A RELATION BETWEEN SCHUR P AND S FUNCTIONS

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Abstract — We define a differential operator of infinite order which sends Schur S-functions to Schur P-functions. Using the properties of Δ we deduce algebraic identities satisfied by the cardinalities of certain sets of tableaux.

Résumé — Nous définissons un opérateur différentiel d'ordre infini qui envoie les fonctions S de Schur sur les fonctions P de Schur. A l'aide des propriétés de Δ, nous déduisons des identités algébriques satisfaites par les cardinalités de certains ensembles de tableaux.

1 Introduction

The subject of this paper is to describe a differential operator Δ which sends a Schur S-function s_λ to a Schur P-function $P_{c(\lambda)}$ where $c(\lambda)$ is a composition associated to the partition λ by an easy combinatorial method. The definition of Δ was motivated by the results of [5]. Using the properties of Δ, we can then deduce algebraic identities satisfied by the cardinalities of certain sets of tableaux (Yamanouchi tableaux, Yamanouchi domino tableaux, "Stembridge" tableaux). It would be interesting to obtain bijective proofs of these identities.

The operator Δ has a natural interpretation in the representation theory of affine Lie algebras [6]. Indeed it is well-known that S-functions form a linear basis of the algebra of symmetric functions Sym and P-functions form a linear basis of $\mathcal{T}^{(2)}$ the subalgebra of Sym generated by odd power sums p_i. Furthermore there are natural actions on Sym and $\mathcal{T}^{(2)}$ of the infinite dimensional Lie algebra b_∞ (see [3] ; we follow Kac's notation for affine Lie algebras). It can be shown that Δ is the unique linear map from Sym to $\mathcal{T}^{(2)}$ sending 1 to 1 and commuting with the lowering operators of b_∞. This viewpoint will not be developed here but see [7].

2 Symmetric functions

We review the necessary background in the theory of symmetric functions (plethysms, differential operators) and recall some definitions concerning partitions (2-quotient, 2-core and 2-sign). We also introduce a linear involution ν on Sym, and we associate to each partition λ a composition $c(\lambda)$ and another partition $J(\lambda)$. Our notations for symmetric functions are as in [8].

A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a weakly decreasing sequence of nonnegative integers. We denote by λ' the conjugate partition of λ, $|\lambda|$ the weight of λ and $l(\lambda)$ the length of λ. To a partition $\lambda = (\alpha_1 \ldots \alpha_l | \beta_1 \ldots \beta_l)$ in Frobenius' notation, we associate the composition

$$c(\lambda) = (\alpha_1 + 1, \beta_1, \alpha_2 + 1, \beta_2, \ldots, \alpha_l + 1, \beta_l).$$

An example is shown in Figure 1.

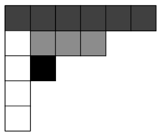

Figure 1: $\lambda = (6,4,2,1,1) = (5,2|4,1), c(\lambda) = (6,4,3,1)$

Products of power sums, Schur S-functions, Schur P-functions are respectively denoted by p_λ, s_λ, P_λ. The Schur P-functions are indexed by strict partitions i.e partitions with distinct parts. We denote by $Sym(A)$ the algebra of symmetric functions in an infinite set of variables $A = \{a_1, a_2, \ldots\}$ with coefficients in $\mathbb{C}$. When there is no danger of confusion we shall omit A and simply write Sym. Sym can also be regarded as the polynomial ring $Sym = \mathbb{C}[p_i;\ i \in \mathbb{N}^*]$. Let $\mathcal{T}^{(2)}$ be the subalgebra of Sym generated by odd power sums $\mathcal{T}^{(2)} = \mathbb{C}[p_{2i+1};\ i \in \mathbb{N}]$. The Schur S-functions give a linear basis of Sym and the Schur P-functions give a linear basis of $\mathcal{T}^{(2)}$.

Let $(\ ,\ )$ be the scalar product of Sym defined by $(s_\lambda, s_\mu) = \delta_{\lambda\mu}$ where λ and μ are partitions and $\delta_{\lambda\mu}$ is Kronecker's symbol. We denote by D_f the adjoint of the multiplication by f, that is,

$$(D_f g, h) = (g, fh), \quad (\ f\ ,\ g\ ,\ h \in Sym)\ .$$

D_f is also called a differential operator since we have for $f = f(p_1, p_2, \ldots)$

$$D_f = f(\frac{\partial}{\partial p_1}, 2\frac{\partial}{\partial p_2}, \ldots, n\frac{\partial}{\partial p_n}, \ldots)\ .$$

In particular we have $D_{s_\mu}(s_\lambda) = s_{\lambda/\mu}$.

We denote by ψ_2 the ring homomorphism of Sym which sends p_i to p_{2i}. ($\psi_2(f)$ is the plethysm of p_2 by f introduced by Littlewood.)

The 2-core $\lambda_{(2)}$ of a partition λ is defined as follows (see [2]). If λ has no 2-hooks $\lambda = \lambda_{(2)}$ otherwise remove as many 2-hooks as possible from the diagram of λ; the partition obtained in this way is $\lambda_{(2)}$. We recall that $\lambda_{(2)}$ is a staircase partition $\rho_k =$

$(k-1,\ldots,1,0)$. We can also compute the 2-core using the algorithm that computes the 2-quotient and the 2-sign as follows. Make λ into a partition of even length $2n$ (where n is the length of λ) by adding if necessary a zero part. Add to λ the staircase partition ρ_{2n}. Reduce modulo 2 from right to left the successive parts of $\mu = \lambda + \rho_{2n}$ without using two times the same representative. This gives a sequence γ, which is put in decreasing order by a permutation σ. We can now compute the 2-core of λ by subtracting the staircase ρ_{2n} from $\sigma(\gamma)$. The 2-sign is $\epsilon_2(\lambda) = sign(\sigma)$. Finally , subtract from the even parts of μ the corresponding residues in γ and divide by 2 to obtain λ^0. The same procedure applied to the odd parts gives the second partition λ^1. There exists a one-to-one correspondence between a partition λ and its 2-core and 2-quotient [2]. We write $\lambda = (\lambda_{(2)}, (\lambda^0, \lambda^1))$. We denote by $Sym_\emptyset$ the subspace spanned by Schur S-functions indexed by partition without 2-core.

We can now define $J(\lambda)$ and λ^*:

$$J(\lambda) = (\emptyset, (\lambda, \emptyset)), \quad \lambda^* = (\emptyset, ((\lambda^0)', \lambda^1)).$$

At last we introduce the linear involution ν of $Sym_\emptyset$ defined by

$$\nu(s_\lambda) = (-1)^{|\lambda^0|} \epsilon_2(\lambda) \epsilon_2(\lambda^*) s_{\lambda^*}.$$

This definition becomes more understandable after the statement of Lemma 6 (ii).

3 Tableaux

In this section we describe the combinatorial objects used in the sequel: Yamanouchi domino tableaux, “Stembridge” tableaux.

A domino diagram of shape λ is a diagram of this shape whose cells are 2×1 or 1×2 rectangles called dominoes. We can see an example on Figure 2.

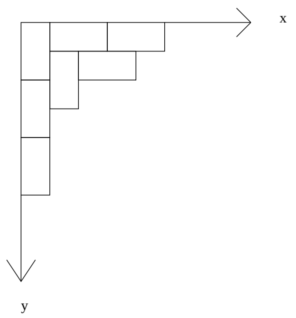

Figure 2: A domino diagram of shape $(5, 4, 2, 1, 1, 1)$

Semistandard domino tableaux are defined as semistandard (ordinary) tableaux, namely, they are obtained by numbering all the dominoes of a domino diagram with nonnegative integers weakly increasing along the rows (from left to right), strictly down the columns (see [1]). To an ordinary tableau T we can associate a word called column reading obtained by reading the successive columns of T from bottom to top and left to right. The column reading of a domino tableau is obtained in the same way except that horizontal dominoes which belong to two successive columns i and $i+1$ are read only when reading column i.

A Yamanouchi word is a word $w = w_1 \dots w_n$ such that each right factor $w_i w_{i+1} \dots w_n$ contains for every j at least as many letters j than $j+1$. For example the word $w = 31423211$ is a Yamanouchi word while $w' = 431223211$ is not. A Yamanouchi tableau (resp. domino tableau) is a tableau (resp. domino tableau) whose column reading is a Yamanouchi word.

A marked tableau T of shape λ is a diagram of this shape whose cells are numbered with a special alphabet $\Sigma = \{1', 1, 2', 2, \dots\}$ (with $1' < 1 < 2' < 2 \dots$) and satisfying the following three rules:

1. $T(i,j) \leq T(i+1,j), \quad T(i,j) \leq T(i,j+1)$,
2. Each column has at most one k $(k = 1, 2, \dots)$,
3. Each row has at most one k' $(k' = 1', 2', \dots)$.

The row reading of a marked tableau T is obtained by reading the successive rows of T from bottom to top and left to right.

Given any word $w = w_1 w_2 \dots w_n$ over Σ we define

$m_i(j)$, the multiplicity of i among $w_{n-j+1} \dots w_n$ $(0 < j \leq n)$,

$m_i(n+j) = m_i(n)+$ the multiplicity of i' among $w_1 w_2 \dots w_j$ $(0 < j \leq n)$.

The word w is said to satisfy the "lattice property" if, whenever $m_i(j) = m_{i-1}(j)$ we have

$$w_{n-j} \neq i, i' \text{ if } (0 \leq j \leq n)$$
$$w_{j-n+1} \neq i-1, i' \text{ if } (n \leq j \leq 2n).$$

The word w is a "Stembridge" word if it satisfies the "lattice property" and if the leftmost i in $|w|$ is unmarked in w (where $|w|$ denotes the word obtained by erasing the marks of w). A "Stembridge" tableau is a tableau whose row reading is a Stembridge word. The content of a "Stembridge" tableau is the content of the corresponding tableau unmarked (see [9]).

On Figure 3, example 1 isn't a "Stembridge" tableau because in
$w = 3\ 3'\ 1'2\ 1'2'22\ 1'11111$ the leftmost occurrence of 1 is marked.

Example 2 isn't a "Stembridge" tableau because $w = 4\ 4'\ 3\ 13'\ 1'\ 23'\ 1'2'22\ 1'1111$ doesn't verify lattice property. Indeed $m_4(18) = 1 = m_3(18)$ and $w_{18-17+1} = w_2 = 4' = i'$. Example 3 is a "Stembridge" tableau.

4 The operator Δ

In [5], a differential operator has been introduced. It sends Schur S-functions on Schur P-functions but with restrictions on the length of the partitions. To improve these results we have considered a new operator Δ, defined in terms of the partition $J(\lambda)$ introduced in Section 2.

Definition 1 *Let Δ be the differential operator on Sym of infinite order :*

$$\Delta = \sum_{\lambda} (-1)^{|\lambda|} \epsilon_2(J(\lambda')) s_{J(\lambda')} D_{\psi_2(s_\lambda)}.$$

Our main result is that:

Theorem 2 *Δ is a projector from Sym to $\mathcal{T}^{(2)}$ and $\Delta(s_\lambda) = P_{c(\lambda)}$.*

The proof of this theorem is in two steps. We first establish the following characterization of Δ:

Proposition 3 *Δ is the unique linear operator satisfying*
(i) $\Delta(fg) = f\Delta(g), (f \in \mathcal{T}^{(2)}, g \in Sym)$,
(ii) $\Delta(\psi_2(g)) = \nu(\psi_2(g)), (\ g \in Sym)$.

We then prove:

Proposition 4 *Let D_1 be the linear map defined by:*

$$\begin{aligned} D_1 : Sym &\rightarrow Sym \\ s_\lambda &\rightarrow P_{c(\lambda)}. \end{aligned}$$

D_1 satisfies (i) and (ii) of Proposition 3.

For the proofs of Propositions 3 and 4 we need to introduce new definitions and objects. Let A_0, A_1 be two infinite sets of independent variables. We shall denote the symmetric functions of A_0 by $Sym(A_0)$, and the symmetric functions of A_1 by $Sym(A_1)$. By $Sym(A_0, A_1)$ we shall mean the functions which are *separately* symmetric in A_0 and A_1. Finally $Sym(A_0 + A_1)$ will denote the symmetric functions of the whole set $A_0 \cup A_1$.

We will use the algebra automorphism w of $Sym(A_0)$ which sends $p_k(A_0)$ to $p_k(-A_0) = -p_k(A_0)$. In general, $w(f(A_0))$ will be denoted by $f(-A_0)$ (λ-ring notation). We recall that

$$s_\lambda(-A_0) = (-1)^{|\lambda|} s_{\lambda'}(A_0), \quad s_\lambda(A_0 + A_1) = \sum_{\nu} s_{\lambda/\nu}(A_0) s_\nu(A_1).$$

Clearly, w can be extended to $Sym(A_0, A_1)$ by setting

$$w(f(A_0)g(A_1)) = f(-A_0)g(A_1),\ f,\ g \in Sym.$$

We shall write $w(Sym(A_0 + A_1)) = Sym(-A_0 + A_1)$. We denote by $\mathcal{S}^{(2)}$ the subalgebra $\psi_2(Sym)$.

We now introduce the main tool of the proof.

Definition 5 *Let Φ_2 be the isomorphism of $\mathbb{C}$-vector spaces:*

$$\begin{aligned} \Phi_2 : Sym_{\emptyset}(A) &\rightarrow Sym(A_0, A_1) \\ s_\lambda(A) &\rightarrow \epsilon_2(\lambda) s_{\lambda^0}(A_0) s_{\lambda^1}(A_1). \end{aligned}$$

We remark that Φ_2 can be generalized at Sym and in this case it is the lifting at the level of symmetric functions of the combinatorial bijection which sends λ to $(\lambda_{(2)}, (\lambda^0, \lambda^1))$.

Lemma 6

$$(i)\ \Phi_2 \circ \psi_2(Sym) = Sym(A_0 + A_1)$$

More precisely, for $f \in Sym$, $\Phi_2(\psi_2(f)) = f(A_0 + A_1)$

$$(ii)\ \nu = \Phi_2^{-1} \circ w \circ \Phi_2$$

Lemma 7 $\nu(\psi_2(Sym)) \subset \mathcal{T}^{(2)}$.

Using these results we can prove the two propositions above. We remark on the one hand that (i) of Proposition 3 is obvious from Definition 1 since Δ involves only derivations with respect to the even power sums p_{2i}. On the other hand (ii) of Proposition 3 is proved using the form $\Phi_2^{-1} \circ w \circ \Phi_2$ of ν and the algebras $Sym(A_0 + A_1)$, $Sym(-A_0 + A_1)$. We then prove that Δ is a projector using (i), (ii) of Proposition 3 and Lemma 7. Proposition 3 is proved. We now give a sequence of lemmas to prove Proposition 4.

Lemma 8 $D_1(fg) = fD_1(g)$, *$f \in \mathcal{T}^{(2)}$, $g \in Sym$, and so D_1 is a projector.*

Lemma 9 $D_1 \circ \nu(s_\lambda) = D_1(s_\lambda)$ *for $s_\lambda \in Sym_{\emptyset}$.*

The proof is totally combinatorial and uses domino diagrams.

Corollary 10 $D_1(\psi_2(s_\lambda)) = \nu(\psi_2(s_\lambda))$ *for $s_\lambda \in Sym$.*

It comes from Lemmas 8, 7, and 9 and the fact that $\psi_2(Sym) \subset Sym_{\emptyset}$. This completes the proof of Proposition 4 and of Theorem 2.

We deduce immediately from Theorem 2 the following expression of P-functions in terms of S-functions.

Corollary 11 *Let $\mu = (\mu_1, \ldots, \mu_k)$ be a strict partition and γ be a partition given by $(\mu_1 - 1, \mu_3 - 1, \ldots | \mu_2, \mu_4 \ldots)$ in Frobenius' notation. Then*

$$P_\mu = \sum_\lambda (-1)^{|\lambda|} \epsilon_2(J(\lambda')) s_{J(\lambda')} D_{\psi_2(s_\lambda)} s_\gamma$$

Examples:

1.

We compute the Schur P-functions indexed by partitions of weight 6. For this it is sufficient to compute in Δ terms indexed with partitions of weight less or equal to 3.

$$\Delta = 1 + s_{(1,1)}(D_{s_{(2)}} - D_{s_{(1,1)}}) + s_{(1,1,1,1)}(D_{s_{(4)}} - D_{s_{(3,1)}} + D_{s_{(2,2)}})$$

$$-s_{(3,1)}(D_{s_{(2,2)}} - D_{s_{(2,1,1)}} + D_{s_{(1,1,1,1)}})+$$

$$s_{(1,1,1,1,1,1)}(D_{s_{(6)}} - D_{s_{(5,1)}} + D_{s_{(4,2)}} - D_{s_{(3,3)}})$$

$$-s_{(3,1,1,1)}(D_{s_{(4,2)}} - D_{s_{(4,1,1)}} - D_{s_{(3,3)}} + D_{s_{(2,2,2)}} - D_{s_{(2,2,1,1)}}$$

$$+D_{s_{(3,1,1,1)}}) + s_{(5,1)}(D_{s_{(2,2,2)}} - D_{s_{(2,2,1,1)}} + D_{s_{(2,1,1,1,1)}} - D_{s_{(1,1,1,1,1,1)}}) + ...$$

- For $\mu = (6)$, $\gamma = (5|0) = (6)$, we have

$$P_{(6)} = \Delta(s_{(6)}) = s_{(6)}.1 + s_{(1,1)}s_{(4)} + s_{(1,1,1,1)}s_{(2)} - 0 + s_{(1,1,1,1,1,1)}.1 + 0 + 0$$

- For $\mu = (5,1)$, $\gamma = (5,1)$, we have

$$P_{(5,1)} = \Delta(s_{(5,1)}) = s_{(5,1)}.1 + s_{(1,1)}s_{(3,1)} + s_{(1,1,1,1)}s_{(4,1)} - 0 + s_{(1,1,1,1,1,1)}.(-1)$$

- For $\mu = (4,2)$, $\gamma = (3|2) = (4,1,1)$, we have

$$P_{(4,2)} = \Delta(s_{(4,1,1)})$$

$$= s_{(4,1,1)}.1+s_{(1,1)}(-s_{(4)}+s_{(2,1,1)})+s_{(1,1,1,1)}(-s_{(2)})-s_{(3,1)}(-s_{(2)})+0-s_{(3,1,1,1)}(-1)+0.$$

2.

For $l(\lambda) \leq k$ we have :

$$D_{\psi_2(s_\mu)} s_{\rho_k+2\lambda} = \sum_\nu c^\lambda_{\mu\nu} s_{\rho_k+2\nu}$$

where $c^\lambda_{\mu\nu}$ is the Littlewood-Richardson coefficient. We then have

$$P_{c(\rho_k+2\lambda)} = \sum_{\gamma\nu} c^\lambda_{\gamma\nu}(-1)^{|\gamma|}\epsilon_2(J(\gamma'))s_{J(\gamma')}s_{\rho_k+2\nu}$$

5 Applications

We now use this operator to prove that various sets of tableaux have the same cardinality.

First we have to recall some formulas on Schur S-functions and P-functions. We have

$$P_\lambda = \sum_\mu g_{\lambda\mu} s_\mu \tag{1}$$

where $g_{\lambda\mu}$ is the number of "Stembridge" tableaux of unshifted shape μ and content λ,

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu \tag{2}$$

where $c^\nu_{\lambda\mu}$ is the Littlewood-Richardson coefficient i.e. the number of Yamanouchi skew-tableaux of shape ν/λ and weight μ,

$$D_{\psi_2(s_\lambda)} s_\mu = \sum_{\nu, |\nu|=|\mu|-2|\lambda|} \epsilon_2(\mu/\nu) d^\mu_{\nu\lambda} s_\nu \tag{3}$$

where $d^\mu_{\nu\lambda}$ is the number of Yamanouchi domino tableaux of shape μ/ν and weight λ, and finally

$$\psi_2(s_\mu) = \sum_{\lambda, |\lambda|=2|\mu|} \epsilon_2(\lambda) d_{\lambda\mu} s_\lambda \tag{4}$$

where $d_{\lambda\mu}$ is the number of Yamanouchi domino tableaux of shape λ and weight μ.

The first formula is proved in [9], the second one is well-known, and the two last formulas are given in [1].

For a strict partition ν, we shall denote by $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}$ all the partitions such that $c(\mu^{(i)}) = w_i(\nu)$ for some permutation w_i.

Theorem 12

$$\sum_{i=1}^k \epsilon(w_i) g_{\lambda\mu^{(i)}} = \left| \begin{array}{l} 1 \ \textit{if}\ \nu = \lambda \\ 0 \ \textit{else} \end{array} \right.$$

The proof is obtained by applying Δ (which is linear and a projection) to equation (1). This corollary shows that the number of marked tableaux with $\epsilon(w_i) > 0$ and the number of those with $\epsilon(w_i) < 0$ are equal. For example, the set 1 and the set 2 on Figure 4 have the same cardinality. This corresponds to the case $\lambda = (8, 4, 3, 2)$ and $\nu = (6, 5, 3, 2, 1)$.

Theorem 13

$$g_{c(\lambda)\mu} = \sum_{\nu,\gamma} (-1)^{|\nu|} \epsilon_2(J(\nu')) \epsilon_2(\lambda/\gamma) d^\lambda_{\gamma\nu} c^\mu_{J(\nu')\gamma}$$

This follows from Definition 1 and relations (2), (3). On Figure 5 we have computed $g_{(8,4,2)(6,4,2,1,1,)}$, drawing all "Stembridge" tableaux of unshifted shape (6,4,2,1,1) and weight (8,4,2).

On Figure 6 we have computed $d^{(8,3,1,1,1)}_{\lambda\mu}$ (where (8,3,1,1,1) is the partition λ such that $c(\lambda) = (8, 4, 2)$) and $c^{(6,4,2,1,1)}_{J(\lambda')\gamma}$ with the sign of the pairs of tableaux. We see that adding the number of pairs of tableaux (being careful of the sign) of Figure 6 we obtain the same number as that of Stembridge tableaux of Figure 5.

6 Generalizations

The operator Δ can be generalised in two ways.

- We can define with a formula similar to that of Definition 1 a family of operators Δ_{ρ_k}, where we put the staircase ρ_k in place of the empty 2-core in $J(\lambda)$.

  These operators satisfy properties analogous to those of Δ. In particular we obtain again the main formula of [5] and give a different proof of it.

- We can also define a differential operator Δ_n which sends Sym to the subalgebra $\mathcal{T}^{(n)} = \mathbb{C}[p_i; i \not\equiv 0(n)]$. For this we have just to put n-core and n-quotient in place of 2-core and 2-quotient.

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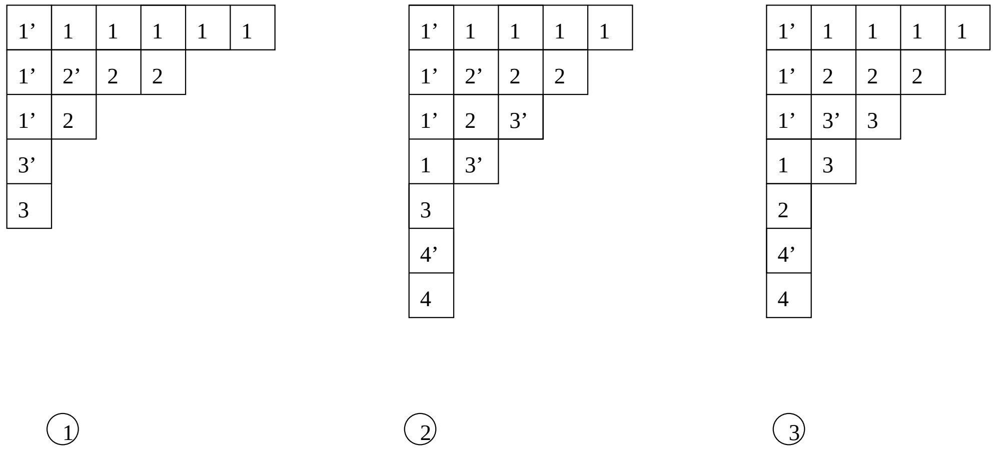

Figure 3:

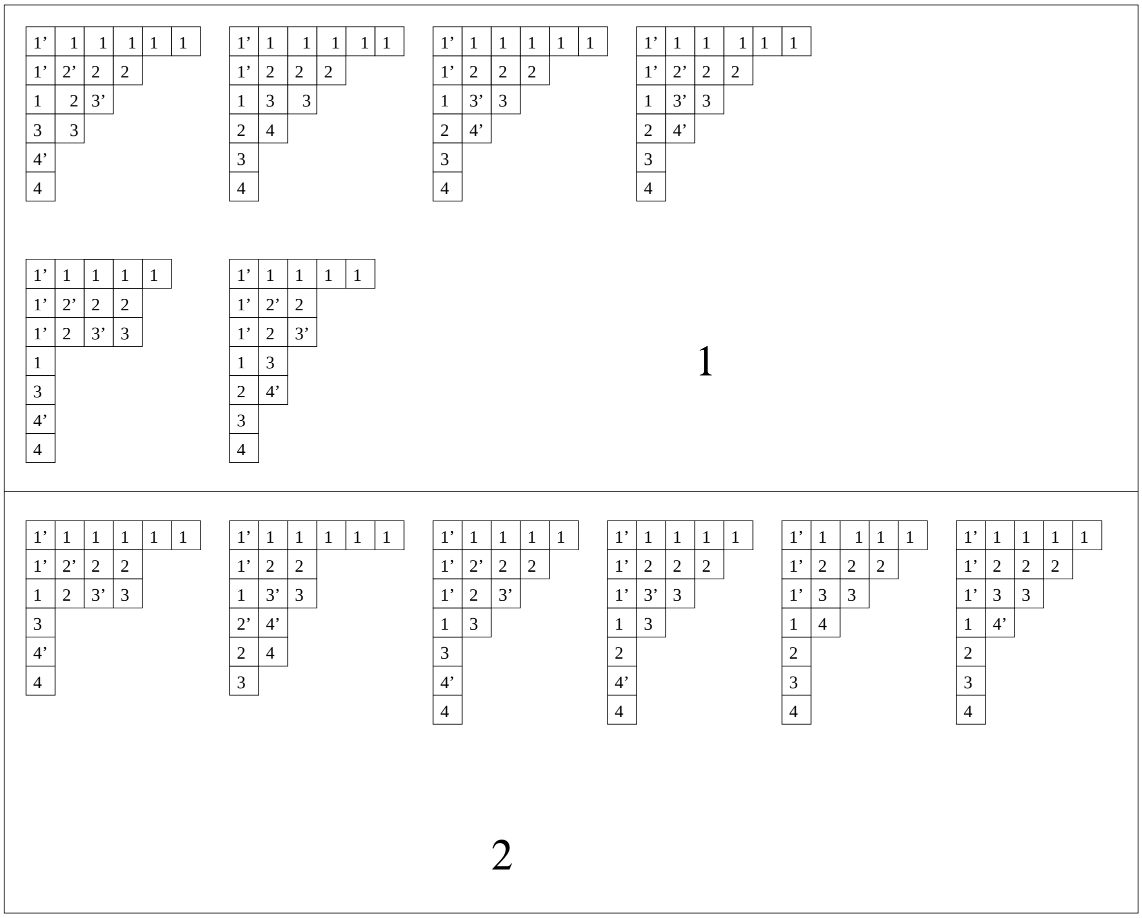

Figure 4:

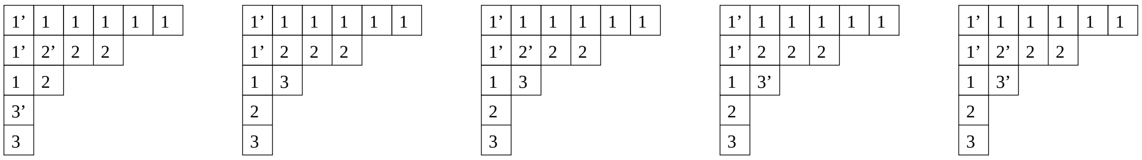

Figure 5:

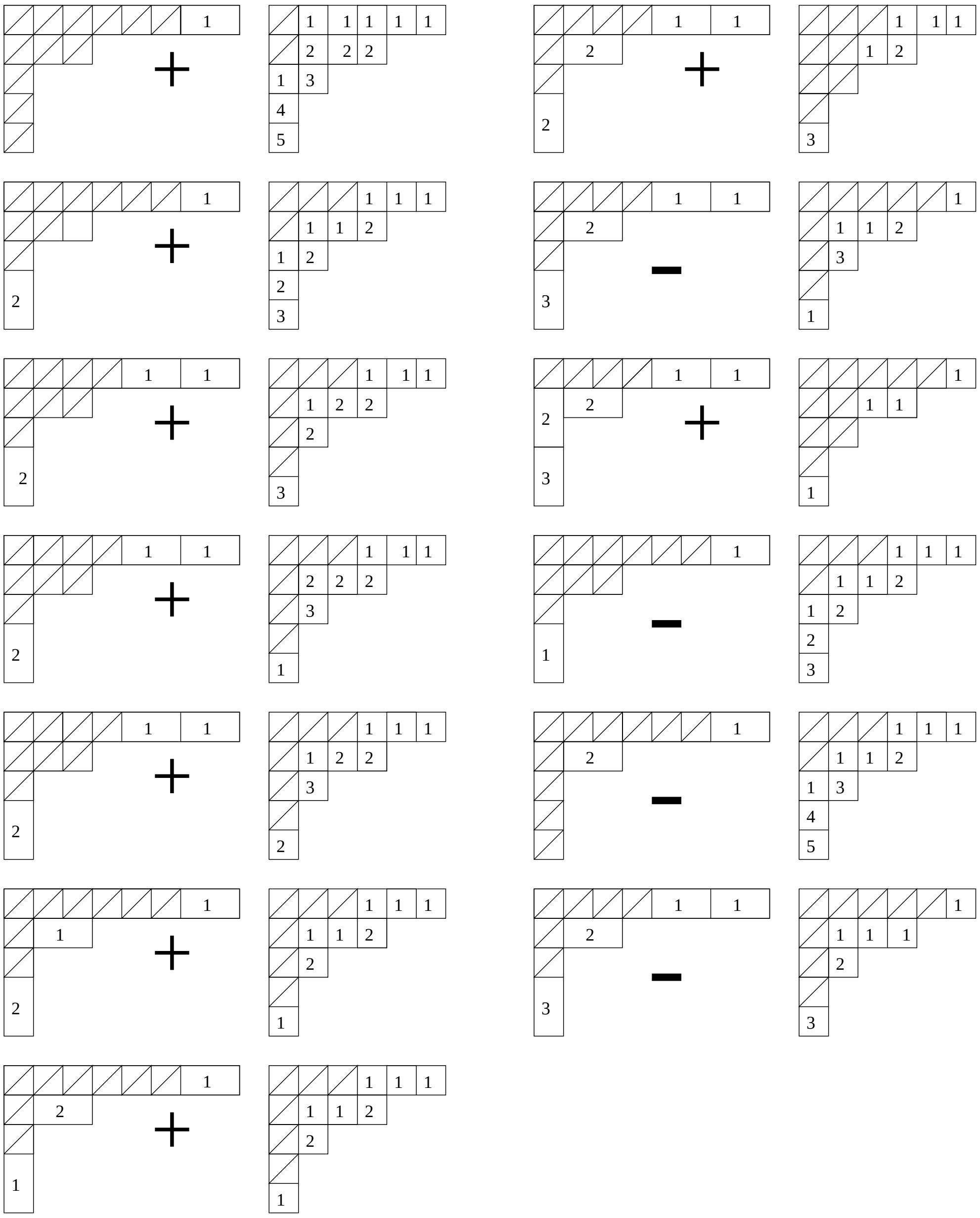

Figure 6: