq^{i+2}} & -1 & \frac{xq^{i+2}}{1-xq^{i+2}} & \cdots & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} & \frac{xq^{i+2}}{1-xq^{i+2}} \\ & \ddots & & & \vdots & \vdots & \vdots & \vdots \\ & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ & & & \ddots & \frac{xy}{1-xq^{j-2}} & -1 & \frac{xq^{j-2}}{1-xq^{j-2}} & \frac{xq^{j-2}}{1-xq^{j-2}} \\ & & & & & \frac{xy}{1-xq^{j-1}} & -1 & \frac{xq^{j-1}(1+q)}{1-xq^{j-1}} \\ & & & & & & -f(x,y,q) & 1 \end{pmatrix}, \quad i \geq 1$$

$\Delta_j(x,y,q) = det\,(A_j(x,y,q))$ and $\beta_j(x,y,q) = det\,(B_j(x,y,q))$, then:

$$S^{j,(2)}(x,y,q) = S^j(x,y,q)\frac{\beta_{j-1}(x,y,q)}{\Delta_j(x,y,q)}. \tag{5}$$

$\Delta_j(x,y,q)$ and $\beta_{j-1}(x,y,q)$ verify the following recursive relations:

$$\begin{cases} \Delta_{j-i}(x,y,q) &=& \beta_{j-(i+1)}(x,y,q) - \frac{xy}{1-xq^{i+2}}\Delta_{j-(i+1)}(x,y,q); & j\geq 3, \;\; 0\leq i\leq (j-3) \\ \beta_{j-i}(x,y,q) &=& -\beta_{j-(i+1)}(x,y,q) - \frac{xy}{1-xq^{i+2}}\frac{xq^{i+1}}{1-xq^{i+1}}\Delta_{j-(i+1)}(x,y,q); & j\geq 3, \;\; 1\leq i\leq (j-3) \\ \Delta_2(x,y,q) &=& 1+f(x,y,q) \\ \beta_2(x,y,q) &=& f(x,y,q)\frac{xq^{j-1}(1+q)}{1-xq^{j-1}} - 1 \end{cases}$$

This means that:

$$\begin{cases} \Delta_{j-i}(x,y,q) + \left(1+\frac{xy}{1-xq^{i+2}}\right)\Delta_{j-(i+1)}(x,y,q) + \frac{xy}{(1-xq^{i+2})(1-xq^{i+3})}\Delta_{j-(i+2)}(x,y,q) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\leq i\leq j-4 \\ \Delta_2(x,y,q) = 1+f(x,y,q) \\ \Delta_3(x,y,q) = f(x,y,q)\frac{xq^{j-1}(1+q)-xy}{1-xq^{j-1}} - \frac{1+xy-xq^{j-1}}{1-xq^{j-1}} \end{cases}$$

The solution of this recursive relation gives the formula for $\Delta_j(x,y,q)$ of the thesis. From Proposition 4.2 and Equation (5) we obtain:

$$S^j(x,y,q) = \frac{xy\Delta_j(x,y,q)}{\beta_{j-1}(x,y,q) - xq\Delta_j(x,y,q)} = \frac{xy}{(1-xq) + \frac{xy}{1-xq^2}\frac{1}{\frac{\Delta_j(x,y,q)}{\Delta_{j-1}(x,y,q)}}}$$

$\square$

Remark 4.3 Let us note that: $\frac{\Delta_{j-i}(x,y,q)}{\Delta_{j-(i+1)}(x,y,q)} = -\frac{1+xy-xq^{i+2}}{1-xq^{i+2}} - \frac{xy}{(1-xq^{i+2})(1-xq^{i+3})\frac{\Delta_{j-(i+1)}(x,y,q)}{\Delta_{j-(i+2)}(x,y,q)}}$

hence we have:

$$\lim_{j\to\infty} S^j(x,y,q) = \cfrac{xy}{1-xq-\cfrac{xy}{1+xy-xq^2-\cfrac{xy}{1+xy-xq^3-\cfrac{xy}{\vdots}}}}.$$

If $y=q=1$ we obtain the continued fraction representing the generating function $C(x)-1 =$ $= \sum_{n\geq 1} C_n(321)x^n$.

Remark 4.4 This continued fraction represents the generating function of Catalan permutations enumerated according to their length, their right minima number and their inversion number and it is alternative to the one that can be obtained by developing the functional equation that appears in [1]. Consequently we obtain the following identities:

$$y\frac{\sum_{n\geq 0}\frac{(-1)^n x^{n+1} q^{\frac{n(n+3)}{2}}}{(q;q)_n(xy;q)_{n+1}}}{\sum_{n\geq 0}\frac{(-1)^n x^n q^{\frac{n(n+1)}{2}}}{(q;q)_n(xy;q)_n}} = \cfrac{xy}{1-xq-\cfrac{xy}{1+xy-xq^2-\cfrac{xy}{1+xy-xq^3-\cfrac{xy}{\vdots}}}} = \cfrac{xy}{1-xq-xy-\cfrac{x^2yq^2}{1-xq^2-xyq-\cfrac{x^2yq^4}{1-xq^3-xyq^2-\cfrac{x^2yq^6}{\vdots}}}}.$$

Let us note that the generating function $S^j(x,y,1)$ is obtained from Theorem 4.5 by setting $q=1$ and by substituting $f(x,y,q)$ with $\tilde{f}(x,y)=\frac{1-xy-\sqrt{(1-xy)^2-4x^2y}}{2x}$ obtained following the steps below described:

1. referring to Proposition 4.4 with some computations we obtain the equalities:
$$\frac{J_1(x,y,q)}{S^{j,(j)}(x,y,q)} - xyJ_0(x,y,q) = \frac{x^2y^2}{1-xy}J_0(xq,y,q),$$
$$(x^2yq^{j+1}+xy-1)\,\frac{J_1(x,y,q)}{S^{j,(j)}(x,y,q)} + xyJ_0(x,y,q) = -\frac{x^3y^2q^{j+1}}{1-xy}\,\frac{J_1(xq,y,q)}{S^{j,(j)}(xq,y,q)};$$

2. from Proposition 4.4 we deduce that:
$$xq^{j+1}S^{j,(k\geq j+1)}(xq,y,q) = \frac{(1-xy-x^2yq^{j+1})\,\frac{S^{j,(k\geq j+1)}(x,y,q)}{S^{j,(j)}(x,y,q)} - xy}{\frac{S^{j,(k\geq j+1)}(x,y,q)}{S^{j,(j)}(x,y,q)} - xy}S^{j,(j)}(xq,y,q); \quad (6)$$

3. if we put $q=1$ in Equation (6) and denote $S^{j,(k\geq j+1)}(x,y,1)$ by $S^{j,(k\geq j+1)}(x,y)$, $S^{j,(j)}(x,y,1)$ by $S^{j,(j)}(x,y)$, then we obtain:
$$x\left(S^{j,(k\geq j+1)}(x,y)\right)^2 - S^{j,(j)}(x,y)(1-xy)S^{j,(k\geq j+1)}(x,y) + xy\left(S^{j,(j)}(x,y)\right)^2 = 0; \quad (7)$$

4. by solving Equation (7), we obtain: $S^{j,(k\geq j+1)}(x,y) = f(x,y)S^{j,(j)}(x,y)$, where $f(x,y)$ is $f(x,y)=\frac{1-xy-\sqrt{(1-xy)^2-4x^2y}}{2x}$.

5 Permutations and trees

The aim of this section is to describe the bijection between the S^j permutations and the subclass of plane trees denoted by $(1-2)^j$ trees, for each $j\geq 1$.

Definition 5.1 *A $(1-2)^j$ tree is a particular plane tree such that each node on level $i\geq j$ has degree at most two (see fig. 4).*

Remark 5.1 If $\boldsymbol{j}=\mathbf{1}$ then we obtain the 1-2 trees, while for $\boldsymbol{j}=\boldsymbol{\infty}$ then we have the plane trees.

The ECO method is applicable to the $(1-2)^j$ trees: let P a $(1-2)^j$ trees; we denote the set of external nodes following the last internal node in the preorder traversal by $\mathcal{F}(P)$ (see fig. 4). Let the parameter p be the number of internal nodes; the operator θ replaces each external node in $\mathcal{F}(P)$ with an internal node (see fig. 5, 6). It is easy to see that θ satisfies Proposition 3.1.

The $(1-2)^j$ trees and the S^j permutations grow in a "parallel" way following the constructions illustrated in fig. 6 and 3 respectively. This give us a constructive bijection between these two classes of combinatorial objects (see fig. 7, 8, 9) with the following correspondences of parameters:

- the length of the permutation is the number of internal nodes minus one,
- the number of active sites is the cardinality of $\mathcal{F}(P)$,
- the number of right minima is the number of internal nodes with degree greater than zero.

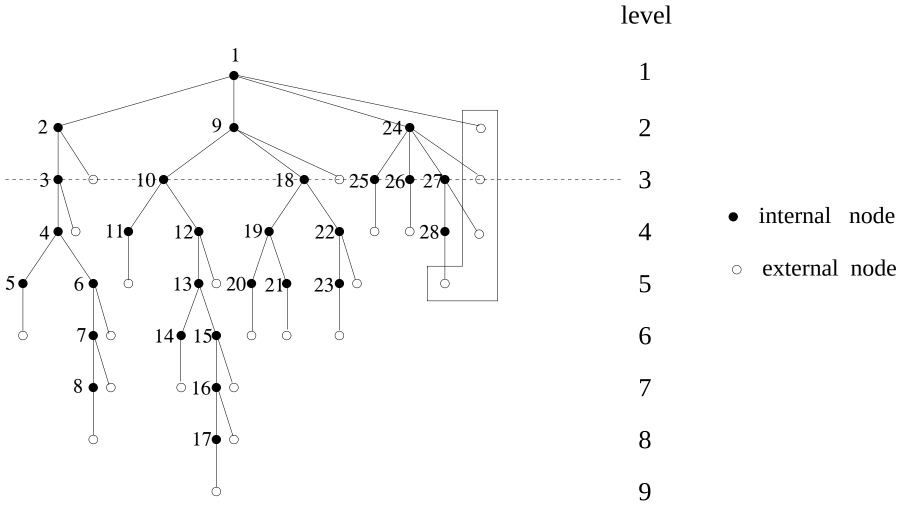

Figure 4: A $(1-2)^3$ tree P with $|\mathcal{F}(P)| = 4$.

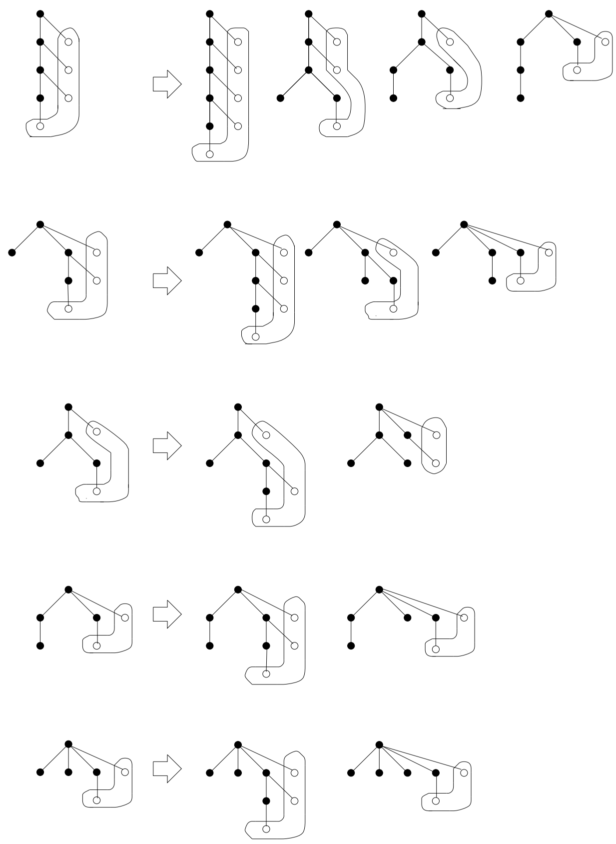

Figure 5: Construction of the $(1-2)^2$ trees with 5 internal nodes starting from the ones with 4 internal nodes.

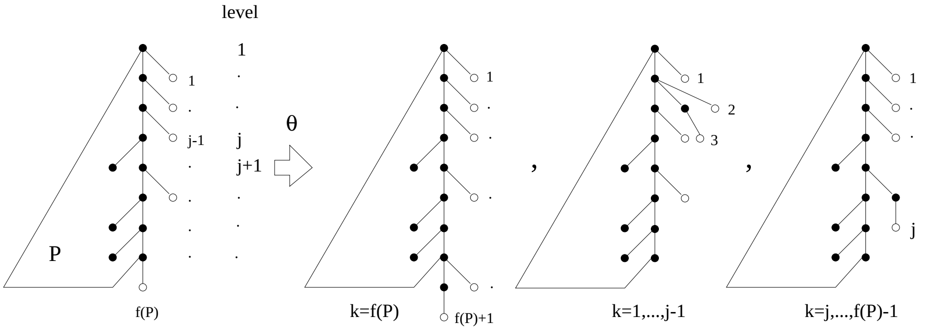

Figure 6: The application θ on a $(1-2)^j$ tree.

6 Examples

In this section we take into account the cases $j=1,2$ and $j=\infty$ involving respectively the Motzkin, S^2 and Catalan permutations.

Motzkin number

The ECO method applied to the Motzkin permutations gives Equation (1). We refer to [1] for its solution and further considerations on the family of permutations $M_n(321, 3\bar{1}42)$. Section 5 shows that ECO is applicable to the 1-2 trees.
We now have a constructive bijection between the Motzkin permutations and the 1-2 trees (see fig. 7).

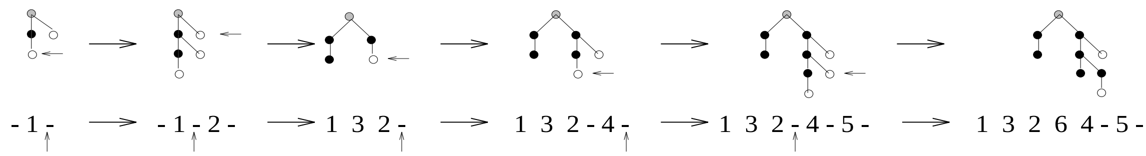

Figure 7: The bijection between the Motzkin permutations and the 1-2 trees.

left-Motzkin factors number

The ECO method applied to the S^2 permutations gives the set of equations whose solution is:

$$S^2(x,y,q) = \frac{xy(J_1(x,y,q)+J_0(x,y,q))}{J_0(x,y,q)(1-xq)-xq(1+q)J_1(x,y,q)},$$

where: $J_1(x,y,q) = \sum_{n\geq 0} \frac{(-1)^n x^{n+1} y q^{3n}}{(xy,q)_{n+1}(q,q)_n}$ and $J_0(x,y,q) = \sum_{n\geq 0} \frac{(-1)^n x^n q^{3n}}{(xy,q)_n (q,q)_n}$.
If $y=q=1$ we obtain the generating function of the left-Motzkin factors numbers: $\frac{1}{2}\left(\sqrt{\frac{1+x}{1-3x}}-1\right)$. Let us examine the generating function of S^2 permutations according to the number of their inversions, denoted by $S^2(q)$. It is easy to see that:

$$S^2(q) = \frac{\bar{J}_1(q)+\bar{J}_0(q)}{(1-q)\bar{J}_0(q)-q(1+q)\bar{J}_1(q)}$$

where: $\bar{J}_1(q) = \sum_{n\geq 0} \frac{(-1)^n q^{3n}}{(q,q)_n(q,q)_n} = \bar{I}_0\left(\frac{q^3}{(1-q)^2}\right)$,

and $\bar{J}_0(q) = \sum_{n\geq 0} \frac{(-1)^{n+1} q^{3(n+1)}}{(q,q)_{n+1}(q,q)_n} = \bar{I}_0\left(\frac{q^3}{(1-q)^2}\right) - \bar{I}_0\left(\frac{q^4}{(1-q)^2}\right)$;

being $\bar{I}_0(x) = \sum_{n\geq 0} \frac{(-1)^n x^n (1-q)^{2n}}{(q,q)_n(q,q)_n}$ the q_0-Bessel function.

Section 5 shows that ECO is applicable to the $(1-2)^2$ trees. The constructive bijection between the S^2 permutations and the $(1-2)^2$ trees and is illustrated in fig. 8.

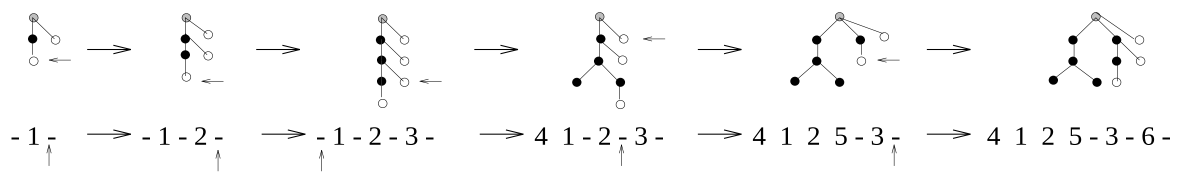

Figure 8: The bijection between the S^2 permutations and the $(1-2)^2$ trees.

Catalan number

The ECO method applied to the Catalan permutations gives Equation (2). We refer to [1] for its solution and further considerations on the family of permutations $C_n(321)$. Section 5 shows that ECO is applicable to the $(1-2)^\infty$ trees. Figure 9 shows the constructive bijection between the Catalan permutations and the planar trees.

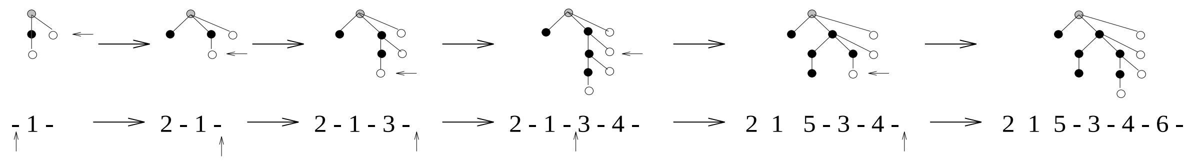

Figure 9: The bijection between the Catalan permutations and the planar trees.

7 Conclusions

The main aim of this work is to completely describe classes of permutations with a barred forbidden subsequence increasing in length, in an orderly way (see fig. 10). The classes of permutations described in this paper are enumerated by numbers lying between the Motzkin and the Catalan numbers. We view the obtained sequences of numbers as providing a "discrete continuity" between the

Motzkin and the Catalan sequences (see fig. 11). Let us note that we find the well-known left-Motzkin factors numbers as a particular case. Presently we are working on the construction of particular mesh lattices on which the underdiagonal directed animals, with area n, are in bijection with the family of permutations $S_n(321, (j+2)\bar{1}(j+3)2...(j+1))$, considering that the Motzkin permutations are in bijection with the underdiagonal directed animals on the square lattice and the Catalan permutations are in bijection with the underdiagonal directed animal on the triangular lattice [5]. A further step is to translate the classical parameters of permutations enumeration into some parameters of directed animals and vice-versa.

Index	Rule for the active sites changes	Family of Permutations
j=1	$\begin{cases}(2) \\ (k) \rightsquigarrow (k-1)\,.....\,(1)\,(k+1)\end{cases}$	$S_n^1(321, 3\bar{1}42) = M_n$
j=2	$\begin{cases}(2) \\ (k) \rightsquigarrow (k-1)\,.....\,(2)\,(2)\,(k+1)\end{cases}$	$S_n^2(321, 4\bar{1}523)$
j=3	$\begin{cases}(2) \\ (k) \rightsquigarrow (k-1)\,.....\,(3)\,(3)\,(2)\,(k+1)\end{cases}$	$S_n^3(321, 5\bar{1}6234)$
$\vdots$	$\vdots$	$\vdots$
j	$\begin{cases}(2) \\ (k) \rightsquigarrow (k-1)\,...\,(j)\,(j)\,...\,(2)\,(k+1)\end{cases}$	$S_n^j(321, (j+2)\bar{1}(j+3)2...(j+1))$
$\vdots$	$\vdots$	$\vdots$
j= ∞	$\begin{cases}(2) \\ (k) \rightsquigarrow (k)\,.....\,(2)\,(k+1)\end{cases}$	$S_n^\infty(321) = C_n$

Figure 10: The permutations S^j and the related rule that describes the changes of the active sites.

Index	Family of Permutations	Numbers
j=1	$S_n^1(321,3\bar{1}42) = M_n$	1 2 4 9 21 . . .
j=2	$S_n^2(321,4\bar{1}523)$	1 2 5 13 35 . . .
j=3	$S_n^3(321,5\bar{1}6234)$	1 2 5 14 41 . . .
j=4	$S_n^4(321,6\bar{1}72345)$	1 2 5 14 42 . . .
$\vdots$	$\vdots$	$\vdots$
$j=\infty$	$S_n^\infty(321) = C_n$	1 2 5 14 42 . . . $\frac{1}{n+1}\binom{2n}{n}$ . . .

Figure 11: First numbers of the sequences related to the permutations in S^j.

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