BINOMIAL FORMULA FOR MACDONALD POLYNOMIALS AND ITS APPLICATIONS

Andrei Okounkov

Abstract. We generalize the binomial formula for Jack polynomials proved in [OO2] and consider some applications.

§1 Binomial formula

Binomial type theorems (that is Taylor and Newton interpolation expansions about various points) are powerful tools for handling special functions. The high-school binomial formula is the Taylor expansion of the function $f(x) = x^l$ about the point $x = 1$. Its q-deformation is the Newton interpolation with knots

$$x = 1, q, q^2, \dots ,$$

which reads

$$x^l = \sum_m \begin{bmatrix} l \\ m \end{bmatrix}_q (x-1)\dots(x-q^{m-1}), \tag{1.1}$$

where

$$\begin{bmatrix} l \\ m \end{bmatrix}_q = \frac{(q^l-1)\dots(q^l-q^{m-1})}{(q^m-1)\dots(q^m-q^{m-1})}$$

is the q-binomial coefficient. Denote the Newton interpolation polynomials in the RHS of (1.1) by

$$P^*_k(x;q) = (x-1)\cdots(x-q^{k-1}), \quad k = 0,1,2,\dots .$$

The formula inverse to (1.1) is the Taylor expansion of $P^*_l(x;q)$ about the point $x = 0$

$$\frac{(x-1)\dots(x-q^{l-1})}{q^{l(l-1)/2}} = \sum_m (-1)^{l-m} \begin{bmatrix} l \\ m \end{bmatrix}_{1/q} \frac{x^m}{q^{m(m-1)/2}}, \tag{1.2}$$

which is essentially the fundamental q-binomial theorem [An,GR].

The formulas (1.1) and (1.2) are limit cases of the formula

$$\frac{P^*_l(ax;q)}{P^*_l(a;q)} = \sum_m a^m \begin{bmatrix} l \\ m \end{bmatrix}_{1/q} \frac{P^*_m(x;1/q)}{P^*_m(a;q)} \tag{1.3}$$

The paper was written during the author's stay at the Institute for Advanced Study in Princeton. I am grateful to IAS for hospitality and to NSF for financial support (grant DMS 9304580).

as $a \to \infty$ and $a \to 0$ respectively. Since

$$\begin{bmatrix} l \\ m \end{bmatrix}_{1/q} = \frac{P^*_m(q^{-l}; 1/q)}{P^*_m(q^{-m}; 1/q)}$$

the formula (1.3) is symmetric in q^{-l} and x. The formula (1.3) is easy to remember in the form

$$\frac{P^*_l(ax;q)}{P^*_l(a;q)} = \sum_m (\dots)\, P^*_m(q^{-l}; 1/q)\, P^*_m(x; 1/q)\,, \tag{1.4}$$

where the factor

$$(\dots) = \frac{a^m}{P^*_m(q^{-m}; 1/q)\, P^*_m(a;q)}$$

can be easily reconstructed using that

(1) this factor does not depend neither on x nor on l, and
(2) the highest degree term of $P^*_l(x;q)$ is x^l.

In this paper we consider a multivariate generalization of the formula (1.4) for symmetric functions. Let q and t be two parameters. Let μ be a partition. Recall the following notations of Macdonald [M]. Set

$$n(\mu) = \sum_i (i-1)\mu_i = \sum_j \mu'_j(\mu'_j - 1)/2\,.$$

For each square $s = (i,j) \in \mu$ the numbers

$$\begin{aligned} a(s) &= \mu_i - j, & a'(s) &= j - 1, \\ l(s) &= \mu'_j - i, & l'(s) &= i - 1, \end{aligned}$$

are called the arm-length, arm-colength, leg-length, and leg-colength respectively. We shall need the following normalization constant

$$H(\mu; q, t) = t^{-2n(\mu)} q^{n(\mu')} \prod_{s \in \mu} \left(q^{a(s)+1} t^{l(s)} - 1 \right) . \tag{1.5}$$

Denote by $\mathcal{P}$ the set of all partitions. This set is partially ordered by inclusion of partitions $\mu \subset \lambda$. Consider the following multivariate Newton interpolation polynomials

Definition ([Kn,S2,Ok3]). Suppose μ has $\leq n$ parts. The *interpolation Macdonald polynomial*

$$P^*_\mu(x_1, \dots, x_n; q, t)$$

is the unique polynomial in $x_1, \dots, x_n$ such that

(1) $P^*_\mu(x_1, \dots, x_n; q, t)$ is symmetric in variables $x_i t^{-i}$,
(2) $\deg P^*_\mu = |\mu|$,
(3) $P^*_\mu(q^\mu; q, t) = H(\mu; q, t)$, and
(4) $P^*_\mu(q^\lambda; q, t) = 0$ for all partitions λ such that $\mu \not\subset \lambda$.

Here $P^*_\mu(q^\lambda;q,t)=P^*_\mu(q^{\lambda_1},\dots,q^{\lambda_n};q,t)$. Another name for the same polynomials is the *shifted Macdonald polynomials.*

The connection with ordinary Macdonald polynomials will be explained below. It is easy to see that the polynomial $P^*_\mu(x;q,t)$ is indeed unique provided it exists (which is not evident). It follows from the definition that $P^*_\mu(x;q,t)$ is stable, that is

$$P^*_\mu(x_1,\dots,x_n,1;q,t)=P^*_\mu(x_1,\dots,x_n;q,t)$$

provided μ has at most n parts. Using this stability one can define polynomials $P^*_\mu(x;q,t)$ in infinitely many variables.

Remark. The condition (4) in the definition of $P^*_\mu(x;q,t)$ can be replaced by a weaker condition

(4′) $P^*_\mu(q^\lambda;q,t)=0$ for all partitions $\lambda\neq\mu$ such that $|\lambda|\le|\mu|$.

The first proof of the fact that (4′) implies the property (4), which is in that case referred to as the *extra vanishing*, was given in [Kn].

The interpolation polynomials in the Schur functions case were studied in [OO] (and, from a different point of view, in earlier papers cited in [OO]). They are called *shifted Schur functions*, have many remarkable properties, and can be interpreted as distinguished basis elements of the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$ (see also [Ok1,2] where explicit formulas for the corresponding central elements were given). We have also conjectured (unpublished) the general combinatorial formula for the polynomials $P^*_\mu(x;q,t)$ (the formula (1.6) below). Then F. Knop and S. Sahi proved a series of results for general (q,t) in [KS,Kn,S2]. In particular, it was proved that the polynomial $P^*_\mu(x;q,t)$ exists and has the fundamental property (1.8) below. They used difference equations for the polynomials $P^*_\mu(x;q,t)$. Later the combinatorial formula (1.6) and also a q-integral representation for $P^*_\mu(x;q,t)$ were proved in [Ok3].

We call a tableau T on μ a *reverse tableau* if its entries strictly decrease down the columns and weakly decrease in the rows. By $T(s)$ denote the entry in the square $s\in\mu$. We have [Ok3]

$$P^*_\mu(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\mu} t^{1-T(s)}\left(x_{T(s)}-q^{a'(s)}t^{-l'(s)}\right), \tag{1.6}$$

where the sum is over all reverse tableau on μ with entries in $\mathbb{N}$ and $\psi_T(q,t)$ is the same (q,t)-weight of a tableau which enters the combinatorial formula for ordinary Macdonald polynomials (see [M],§VI.7)

$$P_\mu(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\mu} x_{T(s)}\,. \tag{1.7}$$

The coefficients $\psi_T(q,t)$ are rational functions of q and t. Explicit formulas for them are given in [M], (VI.7.11.′). It is clear from (1.6) and (1.7) that

$$P^*_\mu(x_1,\dots,x_n;q,t)=P_\mu(x_1,x_2t^{-1},\dots,x_nt^{1-n})+\ \text{lower degree terms}\,. \tag{1.8}$$

In other words, ordinary Macdonald polynomials are the top homogeneous layer of the interpolation Macdonald polynomials.

It follows from the combinatorial formula (1.6) that

$$P^*_\mu(a,\dots,a;q,t) = P_\mu(1,t^{-1},\dots,t^{1-n};q,t) \prod_{s\in\mu} \left(a - q^{a'(s)}t^{-l'(s)}\right), \tag{1.9}$$

where a is repeated n times. Recall that (see [M], (VI.6.11′))

$$P_\mu(1,t^{-1},\dots,t^{1-n};q,t) = t^{n(\mu)+|\mu|(1-n)} \prod_{s\in\mu} \frac{1-q^{a'(s)}t^{n-l'(s)}}{1-q^{a(s)}t^{1+l(s)}}.$$

Now we can state our main formula which is the following generalization of (1.4)

Binomial Theorem.

$$\frac{P^*_\lambda(ax_1,\dots,ax_n;q,t)}{P^*_\lambda(a,\dots,a;q,t)} = \sum_\mu \frac{a^{|\mu|}}{t^{(n-1)|\mu|}} \frac{P^*_\mu(q^{-\lambda};1/q,1/t)}{P^*_\mu(q^{-\mu};1/q,1/t)} \frac{P^*_\mu(x_n,\dots,x_1;1/q,1/t)}{P^*_\mu(a,\dots,a;q,t)}.$$

Remarks.

(1) By definition of $P^*_\mu(x;q,t)$ only summands with $\mu\subset\lambda$ actually enter the sum in the binomial theorem.
(2) By (1.5) and (1.9) all denominators in the binomial theorem have explicit multiplicative expression.
(3) One can easily remember the binomial formula in the same way as the formula (1.4).
(4) The binomial formula calls for notation

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,t} = \frac{P^*_\mu(q^\lambda;q,t)}{P^*_\mu(q^\mu;q,t)}.$$

One can rewrite the binomial formula as an identity for the above (q,t)-binomial coefficients; see the formula (2.5) in the next section. Note that unlike q-binomial coefficients the above (q,t)-binomial coefficients are not polynomials[1].

(5) By (1.9) we have

$$P^*_\mu(a,\dots,a;q,t) = (-a)^{|\mu|} q^{n(\mu')} t^{-n(\mu)-|\mu|(n-1)} P^*_\mu(1/a,\dots,1/a;1/q,1/t).$$

Therefore we can rewrite the binomial formula as follows

$$\frac{P^*_\lambda(ax_1,\dots,ax_n;q,t)}{P^*_\lambda(a,\dots,a;q,t)} = \sum_\mu (-1)^{|\mu|} \frac{t^{n(\mu)}}{q^{n(\mu')}} \frac{P^*_\mu(q^{-\lambda};1/q,1/t)}{P^*_\mu(q^{-\mu};1/q,1/t)} \frac{P^*_\mu(x_n,\dots,x_1;1/q,1/t)}{P^*_\mu(1/a,\dots,1/a;1/q,1/t)}. \tag{1.10}$$

[1] M. Lassalle recently pointed out that in particular case when λ/μ is a single square these binomial coefficients appeared in [Ka2].

Using (1.8) and the well known equality

$$P_\lambda(x;q,t) = P_\lambda(x;1/q,1/t)$$

we obtain as limit cases as $a \to \infty$ or $a \to 0$ binomial formulas involving ordinary Macdonald polynomials. We have

$$\frac{P_\lambda(x_1,\dots,x_n t^{1-n};q,t)}{P_\lambda(1,\dots,t^{1-n};q,t)} = \sum_\mu \frac{P^*_\mu(q^\lambda;q,t)}{P^*_\mu(q^\mu;q,t)} \frac{P^*_\mu(x_1,\dots,x_n;q,t)}{P_\mu(1,\dots,t^{1-n};q,t)}. \tag{1.11}$$

and

$$\frac{P^*_\lambda(x_1,\dots,x_n;q,t)}{P^*_\lambda(0,\dots,0;q,t)} = \sum_\mu \frac{P^*_\mu(q^{-\lambda};1/q,1/t)}{P^*_\mu(q^{-\mu};1/q,1/t)} \frac{P_\mu(x_1,\dots,x_n t^{1-n};q,t)}{P^*_\mu(0,\dots,0;q,t)}, \tag{1.12}$$

as $a \to \infty$ and $a \to 0$ respectively.

If we set $t = q^\theta$ and let $q \to 1$ then (1.11) turns into the binomial formula for Jack polynomials proved in [OO2]. Thus, the above binomial theorem is a (far reaching) generalization of that formula. Various results on binomial coefficients for Jack polynomials were obtained in [Bi,FK,Ka,Lasc,La,OO].

In the next section we list some applications of the binomial formula. Those applications are quite different from the applications of the Jack degeneration of the binomial formula considered in [OO2,OO4,Ok5,Ok6] (see also [KOO]). Then in section 3 we recall some results of F. Knop and S. Sahi. The proof of the theorem is given in section 4. Finally, in Appendix, we discuss an elementary algorithm for multivariate symmetric Newton interpolation (to avoid possible confusion: this is not the interpolation considered by Lascoux and Schützenberger.).

A binomial formula for the Koornwinder polynomials (which generalize Macdonald polynomials for the classical root systems) was proved in [Ok4].

I would like to thank G. Olshanski and S. Sahi for helpful discussions.

§2 Applications

Consider the following matrix whose rows and columns are indexed by partitions with at most n parts

$$\mathbf{S}(a,n;q,t) = \left(\frac{P^*_\lambda(aq^{-\nu_n},\dots,aq^{-\nu_1};q,t)}{P^*_\lambda(a,\dots,a;q,t)} \right)_{\lambda,\nu}.$$

From the binomial theorem we have

Symmetry. *The matrix $\mathbf{S}(a,n;q,t)$ is symmetric for any a.*

This symmetry is a simple yet central result. Since it interchanges the label and the argument of the polynomial P^*_μ all properties of P^*_μ come in pairs, where one is obtained from the other using the above symmetry. For example, one checks that we have two following pairs

$$\boxed{\text{combinatorial formula}} \leftrightarrow \boxed{q\text{-integral representation}}$$

$$\boxed{\text{difference equations}} \leftrightarrow \boxed{\text{Pieri-type rules}}$$

Similarly, it is evident from (1.11) that the matrix

$$\mathbf{S}(\infty, n; 1/q, 1/t) = \left(\frac{P_\lambda(q^{\nu_1}, \dots, q^{\nu_n} t^{1-n}; q, t)}{P_\lambda(1, \dots, t^{1-n}; q, t)} \right)_{\lambda,\nu} \tag{2.1}$$

is symmetric, which was conjectured by I. Macdonald, proved by T. Koornwinder (unpublished, see [M], section VI.6), and, by different methods, in [EK] and [Ch].

The binomial formula is just the Gauss decomposition of the symmetric matrix

$$\mathbf{S}(a, n; q, t) = \mathbf{B}'(1/q, 1/t)\, \mathbf{A}(a, n; 1/q, 1/t)\, \mathbf{B}(1/q, 1/t) \tag{2.2}$$

where prime stands for transposition, the matrix

$$\mathbf{B}(q, t) = \left(\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} \right)_{\mu,\lambda} \tag{2.3}$$

is upper triangular (with respect to the partial ordering of partitions by inclusion) and unipotent, and the matrix

$$\mathbf{A}(a, n; q, t) = \left(\delta_{\lambda\mu} (-1)^{|\mu|} \frac{q^{n(\mu')}}{t^{n(\mu)}} \frac{P^*_\mu(q^\mu; q, t)}{P^*_\mu(1/a, \dots, 1/a; q, t)} \right)_{\mu,\lambda}$$

is diagonal.

The symmetry of the matrix $\mathbf{S}$ has the following combinatorial interpretation. Suppose that for some $b \in \mathbb{N}$

$$\lambda, \nu \subset \beta$$

where β is a box $\beta = (b^n)$. Denote by

$$\beta \setminus \lambda = (b - \lambda_n, \dots, b - \lambda_1)$$

the partition complementary to λ in β. Setting $a = q^b$ we obtain

$$\frac{P^*_\lambda(q^{\beta\setminus\nu}; q, t)}{P^*_\lambda(q^\beta; q, t)} = \frac{P^*_\nu(q^{\beta\setminus\lambda}; q, t)}{P^*_\nu(q^\beta; q, t)} . \tag{2.4}$$

The skew diagram $(\beta \setminus \nu)/\lambda$ is depicted in the following Fig. 1 (where the diagram ν is rotated by $180°$)

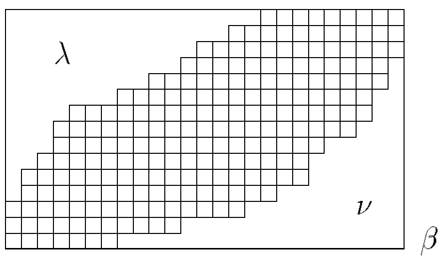

Fig. 1

Introduce the following *trinomial coefficients*

$$\begin{bmatrix} \beta \\ \lambda, \nu \end{bmatrix}_{q,t} = \begin{bmatrix} \beta \\ \nu \end{bmatrix}_{q,t} \begin{bmatrix} \beta \setminus \nu \\ \lambda \end{bmatrix}_{q,t} .$$

Using them one rewrites (2.4) as follows

Symmetry II.

$$\begin{bmatrix} \beta \\ \lambda, \nu \end{bmatrix}_{q,t} = \begin{bmatrix} \beta \\ \nu, \lambda \end{bmatrix}_{q,t}.$$

In particular,

$$\begin{bmatrix} \beta \\ \nu \end{bmatrix}_{q,t} = \begin{bmatrix} \beta \\ \beta \setminus \nu \end{bmatrix}_{q,t}.$$

In the shifted Schur function case the symmetry is obvious (see Fig. 1) from the following formula (see [OO], §8, and also [OO2], §5 for Jack generalization)

$$\begin{bmatrix} \beta \\ \lambda, \nu \end{bmatrix}_{1,1} = \binom{|\beta|}{|\lambda|, |\nu|} \frac{\dim \lambda \, \dim \nu}{\dim \beta} \, \dim(\beta \setminus \nu)/\lambda \,,$$

where dim stands for the number of standard tableaux on a (skew) diagram.

Observe that

$$\mathbf{A}(q^b, n; 1/q, 1/t) = \left(\delta_{\lambda\mu} (-1)^{|\mu|} \frac{t^{n(\mu)}}{q^{n(\mu')}} \begin{bmatrix} \beta \\ \mu \end{bmatrix}_{1/q,1/t}^{-1} \right)_{\mu,\lambda}$$

Therefore, the binomial formula can be rewritten as follows

Binomial Theorem II.

$$\begin{bmatrix} \beta \\ \lambda, \nu \end{bmatrix}_{q,t} = \begin{bmatrix} \beta \\ \lambda \end{bmatrix}_{q,t} \begin{bmatrix} \beta \\ \nu \end{bmatrix}_{q,t} \sum_{\mu} (-1)^{|\mu|} \frac{t^{n(\mu)}}{q^{n(\mu')}} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{1/q,1/t} \begin{bmatrix} \nu \\ \mu \end{bmatrix}_{1/q,1/t} \begin{bmatrix} \beta \\ \mu \end{bmatrix}_{1/q,1/t}^{-1}. \tag{2.5}$$

Note that only summands with $\mu \subset \lambda \cap \nu$ actually enter this sum.

We denote by $\Lambda^t(n)$ the algebra of polynomials in $x_1, \dots, x_n$ which are symmetric in variables

$$x_i t^{-i} .$$

We call such polynomials *shifted symmetric*. The polynomials $P^*_\mu(x; q, t)$, where μ ranges over partitions with at most n parts, form a linear basis in this algebra.

The binomial formula gives transition coefficients between various mutually triangular linear bases of $\Lambda^t(n)$. Combining, for example, (1.11) and (1.12) and taking into account that by (1.9)

$$\frac{P^*_\mu(0, \dots, 0; q, t)}{P_\mu(1, t^{-1}, \dots, t^{1-n}; q, t)} = (-1)^{|\mu|} q^{n(\mu')} t^{-n(\mu)}$$

we obtain

$$\delta_{\lambda\nu} = (-1)^{|\nu|} \frac{t^{n(\nu)}}{q^{n(\nu')}} \sum_{\mu} (-1)^{|\mu|} \frac{q^{n(\mu')}}{t^{n(\mu)}} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} \begin{bmatrix} \mu \\ \nu \end{bmatrix}_{1/q,1/t}.$$

This is equivalent to the following proposition

Inversion. *The inverse matrix of the matrix* (2.3) *is given by*

$$\mathbf{B}^{-1}(q,t) = \mathbf{C}^{-1}(q,t)\,\mathbf{B}(1/q,1/t)\,\mathbf{C}(q,t)\,, \tag{2.6}$$

where $\mathbf{C}(q,t)$ *is the diagonal matrix*

$$\mathbf{C}(q,t) = \left(\delta_{\lambda\mu}(-1)^{|\mu|}\,\frac{q^{n(\mu')}}{t^{n(\mu)}}\right)_{\mu,\lambda}\,.$$

Note that the inverse of the (truncated) matrix (2.1) in the case when q is a root of unity was calculated by A. A. Kirillov, Jr. [Ki] and also by I. Cherednik [Ch].

The algebra $\Lambda^t(n)$ is filtered by degree of polynomials. We denote by Λ^t the projective limit of algebras $\Lambda^t(n)$, $n \in \mathbb{N}$ with respect to homomorphisms

$$f(x_1,\dots,x_{n+1}) \mapsto f(x_1,\dots,x_n,1)\,.$$

Note that $f(q^\lambda)$ is well defined for all $f \in \Lambda^t$ and $\lambda \in \mathcal{P}$. Consider the Newton interpolation of a polynomial $f \in \Lambda^t$ with knots

$$x = q^\lambda\,, \quad \lambda \in \mathcal{P}\,,$$

or, in other words, the expansion of f in the linear basis $\{P^*_\mu(x;q,t)\}_{\mu\in\mathcal{P}}$ of Λ^t. The coefficients $\widehat{f}(\mu;q)$ of this expansion

$$f(x) = \sum_\mu \widehat{f}(\mu;q)\,P^*_\mu(x;q,t)$$

can be found from the following non-degenerate triangular (with respect to ordering of $\mathcal{P}$ by inclusion) system of linear equations

$$f(q^\lambda) = \sum_{\mu\subset\lambda} \widehat{f}(\mu;q)\,P^*_\mu(q^\lambda;q,t)\,, \quad \mu,\lambda\in\mathcal{P}\,. \tag{2.7}$$

Using (2.6) we can solve the equations (2.7) and obtain

Explicit Newton Interpolation. *The coefficients* $\widehat{f}(\mu;q)$ *are given by*

$$\widehat{f}(\mu;q) = \frac{1}{P^*_\mu(q^\mu;q,t)} \sum_{\nu\subset\mu} (-1)^{|\mu/\nu|}\,\frac{q^{n(\mu'/\nu')}}{t^{n(\mu/\nu)}} \begin{bmatrix}\mu\\ \nu\end{bmatrix}_{1/q,1/t} f(q^\nu)\,, \tag{2.8}$$

where $n(\mu/\nu) = n(\mu) - n(\nu)$.

In particular, this formula can be used for expansion of any homogeneous symmetric polynomial in ordinary Macdonald polynomials.

One can think of the function

$$\widehat{f} : \mu \mapsto \widehat{f}(\mu,q) \quad \mu\in\mathcal{P}$$

is an of analog of the *Fourier transform* of the function

$$f : \mu \mapsto f(q^\mu) \quad \mu \in \mathcal{P}\,.$$

The inverse transform is obviously given by (2.7).

Combining the formula (2.8) with the Knop-Sahi difference equation for the polynomials P^*_μ one can obtain a nice multivariate generalization of the classical *algorithm* for Newton interpolation with knots $0, 1, 2, \dots$ (see Appendix).

The binomial theorem provides a natural proof of the **q-integral representation** for interpolation Macdonald polynomials found in [Ok3]. In [Ok3] the integral representation was guessed and checked. One can deduce the integral representation from the formulas (1.7) and (1.11) in exactly the same way as it was done in [OO2] for Jack polynomials.

Notice that the the binomial formula makes certain sense for λ which is not a partition. An example of such *analytic continuation* is the formula (2.10) below. Notice, however, that the RHS may or may not converge to the LHS in the usual analytical sense.

As two particular cases of the binomial formula, one obtains two following generation functions for the 1-column and 1-row interpolation Macdonald polynomials. By definition, set:

$$\begin{aligned} e^*_k(x;t) &= P^*_{(1^k)}(x;q,t)\,, \\ h^*_k(x;q,t) &= P^*_{(k)}(x;q,t)\,, \quad k = 1, 2, \dots\,. \end{aligned}$$

The polynomials $e^*_k(x;t)$ do not depend on q for it follows from the combinatorial formula (1.6) that

$$e^*_k(x;t) = \sum_{i_1 < \dots < i_k} t^{k - \sum i_s} \prod_s (x_i - t^{s-k})\,.$$

Let u be a parameter. We have the following

Generating functions.

(2.9)
$$\prod_{i=1}^{\infty} \frac{1 + x_i t^{1-i}/u}{1 + t^{1-i}/u} = \sum_{k=0}^{\infty} \frac{e^*_k(x;t)}{(u+1)(u+1/t)\dots(u+1/t^{k-1})}\,,$$

(2.10)
$$\prod_{i=1}^{\infty} \frac{(x_i t^{1-i}/u;q)_\infty}{(x_i t^{-i}\ /u;q)_\infty} \frac{(t^{-i}\ /u;q)_\infty}{(t^{1-i}/u;q)_\infty} = \sum_{k=0}^{\infty} t^{-k} \frac{(t;q)_k}{(q;q)_k} \frac{h^*_k(x;q,t)}{(u-1)(u-q)\dots(u-q^{k-1})}\,.$$

Here $(z;q)_r = (1-z)(1-qz)\dots(1-q^{r-1}z)$. Note that above expansions make sense as formal power series in $1/u$ with coefficients in Λ^t.

This generating function (2.9) is a minor modification of the generating function in [OO], theorem 12.1. The analog of the generating function (2.10) for shifted Jack polynomials was found by G. Olshanski (unpublished) and it was used by him to prove the Jack degeneration of the 1-row case of the combinatorial formula (1.6).

It is clear that if

$$x_{n+1} = x_{n+2} = \cdots = 1$$

then (2.9) is just the binomial formula for $\lambda = (1^n)$ and $a = -t^{1-n}/u$, whereas (2.10) is the binomial expansion of

$$\frac{P^*_\lambda(x_n/u, \dots, x_1/u; 1/q, 1/t)}{P^*_\lambda(1/u, \dots, 1/u; 1/q, 1/t)}$$

analytically continued to the point λ satisfying

$$q^\lambda = (1/t, \dots, 1/t).$$

It is easy to check both expansions directly.[2] Since it will be convenient for us to use (2.9) below in section 3, we shall give such a direct proof, which is essentially the argument from [OO], section §12. Same argument applies to (2.10)

Proof of (2.9). Let $e^0_k(x;t)$ be the coefficients in the expansion

$$\prod_{i=1}^{\infty} \frac{1 + x_i t^{1-i}/u}{1 + t^{1-i}/u} = \sum_{k=0}^{\infty} \frac{e^0_k(x;t)}{(u+1)(u+1/t)\dots(u+1/t^{k-1})}. \tag{2.11}$$

We want to prove that $e^0_k(x;t) = e^*_k(x;t)$. It is clear that $e^0_k \in \Lambda^t$. Set $x_i = \xi_i u z$, where ξ_i and z are new variables and let $u \to \infty$. One obtains

$$e^0_k(x;t) = e_k(x_1, x_2 t^{-1}, \dots) + \text{ lower degree terms}.$$

Therefore, it suffices to prove that

$$e^0_n(x_1, \dots, x_{n-1}, 1, \dots, 1; t) = 0, \quad n = 1, 2, \dots .$$

Substitute $x_n = x_{n+1} = \cdots = 1$ in (2.11). Then the RHS becomes

$$\prod_{i=1}^{n-1} \frac{u + x_i t^{1-i}}{u + t^{1-i}}.$$

In its expansion of the form (2.11) all summands with $k \geq n$ vanish. $\square$

The expansions (2.9) and (2.10) are related by the **duality** for the interpolation Macdonald polynomials, see [Ok3], theorem IV. That duality (in particular, the proposition 6.1 from [Ok3]) implies the following duality for (q,t)-binomial coefficients

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q,t} = \begin{bmatrix} \lambda' \\ \mu' \end{bmatrix}_{1/t,1/q}. \tag{2.12}$$

[2] In fact, one can generalize the expansion (2.9) by replacing the sequence $1, 1/t, 1/t^2, \dots$ by an arbitrary sequence. That type of expansions was considered by A. Abderrezzak in [Ab]. However, only very few facts about interpolation Macdonald polynomials admit such a vast generalization, see [Ok7].

§3 Difference equations for $P^*_\mu(x;q,t)$.

In this section we recall some results due to F. Knop and S. Sahi [Kn,S2]. Only the explicit formulas for higher order difference operators are new.

Denote by $T_{x,q}$ the shift operator

$$[T_{x,q}f](x) = f(qx)\,.$$

Observe that for any polynomials p_1, p_2 in x such that

$$p_1(0) = p_2(0)$$

the operator

$$f \longmapsto \frac{p_1(x)f(x) - p_2(x)f(qx)}{x}$$

maps polynomials to polynomials. Set

$$T_i = T_{x_i,1/q}\,.$$

Consider the following n-dimensional difference operator

$$D(u) = V(x)^{-1} \det \left[\frac{(1-ux_i)}{x_i}(1-tx_i)^{j-1} - \frac{(1-x_i)^j}{x_i}\, T_i \right]_{1\le i,j\le n}\,.$$

Here u is a parameter and $V(x)$ is the Vandermonde determinant. The determinant is well defined since multiplication by x_i and the shift T_j commute provided $i \neq j$. It is clear that $D(u)$ maps $\Lambda(n)$ to $\Lambda(n)$ and does not increase degree.

For any subset $I \subset \{1,\dots,n\}$ set

$$T_I = \prod_{i\in I} T_i\,.$$

Using the Vandermonde determinant formula one expands $D(u)$ as follows

$$D(u) = \prod x_i^{-1} \sum_{I\subset\{1,\dots,n\}} (-1)^{|I|} t^{\cdots} \prod_{i\in I}(1-x_i) \prod_{i\notin I}(1-ux_i) \prod_{i\in I,\, j\notin I} \frac{x_i - tx_j}{x_i - x_j}\, T_I\,,$$

where the omitted exponent of t equals $(n-|I|)(n-|I|-1)/2$.

Now shift the variables

$$x_i \mapsto x_i t^{n-i}$$

and consider the operator

$$D^*(u) = \prod x_i^{-1} \sum_{I\subset\{1,\dots,n\}} (-1)^{|I|}\, t^{(n-|I|)(n-|I|-1)/2}$$
$$\prod_{i\in I}(1-x_i t^{n-i}) \prod_{i\notin I}(1-ux_i t^{n-i}) \prod_{i\in I,\, j\notin I} \frac{x_i - x_j t^{i-j+1}}{x_i - x_j t^{i-j}}\, T_I\,,$$

which maps $\Lambda^t(n)$ to $\Lambda^t(n)$.

Below we shall have to evaluate

$$[D^*(u)f]\,(x)\,, \quad f \in \Lambda^t(n)$$

at the points of the form

$$x = (aq^{\xi_1}, \dots, aq^{\xi_n})\,, \quad \xi_i \in \mathbb{Z}\,, \tag{3.1}$$

where $\xi_1 \geq \cdots \geq \xi_n$ and a is a parameter (which will be the same parameter a as in the binomial theorem). Fix a subset I and set

$$\xi'_i = \begin{cases} \xi_i - 1, & i \in I \\ \xi_i, & i \notin I\,. \end{cases}$$

We have the following obvious

Lemma. *Let x be as in* (3.1)*. Then*

$$\prod_{i\in I,\, j\notin I} \frac{x_i - x_j t^{i-j+1}}{x_i - x_j t^{i-j}} \neq 0$$

iff $\xi'_1 \geq \cdots \geq \xi'_n$

This lemma results in the following

Theorem.

$$D^*(u)\, P^*_\mu(x;q,t) = \left(\prod_i (q^{-\mu_i} t^{i-1} - ut^{n-1})\right) P^*_\mu(x;q,t)\,. \tag{3.2}$$

This theorem (without explicit formula for $D^*(u)$) is due to F. Knop and S. Sahi [Kn,S2].

Idea of Proof. Using that

$$\prod_{i\in I}(1 - x_i t^{n-i}) = 0 \quad \text{if} \quad x_n = 1 \quad \text{and} \quad i \in I$$

and the lemma one easily checks that

$$\left[D^*(u)P^*_\mu\right](q^\lambda) = 0, \quad \lambda \in \mathcal{P},\, \lambda \not\subset \mu$$

and that

$$\left[D^*(u)P^*_\mu\right](q^\mu) = \left(\prod_i (q^{-\mu_i} t^{i-1} - ut^{n-1})\right) P^*_\mu(q^\mu)\,. \quad \square$$

In particular, the operators $D^*(u)$ for different values of u commute. Consider the algebra $\mathfrak{D}$ generated by these commuting difference operators.

This commutative algebra is isomorphic to the algebra $\Lambda^{1/t}(n)$ under the Harish-Chandra map which maps an difference operator $D \in \mathfrak{D}$ to the polynomial $d \in \Lambda^{1/t}(n)$ such that

$$D\, P^*_\mu(x;q,t) = d(q^{-\mu})\, P^*_\mu(x;q,t)\,.$$

Let D_k be the coefficient

$$D^*(u) = \sum_{k\le n} (-1)^{n-k} t^{(n-1)(n-k)} (u-1)(u-1/t)\dots(u-1/t^{n-k-1})\, D_k$$

in the Newton interpolation of $D^*(u)$ with knots

$$u = 1, 1/t, 1/t^2, \dots\,.$$

By (3.2) and (2.9) we have

$$D_k\, P^*_\mu(x;q,t) = e^*_k(q^{-\mu}; 1/t)\, P^*_\mu(x;q,t)\,.$$

The operators D_k are generators of the algebra $\mathfrak{D}$.

§4 Proof of the binomial theorem.

We begin with general remark about the proof. As it is usual in the Macdonald polynomials theory, various pieces of the puzzle can be arranged in various combination to give a proof. Below we shall use the approach based on the Knop-Sahi difference equations. However, it will be clear that the formula (4.4), which states that the LHS is a polynomial in $q^{-\lambda}$ with a certain type of symmetry, is the key point of the proof. Instead of using difference equations, one can deduce (4.4) and hence the binomial theorem from the q-integral representation of interpolation Macdonald polynomials [Ok3].

In this section by c_γ we always denote *some* coefficients. This notation is used to show which summands possibly enter the formula.

It is clear from definitions that

$$D_k = \sum_{|I|=k} (-1)^{|I|} t^{\cdots} \prod_{i\in I} \frac{(1-x_i t^{n-i})}{x_i} \prod_{i\in I,\, j\notin I} \frac{x_i - x_j t^{i-j+1}}{x_i - x_j t^{i-j}}\, T_I + \sum_{|I|<k} \dots\,.$$

Suppose

$$x = (aq^{\xi_1}, \dots, aq^{\xi_n})\,, \quad \xi \in \mathbb{Z}\,,$$

where $\xi_1 \ge \dots \ge \xi_n$. Then by the lemma

$$[D_k f]\,(x) = \sum_\eta c_\eta f(aq^{\eta_1}, \dots, aq^{\eta_n}) \tag{4.1}$$

with some coefficients c_η, where

$$\begin{gathered} \eta_1 \ge \dots \ge \eta_n\,, \\ \xi_i - \eta_i \in \{0,1\}\,, \quad i = 1,\dots,n\,, \\ \sum \xi - \sum \eta_i \le k\,. \end{gathered} \tag{4.2}$$

Order the n-tuples η lexicographically as follows

$$\eta > \tilde{\eta}$$

if $\eta_n < \tilde{\eta}_n$, or if $\eta_n = \tilde{\eta}_n$ and $\eta_{n-1} < \tilde{\eta}_{n-1}$, and so on. Then the maximal η satisfying (4.2) is

$$\eta_{\max} = (\xi_1, \dots, \xi_{n-k}, \xi_{n-k+1} - 1, \dots, \xi_n - 1)\,.$$

It is clear that the coefficient

$$c_{\eta_{\max}} \neq 0$$

in (4.1) is a non-zero rational function of a.

Now let μ be a partition and let μ' be the conjugate partition. Iterating the above argument we see that

$$\left[\prod_i D_{\mu_i} f\right](a, \dots, a) = \sum_\nu c_\nu f(aq^{-\nu_n}, \dots, aq^{-\nu_1})\,, \tag{4.3}$$

where c_ν are some coefficients and ν ranges over partitions such that $|\nu| \leq |\mu|$ and

$$\nu \leq \mu'$$

in the lexicographic order of partitions. Observe that the coefficient $c_{\mu'}$ in (4.3) does not vanish. Therefore there exist an operator $D(\mu, a) \in \mathfrak{D}$

$$D(\mu, a) = \sum_{|\nu| \leq |\mu|, \nu \leq \mu} c_\nu \prod_i D_{\nu'_i}$$

such that

$$\left[D(\mu, a) f\right](a, \dots, a) = f(aq^{-\mu_n}, \dots, aq^{-\mu_1})\,.$$

Denote by $d_{\mu,a} \in \Lambda^{1/t}(n)$ the Harish-Chandra image of $D(\mu, a)$

$$D(\mu, a)\, P^*_\lambda(x; q, t) = d_{\mu,a}(q^{-\lambda})\, P^*_\lambda(x; q, t)\,.$$

By definition of $D(\mu, a)$ we have

$$\frac{P^*_\lambda(aq^{-\mu_n}, \dots, aq^{-\mu_1})}{P^*_\lambda(a, \dots, a)} = d_{\mu,a}(q^{-\lambda})\,. \tag{4.4}$$

Since the degree of e^*_k is k we have

$$\deg d_{\mu,a} \leq |\mu|\,.$$

Now consider the Newton interpolation of the function

$$f(x) = \frac{P^*_\lambda(ax_1, \dots, ax_n; q, t)}{P^*_\lambda(a, \dots, a; q, t)} \quad \in \Lambda^t(n)$$

with knots

$$x = (q^{-\mu_n}, \dots, q^{-\mu_1}), \quad \mu \in \mathcal{P}$$

This interpolation has the form

$$\frac{P^*_\lambda(ax_1, \dots, ax_n; q, t)}{P^*_\lambda(a, \dots, a; q, t)} = \sum_\mu b_{\mu,\lambda,a} \, P^*_\mu(x_n, \dots, x_1; 1/q, 1/t) \,. \tag{4.5}$$

The coefficients $b_{\mu,\lambda,a}$ are linear combinations of the values

$$\frac{P^*_\lambda(aq^{-\nu_n}, \dots, aq^{-\nu_1}; q, t)}{P^*_\lambda(a, \dots, a; q, t)}, \quad \nu \subset \mu \,.$$

By (4.4) we have

$$b_{\mu,\lambda,a} = b_{\mu,a}(q^{-\lambda}) \,,$$

for certain polynomials

$$b_{\mu,a} \in \Lambda^{1/t}(n) \,, \quad \deg b_{\mu,a} \le |\mu| \,.$$

Next observe that the highest degree term of the LHS of (4.5) equals

$$\frac{a^{|\lambda|}}{P^*_\lambda(a, \dots, a; q, t)} P_\lambda(x_1, x_2 t^{-1}, \dots, x_n t^{1-n}; q, t)$$

and

$$P^*_\mu(x_n, \dots, x_1; 1/q, 1/t) = t^{(n-1)|\mu|} P_\mu(x_1, x_2 t^{-1}, \dots, x_n t^{1-n}; q, t) + \dots \,,$$

where dots stand for lower degree terms. Therefore

$$b_{\mu,a}(q^{-\lambda}) = \begin{cases} 0\,, & |\lambda| \le |\mu| \quad \text{and} \quad \mu \ne \lambda \,, \\ a^{|\mu|} t^{(1-n)|\mu|} P^*_\mu(a, \dots, a; q, t)^{-1} \,, & \mu = \lambda \,. \end{cases}$$

Since $\deg b_{\mu,a} \le |\mu|$ the polynomial $b_{\mu,a}$ is uniquely determined by its values at the points $x = q^{-\lambda}$, $|\lambda| \le |\mu|$. This implies

$$b_{\mu,a}(x) = \frac{a^{|\mu|} t^{(1-n)|\mu|}}{P^*_\mu(a, \dots, a; q, t)} \frac{P^*_\mu(x; 1/q, 1/t)}{P^*_\mu(q^{-\mu}; 1/q, 1/t)}$$

and concludes the proof of the theorem.

Appendix. Symmetric Newton's algorithm

The formula (2.8) can be turned into a quite efficient algorithm for multivariate symmetric Newton interpolation.

Reccurence relations for generalized binomial coefficients were first used in the Jack polynomial case by M. Lassalle, see [La][3]. Those relations correspond to a

[3] See also M. Lassalle's recent preprint *Coefficients binomiaux généralisés et polynômes de Macdonald*, §10.

Pieri type formula for interpolation Macdonald polynomials (see [OO2], §5). They are related to the Knop-Sahi type relations used below by the *symmetry*, see §2.

Let λ be a diagram and suppose that ν is obtained from λ by removing a corner. We shall denote this by writing

$$\nu \nearrow \lambda \,.$$

Denote the corner being removed by $\bigstar$, see Fig. 2

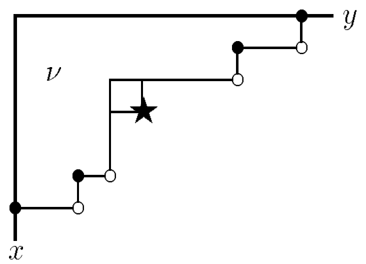

Fig. 2. The diagram ν is obtained from λ by removing the corner $\bigstar$

Consider the outer corners of the diagram ν (denoted by $\circ$ in Fig. 2) and also the inner corner of ν excluding the one where the square λ/ν is attached (those inner corners are denoted by $\bullet$ in Fig. 2). By definition, set

$$\omega(\nu \nearrow \lambda; q, t) = \frac{\displaystyle\prod_{\text{inner corners } \bullet} \left(1 - q^{\Delta y(\bigstar,\bullet)} t^{\Delta x(\bullet,\bigstar)}\right)}{\displaystyle\prod_{\text{outer corners } \circ} \left(1 - q^{\Delta y(\bigstar,\circ)} t^{\Delta x(\circ,\bigstar)}\right)} \,,$$

where

$$\begin{aligned} \Delta y(\bigstar,\bullet) &= y\text{-coordinate } (\bigstar) - y\text{-coordinate } (\bullet) \,, \\ \Delta x(\bullet,\bigstar) &= x\text{-coordinate } (\bullet) - x\text{-coordinate } (\bigstar) \,. \end{aligned}$$

Now let $f \in \Lambda^t$ be the shifted symmetric polynomial to be interpolated. For any pair $\mu \subset \lambda$ set by definition

$$F(\mu, \lambda) = (-1)^{|\lambda/\mu|} \frac{q^{n(\lambda'/\mu')}}{t^{n(\lambda/\mu)}} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{1/q,1/t} f(q^\mu) \,,$$

where $n(\lambda/\mu) = n(\lambda) - n(\mu)$. Then, the interpolation formula (2.8) can be rewritten as follows

$$\widehat{f}(\lambda) = \sum_{\mu \subset \lambda} F(\mu, \lambda) \,. \tag{5.1}$$

On easily computes that the Knop-Sahi difference equation

$$D_1 P^*_\mu(x; q, t) = \left(\sum_i t^{i-1} (q^{-\mu_i} - 1) \right) P^*_\mu(x; q, t)$$

implies that

$$\sum_{\nu\nearrow\lambda} \omega(\nu\nearrow\lambda;q,t) \begin{bmatrix}\nu\\ \mu\end{bmatrix}_{1/q,1/t} = \left(\begin{bmatrix}\lambda\\ 1\end{bmatrix}_{q,t} - \begin{bmatrix}\mu\\ 1\end{bmatrix}_{q,t} \right) \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{1/q,1/t}. \tag{5.2}$$

Therefore, the function $F(\mu,\lambda)$ satisfies the following reccurence relation

$$F(\mu,\lambda) = -\sum_{\mu\subset\nu\nearrow\lambda} \frac{q^{a'(\lambda/\nu)}}{t^{l'(\lambda/\nu)}} \frac{\omega(\nu\nearrow\lambda;q,t)}{\begin{bmatrix}\lambda\\ 1\end{bmatrix}_{q,t} - \begin{bmatrix}\mu\\ 1\end{bmatrix}_{q,t}} F(\mu,\nu),$$

which, together with the obvious initial condition

$$F(\mu,\mu) = f(q^{\mu}),$$

give a simple algorithm for calculation of this function. The summation (5.1) can be performed simultaneously with the computation of $F(\mu,\lambda)$.

Both in formula (2.8) and in the above algorithm one has to run a cycle over all pairs of partitions $\mu\subset\lambda$. This means of order N^2 iterations to interpolate in N points. (The same number of iterations is sufficient for direct solution of the equations (2.7).) However, the above algorithm has one clear advantage:

The function $\omega(\nu\nearrow\lambda;q,t)$, which we have to compute on each step, is much more simple than the coefficients $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{1/q,1/t}$. In other words, one can expand a polynomial f in interpolation Macdonald polynomials without actually knowing those quite complicated polynomials explicitly.

In the Jack limit this algorithm can be made exponentially faster. Using the fact that the LHS of (5.2) depends in the Jack limit only on $|\mu|$, not on particular form of μ, one is able to use on each step some previously computed sums as follows.

We shall use the parameter θ as in [OO2]. Let

$$f(x_1,x_2,\dots)$$

be a polynomial, symmetric in variables $x_i-\theta i$. By definition, set

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{\theta} = \lim_{q\to 1} \begin{bmatrix}\lambda\\ \mu\end{bmatrix}_{q,q^{\theta}}.$$

The polynomials

$$\begin{bmatrix}x\\ \mu\end{bmatrix}_{\theta}$$

form a linear basis in the algebra of polynomials symmetric in variables $x_i-\theta i$. The coefficients $\widehat{f}(\mu)$ in the expansion

$$f = \sum_{\mu} \widehat{f}(\mu) \begin{bmatrix}x\\ \mu\end{bmatrix}_{\theta}$$

can be found from the following specialization of the formula (2.8)

$$\widehat{f}(\lambda)=\sum_{\mu\subset\lambda}(-1)^{|\lambda/\mu|}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_\theta f(\mu)\,.$$

Now, instead of computing the numbers

$$F(\mu,\lambda)=(-1)^{|\lambda/\mu|}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_\theta f(\mu)\,,$$

we shall deal with the sums

$$\begin{aligned}F^{(k)}(\lambda)&=\sum_{\substack{\mu\subset\lambda\\ |\lambda/\mu|\le k}}\binom{|\lambda|}{|\mu|}^{-1}\binom{k}{|\lambda/\mu|}F(\mu,\lambda)\,,\\ &=\sum_{\substack{\mu\subset\lambda\\ |\lambda/\mu|\le k}}(-1)^{|\lambda/\mu|}\binom{|\lambda|}{|\mu|}^{-1}\binom{k}{|\lambda/\mu|}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_\theta f(\mu)\,,\end{aligned}$$

where

$$k=0,\dots,|\lambda|\,.$$

Then we have

$$\begin{aligned}F^{(0)}(\lambda)&=f(\lambda)\,,\\ F^{(|\lambda|)}(\lambda)&=\widehat{f}(\lambda)\,,\end{aligned}$$

and the following reccurence relation

$$F^{(k+1)}(\lambda)=F^{(k)}(\lambda)-\frac{1}{|\lambda|}\sum_{\nu\nearrow\lambda}\omega(\nu\nearrow\lambda;\theta)\,F^{(k)}(\nu)\,,$$

where (see Fig. 2)

$$\begin{aligned}\omega(\nu\nearrow\lambda;\theta)&=\lim_{q\to1}\omega(\nu\nearrow\lambda;q,q^\theta)\\ &=\frac{\displaystyle\prod_{\text{inner corners }\bullet}(\Delta y(\bigstar,\bullet)+\theta\Delta x(\bullet,\bigstar))}{\displaystyle\prod_{\text{outer corners }\circ}(\Delta y(\bigstar,\circ)+\theta\Delta x(\circ,\bigstar))}\,.\end{aligned}$$

The advantage of this algorithm is that instead of a cycle over all pairs of partitions $\mu\subset\lambda$ one has a cycle over all partition λ and all *numbers* k, such that $k\le|\lambda|$, which is much faster.

In the case of one variable this algorithm specializes to the classical Newton's algorithm for interpolation with knots $0,1,2,\dots$ because

$$\omega((l-1)\nearrow(l);\theta)=l\,.$$

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University of Chicago, Dept. of Mathematics, 5734 S. University Ave., Chicago, IL 60637

E-mail address: okounkov@math.uchicago.edu