Lyndon words, polylogarithmic functions and the Riemann ζ function *

Hoang Ngoc Minh
University of Lille II,
59024 Lille, France.
e-mail : `hoang@lifl.fr`

Michel Petitot
University of Lille I,
59655 Villeneuve d'Ascq, France.
e-mail : `petitot@lifl.fr`

Preprint

Abstract : In this work, we explain the functional relations between polylogarithmic functions encoding them by words over non commutative variables. Evaluating these relations at $z = 1$, we found again the relations between the Euler/Zagier sums, especially the decomposition formula. We then compute a Groebner basis of the algebra generated by the Euler/Zagier sums. This computation uses the Lyndon words, which are a transcendence basis of the shuffle algebra.

Résumé : Dans ce travail, on explique les relations fonctionnelles entre les fonctions polylogarithmiques en les codant par des mots en variables non commutatives. En spécialisant ces relations en $z = 1$, nous retrouvons les relations entre les sommes d'Euler/Zagier, en particulier la formule de décomposition. On calcule ensuite une base de Gröbner de l'algèbre engendrée par les sommes d'Euler/Zagier. Ce calcul est organisé en utilisant les mots de Lyndon, qui forment une base de transcendance de l'algèbre de mélange.

*This work is supported by the PRC AMI. The computations were led on the machines of the Medicis group.

1 Introduction

1.1 Polylogarithms

For any multi-index $s = (s_1, s_2, \ldots, s_k)$ made of positive integers, we define the *polylogarithm* function on $[0, 1[$ as follows (see Nielsen [16]):

$$\mathrm{Li}_s(t) \quad = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{t^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}. \tag{1}$$

We used these functions to express the output of dynamical systems with rational entries [12, 14] and to integrate differential equations with meromorphic coefficients [13]. In this work, we show how to generate the *functional relations* between polylogarithmic functions encoding them by words over the non commutative variables x_0 and x_1. To any multi-index $s = (s_1, s_2, \ldots, s_k)$, we associate the word

$$w = x_0^{s_1-1} x_1 x_0^{s_2-1} x_1 \cdots x_0^{s_k-1} x_1 \tag{2}$$

In this way, any polylogarithmic function can be encoded by some word ending with x_1.

We show that the $\mathbf{Q}$-algebra of the polylogarithmic functions is generated by functions which are encoded by some special words called *Lyndon words*. Section 2 deals with the decomposition algorithm in this basis. It is basically the application of the `xtaylor` algorithm, used in [17] to solve a control theory problem in a particular case. We show as well that D. Borwein, J.M. Borwein and R. Girgensohn's *decomposition formula* [2], dealing with Euler sums, holds for polylogarithmic functions.

1.2 Euler sums and their relations

At $z = 1$, the previous polylogarithmic functions give the generalised Riemann ζ function

$$\zeta(s) \quad = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}, \tag{3}$$

which converges when $s_1 > 1$. These generalised Euler sums [11, 9, 10, 4, 3] first appeared in the knots theory, in relation with Feynman diagrams in quantum physics [1, 20]. They appear a lot as well in the study of the fundamental probabilities derived from quadtrees [5]. Ph. Flajolet and B. Salvy approached these Euler sums using their representation by *contour integrals* [6].

Euler showed that, when $n + p \leq 13$ and $n + p$ is odd, $\zeta(n, p)$ can be expressed as a function of the $\zeta(s)$, where s is a simple index (that is to say, a positive integer greater than 2). In 1904, N. Nielsen (see [16], p.192,194), in a deep work, proves in particular the following relations [1]:

$$\zeta(n)\zeta(p) = \zeta(n+p)+\zeta(n,p)+\zeta(p,n), \quad (\text{pour } n>1, p>1), \tag{4}$$

$$\begin{aligned}\zeta(p,q) = {} & (-1)^q\sum_{\nu=0}^{p-2}\binom{q+\nu-1}{q-1}\zeta(p-\nu,q+\nu) \\ & +\sum_{\nu=0}^{q-2}(-1)^\nu\binom{p+\nu-1}{p-1}\zeta(q-\nu)\zeta(p+\nu) \\ & -(-1)^q\binom{p+q-2}{p-1}\left[\zeta(p+q)+\zeta(p+q-1,1)\right] \quad (\text{pour } p>1,\ q>1).\end{aligned} \tag{5}$$

Relation (4), which is called *reflection formula*, does not hold for polylogarithmic functions because the orders of the product $\mathrm{Li}_n(z)\,\mathrm{Li}_p(z)$ and of the sum $\mathrm{Li}_{n,p}(z)+\mathrm{Li}_{p,n}(z)$ are both equal to 2, whereas the order of $\mathrm{Li}_{n+p}(z)$ is equal to 1. This formula (4) can be deduced from the second shuffle product [10], which formulates in a different way the computation of the product of two *quasi–symmetric* functions [19, 8].

Working on relation (5), D. Borwein, J.M. Borwein and R. Girgensohn [2] find explicit formulae giving $\zeta(n,p)$ when $n+p$ is *odd*. We still don't know any explicit expression for $\zeta(n,p)$ when $n+p$ is *even*, except when $n+p=4$ or $n+p=6$.

In this paper, we show how to compute a *Groebner basis* of the ideal of polynomial relations (with coefficients in $\mathbf{Q}$) linking some $\zeta(s)$. We found these relations within a few hours (on a 35 MHz machine), thanks to pure symbolic methods and the computer algebra system Axiom. Our program is still being worked on, and we hope to improve the algorithms we use. The only multi-indices appearing in this basis are Lyndon words. One can easily turn any word in Lyndon words using the algorithm `xtaylor`, which is dealt with in section 2.

The basis, computed for weights up to 10, satisfies Zagier's dimension conjecture [20]:

$$\begin{cases} d_n = d_{n-2}+d_{n-3}, & n\geq 4 \\ d_1=0, \quad d_2=d_3=1, \end{cases} \tag{6}$$

where d_n denotes the dimension of the $\mathbf{Q}$-vector space generated by the $\zeta(s)$ for multi–indices $s=(s_1,s_2,\ldots,s_k)$ the weight $s_1+s_2+\cdots+s_k$ of which equals n.

We can notice as well that this Groebner basis is special: there is no monomial in the left parts of the rules. Hence, such a basis defines a $\mathbf{Q}$-algebra which is freely generated by the $\zeta(s)$ appearing in the right parts of the rules. We can then conjecture:

Conjecture 1.1 *The $\mathbf{Q}$-algebra of the $\zeta(s)$ is a polynomial algebra.*

In the next work, we will show the independence of the polylogarithmic functions indexed by Lyndon words.

[1] Nielsen notes $\zeta(n)$ as s_n.

2 Shuffle algebra

2.1 The shuffle product

Let X be a finite ordered alphabet. Let us note X^* the free monoid generated by X. An element w of X^* is a word the letters of which are in X. The *length* of w, which is the number of its letters, is noted $|w|$ and the *empty word* is noted ε. We note $X^+ = X^* \setminus \varepsilon$.

We consider the free associative algebra $\mathbf{Q}\langle X\rangle$ of the polynomials, with coefficients in $\mathbf{Q}$ and with non commutative variables $x \in X$. The duality, defined on these words by

$$(u|v) = \delta_u^v, \quad u, v \in X^*$$

(where δ denotes the Kronecker symbol) is linearly extended to the polynomials of $\mathbf{Q}\langle X\rangle$. A polynomial $f \in \mathbf{Q}\langle X\rangle$ is a *finite* linear combination of words. It can be written as:

$$f = \sum_{w \in X^*} (f|w)\ w, \qquad (f|w) \in \mathbf{Q}.$$

The *concatenation* product can be obtained extending linearly the combinations of words:

$$\forall f, g \in \mathbf{Q}\langle X\rangle, \quad fg = \sum_{u,v \in X^*} (f|u)(g|v)uv$$

The degree of f, noted $\deg(f)$, is the length of the longest word $w \in X^*$ such that $(f|w) \neq 0$.

The *shuffle product* is first defined on words by the following recursive formulae:

$$\left\{ \begin{array}{lll} \forall w \in X^*, & & \varepsilon ш w = w ш \varepsilon = w, \\ \forall x, y \in X, & u, v \in X^*, & xu ш yv = x(u ш yv) + y(xu ш v). \end{array} \right. \tag{7}$$

and then extended linearly to the polynomials of $\mathbf{Q}\langle X\rangle$. With the shuffle product, $\mathbf{Q}\langle X\rangle$ is a *commutative* and *associative* $\mathbf{Q}$-algebra noted $\mathrm{Sh}_{\mathbf{Q}}(X)$.

2.2 The coding of iterated integrals

Consider a map from X to the set ΩU of the 1-analytical forms defined on an open set $U \subset \mathbf{R}$:

$$x \in X \longrightarrow \omega_x \in \Omega U$$

We define a map α from $\mathrm{Sh}_{\mathbf{Q}}(X)$ to the space $\mathcal{C}^\omega U$ of analytical functions on U with values in $\mathbf{R}$. This map is first defined on words by the following recursive formulae [7]:

$$\left\{ \begin{array}{lcl} \alpha(\varepsilon) & = & 1 \\ \alpha(xu) & = & \displaystyle\int_0^t \omega_x \alpha(u), \quad x \in X, \quad u \in X^* \end{array} \right. \tag{8}$$

We extend the map α to $\mathrm{Sh}_{\mathbf{Q}}(X)$ by linearity.

Theorem 2.1 *The map α is a $\mathbf{Q}$-algebra morphism. Especially:*

$$\begin{cases} \forall p,q \in \mathrm{Sh}_{\mathbf{Q}}(X),\, \forall a,b \in \mathbf{Q}, & \alpha(ap+bq) \;= a\alpha(p)+b\alpha(q) \\ \forall p,q \in \mathrm{Sh}_{\mathbf{Q}}(X), & \alpha(p \sqcup\!\sqcup\, q) \;= \alpha(p)\,\alpha(q) \end{cases} \tag{9}$$

$\mathcal{C}^\omega U$ having the usual sum and product of real functions.

Proof – By induction on the length of words and integrating by parts. □

2.3 Transcendence bases of the shuffle algebra

Let X be an alphabet with a total order $<$. The *lexicographical order* is a total order on words of X^* satisfying:

$$\begin{cases} \forall u,w \in X^*, & w \neq \varepsilon \Rightarrow u < uw \\ \forall u,v_1,v_2 \in X^*,\, \forall x,y \in X, & x < y \Rightarrow uxv_1 < uyv_2 \end{cases} \tag{10}$$

By definition, a *Lyndon word* is a nonempty word of X^* which is less (for the lexicographical order) than any of its proper right factors:

$$\forall u,v \in X^+,\; l = uv \Rightarrow l < v \tag{11}$$

$\mathcal{L}yndon(X)$ denotes the set of all Lyndon words on the alphabet X.

Example 1 *Let $X = \{x_0, x_1\}$ with $x_0 < x_1$, the Lyndon words of length 5 or less on X^* are the following 14 words, in increasing order:*

$$\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2, x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}.$$

We know many *transcendence bases* of the commutative algebra $\mathrm{Sh}_{\mathbf{Q}}(X)$ (with the shuffle product). The Lyndon words are one of them, which we will call *Radford basis*. They are algebraically independent and they generate $\mathrm{Sh}_{\mathbf{Q}}(X)$. More precisely [18, 19]:

Theorem 2.2 (Radford) *Let X be a finite ordered set and let L be the set of all Lyndon words built over letters taken from X. The algebra $\mathrm{Sh}_{\mathbf{Q}}(X)$ is a polynomial algebra which is isomorphic to $\mathbf{Q}[L]$.*

Proof – Any word $w \in X^*$ can be expressed in a unique way as a decreasing (concatenation) product of Lyndon words: $w = l_1^{\alpha_1} l_2^{\alpha_2} \cdots l_k^{\alpha_k}$ with $l_1 > l_2 > \cdots > l_k$ et $k \geq 0$. Let $Q_w = \dfrac{1}{\alpha_1!\alpha_2!\cdots\alpha_k!} l_1^{\alpha_1} \sqcup\!\sqcup\, l_2^{\alpha_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, l_k^{\alpha_k}$, the powers $l_i^{\alpha_i}$ being computed for the shuffle product. We show that $Q_w = w + r$, where polynomial $r \in \mathbf{Q}\langle X \rangle$ does not contain any words strictly smaller than w. Hence, to decompose w in the Radford basis, one only has to decompose r. □

This proof gives an algorithm which is slower than `xtaylor` when the polynomial to be factorized has many words.

2.4 Lie polynomials and differentiations

We know that the *free Lie algebra* $\mathcal{L}ie_{\mathbf{Q}}\langle X\rangle$ is the smaller sub vector space of $\mathbf{Q}\langle X\rangle$ containing X and closed under the Lie bracket operation:

$$[p,q] = pq - qp, \qquad \forall p,q \in \mathbf{Q}\langle X\rangle$$

To any Lyndon word $l \in \mathcal{L}yndon(X)$, we associate the *bracketed form* noted $[l] \in \mathcal{L}ie_{\mathbf{Q}}\langle X\rangle$, which can be defined as follows:

$$\left\{\begin{array}{lcll} [x] & = & x, & \forall x \in X, \\ [l] & = & [[u],[v]], & l,u,v \in \mathcal{L}yndon(X), \end{array}\right. \tag{12}$$

where v denotes the longest Lyndon word such that $l = uv$ with $u \neq \varepsilon$. For example, the bracketed form of $x_0x_1x_0x_1x_1$ is $[[x_0,x_1],[[x_0,x_1],x_1]]$. We know that the set $\{[l] \mid l \in L\}$ is a $\mathbf{Q}$-basis of the vector space $\mathcal{L}ie_{\mathbf{Q}}\langle X\rangle$.

The *right residual* of a polynomial p by another polynomial q, noted $p \triangleright q$, is defined by the following formula:

$$(p \triangleright q|w) \;=\; (p|qw), \qquad \forall p,q \in \mathbf{Q}\langle X\rangle,\ \forall w \in X^* \tag{13}$$

It is little more than a *left simplification* of the words, that is to say:

$$\left\{\begin{array}{ll} (xw) \triangleright x = w, & \forall x \in X,\ \forall w \in X^*, \\ (xw) \triangleright y = 0, & \forall x,y \in X,\ \forall w \in X^*, x \neq y \end{array}\right. \tag{14}$$

We can easily check that this simplification is a *right action* over $\mathbf{Q}\langle X\rangle$, that is to say:

$$p \triangleright (qr) = (p \triangleright q) \triangleright r, \quad \forall p,q,r \in \mathbf{Q}\langle X\rangle.$$

Lemma 2.1 *The right residual by a Lie polynomial is a differentiation for the shuffle product:*

$$(f \sqcup\!\sqcup g) \triangleright p \;=\; (f \triangleright p) \sqcup\!\sqcup g + f \sqcup\!\sqcup (g \triangleright p), \qquad \forall f,g \in \mathbf{Q}\langle X\rangle,\ \forall p \in \mathcal{L}ie_{\mathbf{Q}}\langle X\rangle. \tag{15}$$

Proof – We first prove this lemma when p is a letter of the alphabet X. Then we notice that the Lie bracket of two differentiations is a differentiation. □

2.4.1 Decomposition algorithm in the Radford basis

The decomposition algorithm `xtaylor` [17] of a polynomial $f \in \mathbf{Q}\langle X\rangle$ in the Radford basis lies on the following *triangular* property:

Lemma 2.2 *Let l be a Lyndon word and let $[l]$ be its bracketed form. l is the smallest word that appears in the support of $[l]$, and has coefficient 1, that is to say:*

$$[l] = l + \sum_{w \in X^*} \alpha_w w, \quad \textit{with } |w| = |l|,\ w > l,\ \alpha_w \in \mathbf{Z}.$$

Corollary 2.1 *The Lyndon words, compared according to the lexicographical order by length first* [2]*, satisfy the following triangular property:*

$$\begin{cases} \forall l, l' \in \mathcal{L}yndon(X), \quad l \triangleright [l'] & = 1 \qquad \text{if } l = l', \\ & = 0 \qquad \text{if } l < l' \end{cases} \tag{16}$$

Let d be the degree of polynomial $f \in \mathbf{Q}\langle X\rangle$. We compute the set L_d of all Lyndon words of length less than d. We can show that the decomposition of f in the Radford basis will only need words from L_d. Let l be the longer word of L_d. We are looking for a decomposition of the form:

$$f = \sum_{i=0}^{n} A_i \sqcup\!\sqcup\, l^{\sqcup\!\sqcup\, i}, \qquad A_i \in \mathbf{Q}\langle X\rangle.$$

The decomposition of all A_i will take place in $L'_d = L_d \setminus \{l\}$, without l appearing. To simplify the discussion, let us choose $n = 2$ for example. From

$$f = A_2 \sqcup\!\sqcup\, l \sqcup\!\sqcup\, l + A_1 \sqcup\!\sqcup\, l + A_0$$

we get the unknown polynomials A_0, A_1, A_2 iterating on f the differentiation $\triangleright[l]$. Because of the corollary 2.1, applying this differentiation to the A_i gives 0. We then obtain:

$$\begin{cases} f_1 = f \triangleright [l] & = & 2A_2 \sqcup\!\sqcup\, l + A_1, \\ f_2 = f_1 \triangleright [l] & = & 2A_2, \\ f_3 = f_2 \triangleright [l] & = & 0. \end{cases} \tag{17}$$

Knowing A_2 thanks to f_2, we then get A_1 thanks to f_1, and A_0 thanks to f. The decomposition of the A_i in the Radford basis will be computed by a recursive call of the same algorithm using the list L'_d. Given that the length of L'_d is strictly less than the length of L_d, this computation eventually comes to an end.

This algorithm is executed faster when one has access to a good implementation of polynomials in non commutative variables. Experience shows that a *recursive* [17] representation, based upon a (non commutative) Horner scheme, is especially efficient. Such a representation lies on the reconstruction formula:

$$p = (p|\varepsilon) + \sum_{x \in X} x \ (p \triangleright x), \quad \forall p \in \mathbf{Q}\langle X\rangle.$$

Unfortunately, some computer algebra systems will not allow one to control closely the representation of data in memory.

[2] Lyndon words are first sorted by increasing length, then, when they have equal lengths, according to the lexicographical order.

3 ζ function, polylogarithmic functions and iterated integrals

3.1 Reminder

The first computations on the ζ function are as old as Euler. This function is defined for $k \in \mathbf{N}$:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \tag{18}$$

This series converges for $k \geq 2$. We know that when $k = 2p$ is even, $\zeta(k)$ is a (rational) multiple of π^k, that is to say $\zeta(2p)/\pi^{2p} \in \mathbf{Q}$. Thus $\zeta(2) = \pi^2/6$.

The function $\zeta(k)$ can be computed evaluating at $t = 1$ the polylogarithmic function $\mathrm{Li}_k(t)$ with $k \geq 1$. This function is recursively defined with an iterated integral ($t \in \mathbf{R}$):

$$\begin{cases} \mathrm{Li}_1(t) = \displaystyle\int_0^t \frac{dz}{1-z} & = \displaystyle\sum_{n=1}^{\infty} \frac{t^n}{n} \\ \mathrm{Li}_k(t) = \displaystyle\int_0^t \frac{dz}{z} \mathrm{Li}_{k-1}(z) & = \displaystyle\sum_{n=1}^{\infty} \frac{t^n}{n^k}, \quad k > 1 \end{cases} \tag{19}$$

These formulae can be generalised for a multi-index $s = (s_1, s_2, \ldots, s_k)$ thanks to the definitions (1) and (3).

3.2 Notations and principle of the coding

Let $X = \{x_0, x_1\}$ the coding alphabet be ordered such that $x_0 < x_1$. Let $Y = \{y_i \mid i > 0\}$ be ordered such that $y_i < y_j$ when $i > j$. To any multi-index s, we associate in a one-to-one correspondence the words $w \in X^* x_1$ and $w' \in Y^*$:

$$\begin{cases} w = x_0^{s_1-1} x_1 x_0^{s_2-1} x_1 \cdots x_0^{s_k-1} x_1 \\ w' = y_{s_1} y_{s_2} \ldots y_{s_k} \end{cases} \tag{20}$$

This way, we get an increasing isomorphism between the free monoids $\varepsilon + X^* x_1$ et Y^* posing

$$y_i = x_0^{i-1} x_1, \tag{21}$$

which will allow us to identify the words s, w and w'. In what follows, we will use both notations Li_s and Li_w (respectively, $\zeta(s)$ and $\zeta(w)$). We can show that the formulae (19) can be generalised for all words. Li_w, especially, is an iterated integral as defined in section 2.2: x_0 denotes the form $\omega_0 = dz/z$ and x_1 denotes the form $\omega_1 = dz/(1-z)$. We have:

$$\begin{cases} \mathrm{Li}_{x_0 w}(t) = \displaystyle\int_0^t \frac{dz}{z} \mathrm{Li}_w(z), \quad \forall w \in X^* x_1 \\ \mathrm{Li}_{x_1 w}(t) = \displaystyle\int_0^t \frac{dz}{1-z} \mathrm{Li}_w(z) \quad \forall w \in X^* x_1 \end{cases} \tag{22}$$

Lemma 3.1 *Let w and w' be two words, both associated to the same multi-index s for their coding. Then*

$$w \in X^* x_1,\, w \in \mathcal{L}yndon(X) \quad \Leftrightarrow \quad w' \in \mathcal{L}yndon(Y) \tag{23}$$

PROOF – From the definition of Lyndon words (11) and the increasing isomorphism defined by 21). □

Pose

$$\left\{ \begin{array}{lcl} \mathcal{H}_1 & = & \mathbf{Q} \oplus \mathbf{Q}\langle X \rangle x_1 \simeq \mathbf{Q}\langle Y \rangle, \\ \mathcal{H}_2 & = & \mathbf{Q} \oplus x_0 \mathbf{Q}\langle X \rangle x_1, \\ L_1 & = & \mathcal{L}yndon(X) \setminus \{x_0\}, \\ L_2 & = & \mathcal{L}yndon(X) \setminus \{x_0, x_1\}. \end{array} \right. \tag{24}$$

Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are stable by concatenation product and by shuffle product. We have as well that $\mathcal{H}_1$ and $\mathcal{H}_2$, with the shuffle product $ш$, are $\mathbf{Q}$-algebras of polynomials which are respectively isomorphic to $\mathbf{Q}[L_1]$ and to $\mathbf{Q}[L_2]$.

PROOF – It is clear that $L_1 \subset \mathcal{H}_1$ and $L_2 \subset \mathcal{H}_2$. The proof of the Radford theorem can be used again. To show that $\mathcal{H}_1 \simeq \mathbf{Q}[L_1]$ (respectively that $\mathcal{H}_2 \simeq \mathbf{Q}[L_2]$), notice that the decomposition of a word $w \in X^* x_1$ (respectively $w \in x_0 X^* x_1$) in a decreasing (concatenation) product of Lyndon words cannot contain the Lyndon word x_0 (respectively the Lyndon word x_1). □

Extending $\mathbf{Q}$-linearly the maps $s \to \zeta(s)$ and $s \to \mathrm{Li}_s$, we build the maps:

$$\left\{ \begin{array}{lll} \mathrm{Li}: & p \in \mathcal{H}_1 & \to \mathrm{Li}_p \in \mathcal{C}^\omega[0,1[\\ \zeta: & p \in \mathcal{H}_2 & \to \zeta(p) \in \mathbf{R} \end{array} \right. \tag{25}$$

Theorem 3.1 *The applications* Li *and ζ, as defined in (25), are morphisms of $\mathbf{Q}$-algebras. In particular, $\forall p, q \in \mathcal{H}_1$, $\mathrm{Li}_{p\, ш\, q} = \mathrm{Li}_p \mathrm{Li}_q$ and $\forall p, q \in \mathcal{H}_2$, $\zeta(p\, ш\, q) = \zeta(p)\zeta(q)$. The $\mathbf{Q}$-algebra of the polylogarithmic functions is generated by the Li_w such that $w \in L_1$. The $\mathbf{Q}$-algebra of the $\zeta(p)$ is generated by the $\zeta(w)$ such that $w \in L_2$.*

PROOF – This comes directly from $\zeta(p) = \mathrm{Li}_p(1)$, in theorem 2.1, and from the Radford theorem 2.2. □

Theorem 3.2 *The application* Li*, defined in (25), is a one-to-one correspondence (that is to say,* $\ker Li = (0)$*)..*

PROOF – To appear. □

3.3 Decomposition formulae

Lemma 3.2 *Let a, x, y be three random letters and $p, q \in \mathbf{N}$. We have:*

$$a^n \sqcup\!\sqcup\, a^p = \binom{n+p}{n} a^{n+p} \tag{26}$$

$$a^n \sqcup\!\sqcup\,(a^p y) = \sum_{i=0}^{n} (a^i \sqcup\!\sqcup\, a^p) y a^{n-i} \tag{27}$$

$$(a^n x) \sqcup\!\sqcup\,(a^p y) = \sum_{i=0}^{n} \binom{p+i}{i} a^{p+i} y a^{n-i} x + \sum_{i=0}^{p} \binom{n+i}{i} a^{n+i} x a^{p-i} y \tag{28}$$

Proof – First two identities are easy; to show the third one, we can use the recursive definition $a^n x \sqcup\!\sqcup\, a^p y = (a^n \sqcup\!\sqcup\, a^p y)x + (a^n x \sqcup\!\sqcup\, a^p)y$. □

Applying the formula (28) in the case $a = x_0, x = y = x_1$ and replacing n with $n-1$ and p with $p-1$, we can deduce the formula:

$$\mathrm{Li}_n(t)\,\mathrm{Li}_p(t) = \sum_{i=0}^{n-1} \binom{p+i-1}{i} \mathrm{Li}_{p+i,n-i}(t) + \sum_{i=0}^{p-1} \binom{n+i-1}{i} \mathrm{Li}_{n+i,p-i}(t). \tag{29}$$

Evaluating this last equality at $t = 1$ gives a formula analogous to [2] to decompose $\zeta(n)\zeta(p)$.

4 Relations between the values of ζ

We are interested in the kernel $\ker\zeta$ of the morphism of (commutative) $\mathbf{Q}$-algebras defined in (25):

$$\zeta : p \in \mathcal{H}_2 \to \zeta_p \in \mathbf{R}$$

4.1 Quasi-symmetric functions

Let us note $T = \{t_1, t_2, \ldots\}$ the infinite alphabet indexed by positive integers and $\mathbf{Q}[[T]]$ the algebra of formal series (in commutative variables) defined over T and with coefficients in $\mathbf{Q}$. A series $F \in \mathbf{Q}[[T]]$ is called *quasi–symmetric* [19] if for all multi-indices $s = (s_1, s_2, \cdots, s_k)$, the coefficient in F of any monomial of the form $t_{i_1}^{s_1} t_{i_2}^{s_2} \cdots t_{i_k}^{s_k}$ with $i_1 > i_2 > \cdots > i_k$ does not depend on the multi-index $(i_1, i_2, \cdots, i_k)$. We can show that the product of two quasi-symmetric series is quasi-symmetric.

Let us consider the $\mathbf{Q}$-linear map F: $\mathcal{H}_1 \to \mathbf{Q}[[T]]$ which maps any word $w = y_{s_1} y_{s_2} \cdots y_{s_k}$ to the formal quasi-symmetric series:

$$\mathrm{F}_w = \sum_{n_1 > n_2 > \cdots > n_k > 0} t_{n_1}^{s_1} t_{n_2}^{s_2} \cdots t_{n_k}^{s_k} \tag{30}$$

4.1.1 Second shuffle product

The second shuffle product $* : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_1$ is recursively defined on the words of $\mathbf{Q}\langle Y\rangle$ with:

$$\begin{cases} \forall w \in \mathbf{Q}\langle Y\rangle, & \varepsilon * w & = w * \varepsilon = w \\ \forall u, v \in \mathbf{Q}\langle Y\rangle, \forall i, j > 0, & (y_i u) * (y_j v) & = y_i(u * y_j v) + y_j(y_i u * v) + y_{i+j}(u * v) \end{cases} \tag{31}$$

Example 2 $\forall i, j > 0, \quad y_i * y_j = y_i y_j + y_j y_i + y_{i+j}$

We show by induction [10] that this product is associative and commutative. Furthermore, there exists an analogous to Radford theorem: the algebra $\mathcal{H}_1 \simeq \mathbf{Q}\langle Y\rangle$, with the product $*$, is a polynomial algebra freely generated [10] by L_1.

Theorem 4.1

$$\forall p, q \in \mathcal{H}_1, \quad \mathrm{F}_{p*q} \;=\; \mathrm{F}_p \mathrm{F}_q \tag{32}$$

PROOF – See [10]. □

Corollary 4.1

$$\forall p, q \in \mathcal{H}_2, \quad \zeta(p * q) = \zeta(p)\zeta(q) \tag{33}$$

PROOF – $\zeta(p)$ can be computed substituting $(1, 1/2, 1/3, \ldots)$ to the indeterminate variables $(t_1, t_2, t_3, \ldots)$ in the series F_p. □

Hence, the kernel $\ker\zeta$ is an ideal of $\mathcal{H}_2$ for each of both shuffle products $\sqcup\!\sqcup$ and $*$.

4.1.2 Differentiation operator

Let us consider the $\mathbf{Q}$–linear operator D defined over $\mathcal{H}_2$ by

$$D(p) = x_1 \sqcup\!\sqcup\, p - x_1 * p, \quad p \in \mathcal{H}_2. \tag{34}$$

Lemma 4.1 *One has $D(\mathcal{H}_2) \subset \mathcal{H}_2$. D is also the restriction on $\mathcal{H}_2$ of the unique differentiation operator (for the concatenation product) $\bar{D} : \mathbf{Q}\langle X\rangle \to \mathbf{Q}\langle X\rangle$,*

$$\forall p, q \in \mathbf{Q}\langle X\rangle, \quad \bar{D}(pq) \;=\; (\bar{D}p)q + p(\bar{D}q) \tag{35}$$

and the action over the letters x_0, x_1 given by $\bar{D}(x_0) = x_0 x_1$ and $\bar{D}(x_1) = -x_0 x_1$.

Theorem 4.2 *For any polynomial $p \in \mathcal{H}_2$, $D(p)$ are elements of the* kernel *of ζ, i.e.*

$$\zeta_{D(p)} = 0.$$

PROOF – It follows essentially from the fact $D(\mathcal{H}_2) \subset D(\mathcal{H}_2)$ and the divergent terms on $\zeta_{x_1} = \zeta(1) = \sum_{n\geq 1} 1/n$ disappear (see [9, 10]). □

4.2 Generation of a Groebner basis of the relations

Let I be the ideal of $\mathcal{H}_2$ (with the shuffle product ш) generated by the following relations:

$$\begin{cases} \forall l_1, l_2 \in L_2, & l_1 * l_2 = l_1 \,\text{ш}\, l_2 \\ \forall l \in L_2, & D(l) = 0 \end{cases} \tag{36}$$

We showed previously that $I \subset \ker \zeta$.

Thanks to the function `xtaylor` which decomposes a random polynomial of $\mathbf{Q}\langle X \rangle$ in the Radford basis, we can compute a Groebner basis of the ideal I up to any order n. We use the following algorithm:

- Initially $G := \emptyset$.
- For any two Lyndon words $l_1, l_2 \in L_2$, $l_1 \leq l_2$ et $|l_1| + |l_2| \leq N$ add the relation $l_1 \,\text{ш}\, l_2 = \texttt{xtaylor}(l_1 * l_2)$ to G.
- For any Lyndon word $l \in x_0 X^* x_1$, $|w| < N$, add the relation $\texttt{xtaylor}(D(l)) = 0$ to G.
- Return the Groebner basis of $G \subset \mathbf{Q}[L_2]$ where the set L_2 is considered as a set of commutative indeterminate variables.

Let us recall that a Groebner basis B of an ideal I is a set of rewriting rules such that:

$$p \in I \iff p \to 0 \bmod B \tag{37}$$

Completely reduced, it only depends on the ideal I and on the order that has been chosen upon the terms.

In the basis computed up to $N = 10$ in the annex, the set L_2 is sorted according to the lexicographical order by length first. To any term of the form $l_1^{\alpha_1} \,\text{ш}\, l_2^{\alpha_2} \cdots$ corresponds the multi–index $(\alpha_1, \alpha_2, \cdots)$. The terms are compared according to an *elimination order*. In other words, two terms are compared using the lexicographical order on the multi-indices which correspond to them.

We can check (computing the *Hilbert function* associated to this Groebner basis) that the basis we obtain satisfies the dimension conjecture (6).

4.3 Experimental result

Conjecture 4.1 *The ideal defined by the relations (36) is equal to* $\ker \zeta$.

This conjecture is checked up to order 10 if one admits Zagier's dimension conjecture (6).

We notice that the Groebner basis defines a polynomial $\mathbf{Q}$-algebra generated by the numbers $\zeta(2), \zeta(6,2), \zeta(8,2)$ and by the $\zeta(s)$ for the odd $s \geq 3$.

Thanks We thank Ph. Flajolet, J. Borwein and M. Hoffman for having communicated their papers to us, hence encouraging us in the pursuit of this study. The Groebner basis has been verified thanks to the `EZ-FACE` piece of software, which J. Borwein has made available on the WEB. Thanks to J. Marchand, system administrator for the group Medicis in the GAGE lab, at the École Polytechnique in Palaiseau. Sébastien Blondeel, from LIFL, has translated the original article into English.

5 General basis up to order 10

5.1 Order 3

$$\zeta(2, 1) = \zeta(3) \tag{38}$$

5.2 Order 4

$$\zeta(4) = \frac{2}{5}\,\zeta(2)^2 \tag{39}$$
$$\zeta(3, 1) = \frac{1}{10}\,\zeta(2)^2 \tag{40}$$
$$\zeta(2, 1, 1) = \frac{2}{5}\,\zeta(2)^2 \tag{41}$$

5.3 Order 5

$$\zeta(4, 1) = 2\,\zeta(5) - \zeta(2)\,\zeta(3) \tag{42}$$
$$\zeta(3, 2) = -\frac{11}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3) \tag{43}$$
$$\zeta(3, 1, 1) = 2\,\zeta(5) - \zeta(2)\,\zeta(3) \tag{44}$$
$$\zeta(2, 2, 1) = -\frac{11}{2}\,\zeta(5) + 3\,\zeta(2)\,\zeta(3) \tag{45}$$
$$\zeta(2, 1, 1, 1) = \zeta(5) \tag{46}$$

5.4 Order 6

$$\zeta(6) = \frac{8}{35}\,\zeta(2)^3 \tag{47}$$
$$\zeta(5, 1) = -\frac{1}{2}\,\zeta(3)^2 + \frac{6}{35}\,\zeta(2)^3 \tag{48}$$
$$\zeta(4, 2) = \zeta(3)^2 - \frac{32}{105}\,\zeta(2)^3 \tag{49}$$
$$\zeta(4, 1, 1) = -\zeta(3)^2 + \frac{23}{70}\,\zeta(2)^3 \tag{50}$$
$$\zeta(3, 2, 1) = 3\,\zeta(3)^2 - \frac{29}{30}\,\zeta(2)^3 \tag{51}$$

$$\zeta(3,\,1,\,2) = -\frac{3}{2}\,\zeta(3)^2 + \frac{53}{105}\,\zeta(2)^3 \tag{52}$$

$$\zeta(3,\,1,\,1,\,1) = -\frac{1}{2}\,\zeta(3)^2 + \frac{6}{35}\,\zeta(2)^3 \tag{53}$$

$$\zeta(2,\,2,\,1,\,1) = \zeta(3)^2 - \frac{32}{105}\,\zeta(2)^3 \tag{54}$$

$$\zeta(2,\,1,\,1,\,1,\,1) = \frac{8}{35}\,\zeta(2)^3 \tag{55}$$

5.5 Order 7

$$\zeta(6,\,1) = 3\,\zeta(7) - \zeta(2)\,\zeta(5) - \frac{2}{5}\,\zeta(2)^2\,\zeta(3) \tag{56}$$

$$\zeta(5,\,2) = -11\,\zeta(7) + 5\,\zeta(2)\,\zeta(5) + \frac{4}{5}\,\zeta(2)^2\,\zeta(3) \tag{57}$$

$$\zeta(5,\,1,\,1) = 5\,\zeta(7) - 2\,\zeta(2)\,\zeta(5) - \frac{1}{2}\,\zeta(2)^2\,\zeta(3) \tag{58}$$

$$\zeta(4,\,3) = 17\,\zeta(7) - 10\,\zeta(2)\,\zeta(5) \tag{59}$$

$$\zeta(4,\,2,\,1) = -\frac{221}{16}\,\zeta(7) + \frac{11}{2}\,\zeta(2)\,\zeta(5) + \frac{7}{5}\,\zeta(2)^2\,\zeta(3) \tag{60}$$

$$\zeta(4,\,1,\,2) = \frac{5}{8}\,\zeta(7) + \frac{5}{2}\,\zeta(2)\,\zeta(5) - \frac{3}{2}\,\zeta(2)^2\,\zeta(3) \tag{61}$$

$$\zeta(4,\,1,\,1,\,1) = 5\,\zeta(7) - 2\,\zeta(2)\,\zeta(5) - \frac{1}{2}\,\zeta(2)^2\,\zeta(3) \tag{62}$$

$$\zeta(3,\,3,\,1) = \frac{61}{8}\,\zeta(7) - \frac{9}{2}\,\zeta(2)\,\zeta(5) \tag{63}$$

$$\zeta(3,\,2,\,2) = \frac{157}{16}\,\zeta(7) - \frac{15}{2}\,\zeta(2)\,\zeta(5) + \frac{9}{10}\,\zeta(2)^2\,\zeta(3) \tag{64}$$

$$\zeta(3,\,2,\,1,\,1) = -\frac{221}{16}\,\zeta(7) + \frac{11}{2}\,\zeta(2)\,\zeta(5) + \frac{7}{5}\,\zeta(2)^2\,\zeta(3) \tag{65}$$

$$\zeta(3,\,1,\,2,\,1) = \frac{61}{8}\,\zeta(7) - \frac{9}{2}\,\zeta(2)\,\zeta(5) \tag{66}$$

$$\zeta(3,\,1,\,1,\,2) = -\frac{109}{16}\,\zeta(7) + 5\,\zeta(2)\,\zeta(5) - \frac{1}{2}\,\zeta(2)^2\,\zeta(3) \tag{67}$$

$$\zeta(3,\,1,\,1,\,1,\,1) = 3\,\zeta(7) - \zeta(2)\,\zeta(5) - \frac{2}{5}\,\zeta(2)^2\,\zeta(3) \tag{68}$$

$$\zeta(2,\,2,\,2,\,1) = \frac{157}{16}\,\zeta(7) - \frac{15}{2}\,\zeta(2)\,\zeta(5) + \frac{9}{10}\,\zeta(2)^2\,\zeta(3) \tag{69}$$

$$\zeta(2,\,2,\,1,\,1,\,1) = -11\,\zeta(7) + 5\,\zeta(2)\,\zeta(5) + \frac{4}{5}\,\zeta(2)^2\,\zeta(3) \tag{70}$$

$$\zeta(2,\,1,\,2,\,1,\,1) = 17\,\zeta(7) - 10\,\zeta(2)\,\zeta(5) \tag{71}$$

$$\zeta(2,\,1,\,1,\,1,\,1,\,1) = \zeta(7) \tag{72}$$

5.6 Order 8

$$\zeta(8) = \frac{24}{175}\,\zeta(2)^4 \tag{73}$$

$$\zeta(7,\,1) = -\zeta(3)\,\zeta(5) + \frac{6}{35}\,\zeta(2)^4 \tag{74}$$

$$\zeta(6,\,1,\,1) = -3\,\zeta(3)\,\zeta(5) + \frac{1}{2}\,\zeta(2)\,\zeta(3)^2 + \frac{61}{175}\,\zeta(2)^4 \tag{75}$$

$$\zeta(5,\,3) = -\frac{5}{2}\,\zeta(6,\,2) + 5\,\zeta(3)\,\zeta(5) - \frac{21}{25}\,\zeta(2)^4 \tag{76}$$

$$\zeta(5,\,2,\,1) = \frac{7}{4}\,\zeta(6,\,2) + \frac{7}{2}\,\zeta(3)\,\zeta(5) - \zeta(2)\,\zeta(3)^2 - \frac{289}{1050}\,\zeta(2)^4 \tag{77}$$

$$\zeta(5,\,1,\,2) = -\zeta(6,\,2) + \frac{9}{2}\,\zeta(3)\,\zeta(5) - \frac{3}{2}\,\zeta(2)\,\zeta(3)^2 - \frac{29}{105}\,\zeta(2)^4 \tag{78}$$

$$\zeta(5,\,1,\,1,\,1) = -4\,\zeta(3)\,\zeta(5) + \zeta(2)\,\zeta(3)^2 + \frac{499}{1400}\,\zeta(2)^4 \tag{79}$$

$$\zeta(4,\,3,\,1) = -\frac{25}{4}\,\zeta(6,\,2)+\frac{21}{2}\,\zeta(3)\,\zeta(5)+\frac{1}{2}\,\zeta(2)\,\zeta(3)^2-\frac{677}{350}\,\zeta(2)^4 \tag{80}$$
$$\zeta(4,\,2,\,2) = \frac{9}{2}\,\zeta(6,\,2)-20\,\zeta(3)\,\zeta(5)+3\,\zeta(2)\,\zeta(3)^2+\frac{1271}{525}\,\zeta(2)^4 \tag{81}$$
$$\zeta(4,\,2,\,1,\,1) = \frac{3}{2}\,\zeta(6,\,2)+5\,\zeta(3)\,\zeta(5)-2\,\zeta(2)\,\zeta(3)^2-\frac{863}{4200}\,\zeta(2)^4 \tag{82}$$
$$\zeta(4,\,1,\,3) = \frac{5}{2}\,\zeta(6,\,2)-\frac{15}{2}\,\zeta(3)\,\zeta(5)+\frac{1}{2}\,\zeta(2)\,\zeta(3)^2+\frac{583}{525}\,\zeta(2)^4 \tag{83}$$
$$\zeta(4,\,1,\,2,\,1) = -\frac{15}{4}\,\zeta(6,\,2)+13\,\zeta(3)\,\zeta(5)-\frac{3}{2}\,\zeta(2)\,\zeta(3)^2-\frac{7211}{4200}\,\zeta(2)^4 \tag{84}$$
$$\zeta(4,\,1,\,1,\,2) = 3\,\zeta(6,\,2)-9\,\zeta(3)\,\zeta(5)+\frac{1}{2}\,\zeta(2)\,\zeta(3)^2+\frac{229}{168}\,\zeta(2)^4 \tag{85}$$
$$\zeta(4,\,1,\,1,\,1,\,1) = -3\,\zeta(3)\,\zeta(5)+\frac{1}{2}\,\zeta(2)\,\zeta(3)^2+\frac{61}{175}\,\zeta(2)^4 \tag{86}$$
$$\zeta(3,\,3,\,2) = \frac{13}{4}\,\zeta(6,\,2)-23\,\zeta(3)\,\zeta(5)+\frac{9}{2}\,\zeta(2)\,\zeta(3)^2+\frac{857}{350}\,\zeta(2)^4 \tag{87}$$
$$\zeta(3,\,3,\,1,\,1) = -\frac{15}{4}\,\zeta(6,\,2)+13\,\zeta(3)\,\zeta(5)-\frac{3}{2}\,\zeta(2)\,\zeta(3)^2-\frac{7211}{4200}\,\zeta(2)^4 \tag{88}$$
$$\zeta(3,\,2,\,2,\,1) = 9\,\zeta(6,\,2)-51\,\zeta(3)\,\zeta(5)+9\,\zeta(2)\,\zeta(3)^2+\frac{1607}{280}\,\zeta(2)^4 \tag{89}$$
$$\zeta(3,\,2,\,1,\,2) = -\frac{27}{4}\,\zeta(6,\,2)+27\,\zeta(3)\,\zeta(5)-3\,\zeta(2)\,\zeta(3)^2-\frac{15143}{4200}\,\zeta(2)^4 \tag{90}$$
$$\zeta(3,\,2,\,1,\,1,\,1) = \frac{7}{4}\,\zeta(6,\,2)+\frac{7}{2}\,\zeta(3)\,\zeta(5)-\zeta(2)\,\zeta(3)^2-\frac{289}{1050}\,\zeta(2)^4 \tag{91}$$
$$\zeta(3,\,1,\,2,\,2) = -\frac{9}{2}\,\zeta(6,\,2)+\frac{51}{2}\,\zeta(3)\,\zeta(5)-\frac{9}{2}\,\zeta(2)\,\zeta(3)^2-\frac{803}{280}\,\zeta(2)^4 \tag{92}$$
$$\zeta(3,\,1,\,2,\,1,\,1) = -\frac{25}{4}\,\zeta(6,\,2)+\frac{21}{2}\,\zeta(3)\,\zeta(5)+\frac{1}{2}\,\zeta(2)\,\zeta(3)^2-\frac{677}{350}\,\zeta(2)^4 \tag{93}$$
$$\zeta(3,\,1,\,1,\,2,\,1) = \frac{15}{4}\,\zeta(6,\,2)-2\,\zeta(2)\,\zeta(3)^2+\frac{673}{1050}\,\zeta(2)^4 \tag{94}$$
$$\zeta(3,\,1,\,1,\,1,\,2) = \frac{7}{4}\,\zeta(6,\,2)-12\,\zeta(3)\,\zeta(5)+2\,\zeta(2)\,\zeta(3)^2+\frac{487}{350}\,\zeta(2)^4 \tag{95}$$
$$\zeta(3,\,1,\,1,\,1,\,1,\,1) = -\zeta(3)\,\zeta(5)+\frac{6}{35}\,\zeta(2)^4 \tag{96}$$
$$\zeta(2,\,2,\,2,\,1,\,1) = \frac{9}{2}\,\zeta(6,\,2)-20\,\zeta(3)\,\zeta(5)+3\,\zeta(2)\,\zeta(3)^2+\frac{1271}{525}\,\zeta(2)^4 \tag{97}$$
$$\zeta(2,\,2,\,1,\,2,\,1) = \frac{13}{4}\,\zeta(6,\,2)-23\,\zeta(3)\,\zeta(5)+\frac{9}{2}\,\zeta(2)\,\zeta(3)^2+\frac{857}{350}\,\zeta(2)^4 \tag{98}$$
$$\zeta(2,\,2,\,1,\,1,\,1,\,1) = \zeta(6,\,2) \tag{99}$$
$$\zeta(2,\,1,\,2,\,1,\,1,\,1) = -\frac{5}{2}\,\zeta(6,\,2)+5\,\zeta(3)\,\zeta(5)-\frac{21}{25}\,\zeta(2)^4 \tag{100}$$
$$\zeta(2,\,1,\,1,\,1,\,1,\,1,\,1) = \frac{24}{175}\,\zeta(2)^4 \tag{101}$$

5.7 Order 9

$$\zeta(8,\,1) = 4\,\zeta(9)-\zeta(2)\,\zeta(7)-\frac{2}{5}\,\zeta(2)^2\,\zeta(5)-\frac{8}{35}\,\zeta(2)^3\,\zeta(3) \tag{102}$$
$$\zeta(7,\,2) = -\frac{37}{2}\,\zeta(9)+7\,\zeta(2)\,\zeta(7)+\frac{8}{5}\,\zeta(2)^2\,\zeta(5)+\frac{16}{35}\,\zeta(2)^3\,\zeta(3) \tag{103}$$
$$\zeta(7,\,1,\,1) = \frac{28}{3}\,\zeta(9)-3\,\zeta(2)\,\zeta(7)-\frac{9}{10}\,\zeta(2)^2\,\zeta(5)+\frac{1}{6}\,\zeta(3)^3-\frac{2}{5}\,\zeta(2)^3\,\zeta(3) \tag{104}$$
$$\zeta(6,\,3) = \frac{83}{2}\,\zeta(9)-21\,\zeta(2)\,\zeta(7)-\frac{12}{5}\,\zeta(2)^2\,\zeta(5) \tag{105}$$
$$\zeta(6,\,2,\,1) = -\frac{2189}{72}\,\zeta(9)+11\,\zeta(2)\,\zeta(7)+\frac{13}{5}\,\zeta(2)^2\,\zeta(5)-\frac{1}{3}\,\zeta(3)^3+\frac{36}{35}\,\zeta(2)^3\,\zeta(3) \tag{106}$$
$$\zeta(6,\,1,\,2) = -\frac{313}{36}\,\zeta(9)+7\,\zeta(2)\,\zeta(7)-\frac{1}{10}\,\zeta(2)^2\,\zeta(5)-\frac{1}{3}\,\zeta(3)^3-\frac{8}{21}\,\zeta(2)^3\,\zeta(3) \tag{107}$$
$$\zeta(6,\,1,\,1,\,1) = 14\,\zeta(9)-5\,\zeta(2)\,\zeta(7)-\frac{7}{5}\,\zeta(2)^2\,\zeta(5)+\frac{1}{2}\,\zeta(3)^3-\frac{1}{2}\,\zeta(2)^3\,\zeta(3) \tag{108}$$
$$\zeta(5,\,4) = -\frac{127}{2}\,\zeta(9)+35\,\zeta(2)\,\zeta(7)+2\,\zeta(2)^2\,\zeta(5) \tag{109}$$

$$\zeta(5,3,1) = \frac{845}{24}\zeta(9) - 17\,\zeta(2)\,\zeta(7) - \frac{23}{10}\zeta(2)^2\,\zeta(5) + \frac{1}{6}\zeta(3)^3 - \frac{6}{35}\zeta(2)^3\,\zeta(3) \tag{110}$$

$$\zeta(5,2,2) = \frac{2513}{72}\zeta(9) - 21\,\zeta(2)\,\zeta(7) + \frac{7}{10}\zeta(2)^2\,\zeta(5) + \frac{2}{3}\zeta(3)^3 - \frac{64}{105}\zeta(2)^3\,\zeta(3) \tag{111}$$

$$\zeta(5,2,1,1) = -\frac{1909}{48}\zeta(9) + \frac{221}{16}\zeta(2)\,\zeta(7) + \frac{19}{4}\zeta(2)^2\,\zeta(5) - \zeta(3)^3 + \zeta(2)^3\,\zeta(3) \tag{112}$$

$$\zeta(5,1,3) = \frac{121}{12}\zeta(9) - 7\,\zeta(2)\,\zeta(7) + \frac{1}{5}\zeta(2)^2\,\zeta(5) - \frac{1}{3}\zeta(3)^3 + \frac{2}{7}\zeta(2)^3\,\zeta(3) \tag{113}$$

$$\zeta(5,1,2,1) = \frac{361}{144}\zeta(9) - \frac{5}{8}\zeta(2)\,\zeta(7) - \frac{7}{4}\zeta(2)^2\,\zeta(5) - \zeta(3)^3 + \frac{29}{30}\zeta(2)^3\,\zeta(3) \tag{114}$$

$$\zeta(5,1,1,2) = -\frac{2765}{144}\zeta(9) + \frac{189}{16}\zeta(2)\,\zeta(7) + \frac{3}{5}\zeta(2)^2\,\zeta(5) + \frac{1}{2}\zeta(3)^3 - \frac{19}{35}\zeta(2)^3\,\zeta(3) \tag{115}$$

$$\zeta(5,1,1,1,1) = 14\,\zeta(9) - 5\,\zeta(2)\,\zeta(7) - \frac{7}{5}\zeta(2)^2\,\zeta(5) + \frac{1}{2}\zeta(3)^3 - \frac{1}{2}\zeta(2)^3\,\zeta(3) \tag{116}$$

$$\zeta(4,4,1) = -\frac{328}{9}\zeta(9) + 18\,\zeta(2)\,\zeta(7) + 2\,\zeta(2)^2\,\zeta(5) - \frac{1}{3}\zeta(3)^3 + \frac{32}{105}\zeta(2)^3\,\zeta(3) \tag{117}$$

$$\zeta(4,3,2) = -\frac{53}{36}\zeta(9) + 14\,\zeta(2)\,\zeta(7) - 7\,\zeta(2)^2\,\zeta(5) + \frac{2}{3}\zeta(3)^3 - \frac{64}{105}\zeta(2)^3\,\zeta(3) \tag{118}$$

$$\zeta(4,3,1,1) = \frac{3227}{144}\zeta(9) - \frac{61}{8}\zeta(2)\,\zeta(7) - \frac{69}{20}\zeta(2)^2\,\zeta(5) - \frac{1}{42}\zeta(2)^3\,\zeta(3) \tag{119}$$

$$\zeta(4,2,3) = -59\,\zeta(9) + 28\,\zeta(2)\,\zeta(7) + 4\,\zeta(2)^2\,\zeta(5) - \frac{1}{3}\zeta(3)^3 + \frac{8}{21}\zeta(2)^3\,\zeta(3) \tag{120}$$

$$\zeta(4,2,2,1) = \frac{845}{24}\zeta(9) - \frac{157}{16}\zeta(2)\,\zeta(7) - \frac{16}{5}\zeta(2)^2\,\zeta(5) + 3\,\zeta(3)^3 - \frac{299}{105}\zeta(2)^3\,\zeta(3) \tag{121}$$

$$\zeta(4,2,1,2) = \frac{1501}{36}\zeta(9) - \frac{483}{16}\zeta(2)\,\zeta(7) + \frac{33}{20}\zeta(2)^2\,\zeta(5) - \zeta(3)^3 + \frac{106}{105}\zeta(2)^3\,\zeta(3) \tag{122}$$

$$\zeta(4,2,1,1,1) = -\frac{1909}{48}\zeta(9) + \frac{221}{16}\zeta(2)\,\zeta(7) + \frac{19}{4}\zeta(2)^2\,\zeta(5) - \zeta(3)^3 + \zeta(2)^3\,\zeta(3) \tag{123}$$

$$\zeta(4,1,3,1) = \frac{103}{144}\zeta(9) + \frac{1}{4}\zeta(2)\,\zeta(7) + \frac{1}{4}\zeta(2)^2\,\zeta(5) + \frac{1}{2}\zeta(3)^3 - \frac{53}{105}\zeta(2)^3\,\zeta(3) \tag{124}$$

$$\zeta(4,1,2,2) = -\frac{1187}{36}\zeta(9) + \frac{189}{16}\zeta(2)\,\zeta(7) + \frac{15}{4}\zeta(2)^2\,\zeta(5) - \zeta(3)^3 + \frac{61}{70}\zeta(2)^3\,\zeta(3) \tag{125}$$

$$\zeta(4,1,2,1,1) = \frac{3227}{144}\zeta(9) - \frac{61}{8}\zeta(2)\,\zeta(7) - \frac{69}{20}\zeta(2)^2\,\zeta(5) - \frac{1}{42}\zeta(2)^3\,\zeta(3) \tag{126}$$

$$\zeta(4,1,1,3) = \frac{3721}{144}\zeta(9) - \frac{231}{16}\zeta(2)\,\zeta(7) - \zeta(2)^2\,\zeta(5) - \frac{1}{2}\zeta(3)^3 + \frac{34}{105}\zeta(2)^3\,\zeta(3) \tag{127}$$

$$\zeta(4,1,1,2,1) = -\frac{4067}{144}\zeta(9) + \frac{109}{16}\zeta(2)\,\zeta(7) + \frac{43}{10}\zeta(2)^2\,\zeta(5) - \frac{3}{2}\zeta(3)^3 + \frac{148}{105}\zeta(2)^3\,\zeta(3) \tag{128}$$

$$\zeta(4,1,1,1,2) = \frac{265}{144}\zeta(9) + 7\,\zeta(2)\,\zeta(7) - \frac{13}{5}\zeta(2)^2\,\zeta(5) + \frac{3}{2}\zeta(3)^3 - \frac{172}{105}\zeta(2)^3\,\zeta(3) \tag{129}$$

$$\zeta(4,1,1,1,1,1) = \frac{28}{3}\zeta(9) - 3\,\zeta(2)\,\zeta(7) - \frac{9}{10}\zeta(2)^2\,\zeta(5) + \frac{1}{6}\zeta(3)^3 - \frac{2}{5}\zeta(2)^3\,\zeta(3) \tag{130}$$

$$\zeta(3,3,2,1) = \frac{1009}{36}\zeta(9) - \frac{75}{8}\zeta(2)\,\zeta(7) - \frac{9}{20}\zeta(2)^2\,\zeta(5) + \frac{7}{2}\zeta(3)^3 - \frac{227}{70}\zeta(2)^3\,\zeta(3) \tag{131}$$

$$\zeta(3,3,1,2) = -\frac{1205}{72}\zeta(9) + \frac{189}{16}\zeta(2)\,\zeta(7) - \frac{27}{20}\zeta(2)^2\,\zeta(5) - \frac{1}{2}\zeta(3)^3 + \frac{12}{35}\zeta(2)^3\,\zeta(3) \tag{132}$$

$$\zeta(3,3,1,1,1) = \frac{361}{144}\zeta(9) - \frac{5}{8}\zeta(2)\,\zeta(7) - \frac{7}{4}\zeta(2)^2\,\zeta(5) - \zeta(3)^3 + \frac{29}{30}\zeta(2)^3\,\zeta(3) \tag{133}$$

$$\zeta(3,2,3,1) = -\frac{367}{8}\zeta(9) + \frac{291}{16}\zeta(2)\,\zeta(7) - \frac{9}{2}\zeta(3)^3 + \frac{309}{70}\zeta(2)^3\,\zeta(3) \tag{134}$$

$$\zeta(3,2,2,2) = -\frac{223}{16}\zeta(9) + \frac{189}{16}\zeta(2)\,\zeta(7) - \frac{9}{4}\zeta(2)^2\,\zeta(5) + \frac{9}{70}\zeta(2)^3\,\zeta(3) \tag{135}$$

$$\zeta(3,2,2,1,1) = \frac{845}{24}\zeta(9) - \frac{157}{16}\zeta(2)\,\zeta(7) - \frac{16}{5}\zeta(2)^2\,\zeta(5) + 3\,\zeta(3)^3 - \frac{299}{105}\zeta(2)^3\,\zeta(3) \tag{136}$$

$$\zeta(3,2,1,2,1) = \frac{1009}{36}\zeta(9) - \frac{75}{8}\zeta(2)\,\zeta(7) - \frac{9}{20}\zeta(2)^2\,\zeta(5) + \frac{7}{2}\zeta(3)^3 - \frac{227}{70}\zeta(2)^3\,\zeta(3) \tag{137}$$

$$\zeta(3,2,1,1,2) = \frac{989}{72}\zeta(9) - 21\,\zeta(2)\,\zeta(7) + \frac{33}{20}\zeta(2)^2\,\zeta(5) - 4\,\zeta(3)^3 + \frac{153}{35}\zeta(2)^3\,\zeta(3) \tag{138}$$

$$\zeta(3,2,1,1,1,1) = -\frac{2189}{72}\zeta(9) + 11\,\zeta(2)\,\zeta(7) + \frac{13}{5}\zeta(2)^2\,\zeta(5) - \frac{1}{3}\zeta(3)^3 + \frac{36}{35}\zeta(2)^3\,\zeta(3) \tag{139}$$

$$\zeta(3,1,3,1,1) = \frac{103}{144}\zeta(9) + \frac{1}{4}\zeta(2)\,\zeta(7) + \frac{1}{4}\zeta(2)^2\,\zeta(5) + \frac{1}{2}\zeta(3)^3 - \frac{53}{105}\zeta(2)^3\,\zeta(3) \tag{140}$$

$$\zeta(3,1,2,2,1) = -\frac{367}{8}\zeta(9)+\frac{291}{16}\zeta(2)\zeta(7)-\frac{9}{2}\zeta(3)^3+\frac{309}{70}\zeta(2)^3\zeta(3) \tag{141}$$

$$\zeta(3,1,2,1,2) = -\frac{91}{6}\zeta(9)+14\zeta(2)\zeta(7)-\frac{9}{10}\zeta(2)^2\zeta(5)+\frac{3}{2}\zeta(3)^3-\frac{53}{35}\zeta(2)^3\zeta(3) \tag{142}$$

$$\zeta(3,1,2,1,1,1) = \frac{845}{24}\zeta(9)-17\zeta(2)\zeta(7)-\frac{23}{10}\zeta(2)^2\zeta(5)+\frac{1}{6}\zeta(3)^3-\frac{6}{35}\zeta(2)^3\zeta(3) \tag{143}$$

$$\zeta(3,1,1,2,2) = \frac{803}{36}\zeta(9)-14\zeta(2)\zeta(7)+\frac{17}{10}\zeta(2)^2\zeta(5)+\zeta(3)^3-\frac{221}{210}\zeta(2)^3\zeta(3) \tag{144}$$

$$\zeta(3,1,1,2,1,1) = -\frac{328}{9}\zeta(9)+18\zeta(2)\zeta(7)+2\zeta(2)^2\zeta(5)-\frac{1}{3}\zeta(3)^3+\frac{32}{105}\zeta(2)^3\zeta(3) \tag{145}$$

$$\zeta(3,1,1,1,2,1) = \frac{341}{24}\zeta(9)-10\zeta(2)\zeta(7)+\frac{1}{2}\zeta(2)^2\zeta(5)-\frac{1}{3}\zeta(3)^3+\frac{2}{7}\zeta(2)^3\zeta(3) \tag{146}$$

$$\zeta(3,1,1,1,1,2) = -\frac{461}{72}\zeta(9)+7\zeta(2)\zeta(7)-\frac{7}{10}\zeta(2)^2\zeta(5)+\frac{2}{3}\zeta(3)^3-\frac{86}{105}\zeta(2)^3\zeta(3) \tag{147}$$

$$\zeta(3,1,1,1,1,1,1) = 4\zeta(9)-\zeta(2)\zeta(7)-\frac{2}{5}\zeta(2)^2\zeta(5)-\frac{8}{35}\zeta(2)^3\zeta(3) \tag{148}$$

$$\zeta(2,2,2,2,1) = -\frac{223}{16}\zeta(9)+\frac{189}{16}\zeta(2)\zeta(7)-\frac{9}{4}\zeta(2)^2\zeta(5)+\frac{9}{70}\zeta(2)^3\zeta(3) \tag{149}$$

$$\zeta(2,2,2,1,1,1) = \frac{2513}{72}\zeta(9)-21\zeta(2)\zeta(7)+\frac{7}{10}\zeta(2)^2\zeta(5)+\frac{2}{3}\zeta(3)^3-\frac{64}{105}\zeta(2)^3\zeta(3) \tag{150}$$

$$\zeta(2,2,1,2,1,1) = -\frac{53}{36}\zeta(9)+14\zeta(2)\zeta(7)-7\zeta(2)^2\zeta(5)+\frac{2}{3}\zeta(3)^3-\frac{64}{105}\zeta(2)^3\zeta(3) \tag{151}$$

$$\zeta(2,2,1,1,2,1) = \frac{593}{36}\zeta(9)-14\zeta(2)\zeta(7)+3\zeta(2)^2\zeta(5)+\frac{2}{3}\zeta(3)^3-\frac{8}{15}\zeta(2)^3\zeta(3) \tag{152}$$

$$\zeta(2,2,1,1,1,1,1) = -\frac{37}{2}\zeta(9)+7\zeta(2)\zeta(7)+\frac{8}{5}\zeta(2)^2\zeta(5)+\frac{16}{35}\zeta(2)^3\zeta(3) \tag{153}$$

$$\zeta(2,1,2,1,1,1,1) = \frac{83}{2}\zeta(9)-21\zeta(2)\zeta(7)-\frac{12}{5}\zeta(2)^2\zeta(5) \tag{154}$$

$$\zeta(2,1,1,2,1,1,1) = -\frac{127}{2}\zeta(9)+35\zeta(2)\zeta(7)+2\zeta(2)^2\zeta(5) \tag{155}$$

$$\zeta(2,1,1,1,1,1,1,1) = \zeta(9) \tag{156}$$

5.8 Order 10

$$\zeta(10) = \frac{32}{385}\zeta(2)^5 \tag{157}$$

$$\zeta(9,1) = -\zeta(3)\zeta(7)-\frac{1}{2}\zeta(5)^2+\frac{8}{55}\zeta(2)^5 \tag{158}$$

$$\zeta(8,1,1) = -4\zeta(3)\zeta(7)-2\zeta(5)^2+\zeta(2)\zeta(3)\zeta(5)+\frac{1}{5}\zeta(2)^2\zeta(3)^2+\frac{666}{1925}\zeta(2)^5 \tag{159}$$

$$\zeta(7,3) = -\frac{7}{2}\zeta(8,2)+7\zeta(3)\zeta(7)+4\zeta(5)^2-\frac{408}{385}\zeta(2)^5 \tag{160}$$

$$\zeta(7,2,1) = \frac{9}{4}\zeta(8,2)-\zeta(2)\zeta(6,2)+\frac{9}{2}\zeta(3)\zeta(7)+\frac{9}{4}\zeta(5)^2-\frac{2}{5}\zeta(2)^2\zeta(3)^2-\frac{3016}{5775}\zeta(2)^5 \tag{161}$$

$$\begin{aligned}\zeta(7,1,2) &= -\zeta(8,2)+\zeta(2)\zeta(6,2)+10\zeta(3)\zeta(7)+5\zeta(5)^2-7\zeta(2)\zeta(3)\zeta(5)-\frac{2}{5}\zeta(2)^2\zeta(3)^2\\ &- \frac{2}{15}\zeta(2)^5\end{aligned} \tag{162}$$

$$\zeta(7,1,1,1) = -8\zeta(3)\zeta(7)-4\zeta(5)^2+3\zeta(2)\zeta(3)\zeta(5)+\frac{9}{20}\zeta(2)^2\zeta(3)^2+\frac{139}{275}\zeta(2)^5 \tag{163}$$

$$\zeta(6,4) = \frac{7}{2}\zeta(8,2)-7\zeta(3)\zeta(7)-5\zeta(5)^2+\frac{2216}{1925}\zeta(2)^5 \tag{164}$$

$$\begin{aligned}\zeta(6,3,1) &= -\frac{35}{4}\zeta(8,2)+\frac{5}{2}\zeta(2)\zeta(6,2)+\frac{29}{2}\zeta(3)\zeta(7)+\frac{37}{4}\zeta(5)^2-4\zeta(2)\zeta(3)\zeta(5)\\ &+ \frac{1}{5}\zeta(2)^2\zeta(3)^2-\frac{3219}{1925}\zeta(2)^5\end{aligned} \tag{165}$$

$$\begin{aligned}\zeta(6,2,2) &= 3\zeta(8,2)+\zeta(2)\zeta(6,2)-28\zeta(3)\zeta(7)-18\zeta(5)^2+8\zeta(2)\zeta(3)\zeta(5)+\frac{4}{5}\zeta(2)^2\zeta(3)^2\\ &+ \frac{14}{5}\zeta(2)^5\end{aligned} \tag{166}$$

$$\begin{aligned}
\zeta(6,\,2,\,1,\,1) &= \frac{141}{64}\,\zeta(8,\,2) - \frac{7}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{371}{32}\,\zeta(3)\,\zeta(7) + \frac{251}{32}\,\zeta(5)^2 - \frac{5}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{9}{10}\,\zeta(2)^2\,\zeta(3)^2 - \frac{8823}{7700}\,\zeta(2)^5 \qquad (167)\\
\zeta(6,\,1,\,3) &= \frac{7}{2}\,\zeta(8,\,2) - \frac{7}{2}\,\zeta(2)\,\zeta(6,\,2) - 21\,\zeta(3)\,\zeta(7) - 9\,\zeta(5)^2 + 16\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{1}{5}\,\zeta(2)^2\,\zeta(3)^2 + \frac{73}{275}\,\zeta(2)^5 \qquad (168)\\
\zeta(6,\,1,\,2,\,1) &= -\frac{119}{64}\,\zeta(8,\,2) + \zeta(2)\,\zeta(6,\,2) + \frac{561}{32}\,\zeta(3)\,\zeta(7) + \frac{151}{32}\,\zeta(5)^2 - \frac{15}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{7}{10}\,\zeta(2)^2\,\zeta(3)^2 - \frac{15787}{23100}\,\zeta(2)^5 \qquad (169)\\
\zeta(6,\,1,\,1,\,2) &= \frac{39}{64}\,\zeta(8,\,2) + \frac{3}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{101}{32}\,\zeta(3)\,\zeta(7) + \frac{113}{32}\,\zeta(5)^2 - 7\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{3}{4}\,\zeta(2)^2\,\zeta(3)^2 + \frac{7237}{23100}\,\zeta(2)^5 \qquad (170)\\
\zeta(6,\,1,\,1,\,1,\,1) &= -10\,\zeta(3)\,\zeta(7) - 5\,\zeta(5)^2 + 4\,\zeta(2)\,\zeta(3)\,\zeta(5) + \frac{1}{2}\,\zeta(2)^2\,\zeta(3)^2 + \frac{1341}{2200}\,\zeta(2)^5 \qquad (171)\\
\zeta(5,\,4,\,1) &= \frac{21}{2}\,\zeta(8,\,2) - 10\,\zeta(3)\,\zeta(7) - \frac{27}{2}\,\zeta(5)^2 - 5\,\zeta(2)\,\zeta(3)\,\zeta(5) - \frac{2}{5}\,\zeta(2)^2\,\zeta(3)^2 \\
&+ \frac{18418}{5775}\,\zeta(2)^5 \qquad (172)\\
\zeta(5,\,3,\,2) &= \frac{19}{4}\,\zeta(8,\,2) - \frac{25}{2}\,\zeta(2)\,\zeta(6,\,2) - \frac{63}{2}\,\zeta(3)\,\zeta(7) - \frac{3}{4}\,\zeta(5)^2 + 35\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2 - \frac{5683}{1925}\,\zeta(2)^5 \qquad (173)\\
\zeta(5,\,3,\,1,\,1) &= -\frac{245}{32}\,\zeta(8,\,2) + \frac{25}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{341}{16}\,\zeta(3)\,\zeta(7) + \frac{113}{16}\,\zeta(5)^2 - \frac{31}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{3}{20}\,\zeta(2)^2\,\zeta(3)^2 - \frac{83}{825}\,\zeta(2)^5 \qquad (174)\\
\zeta(5,\,2,\,3) &= -\frac{49}{4}\,\zeta(8,\,2) + 10\,\zeta(2)\,\zeta(6,\,2) + \frac{161}{2}\,\zeta(3)\,\zeta(7) + \frac{163}{4}\,\zeta(5)^2 - 50\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2 - \frac{5752}{1925}\,\zeta(2)^5 \qquad (175)\\
\zeta(5,\,2,\,2,\,1) &= \frac{477}{64}\,\zeta(8,\,2) - \frac{9}{2}\,\zeta(2)\,\zeta(6,\,2) - \frac{2417}{32}\,\zeta(3)\,\zeta(7) - \frac{1101}{32}\,\zeta(5)^2 + 35\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{13}{5}\,\zeta(2)^2\,\zeta(3)^2 + \frac{29871}{7700}\,\zeta(2)^5 \qquad (176)\\
\zeta(5,\,2,\,1,\,2) &= -\frac{141}{32}\,\zeta(8,\,2) + \frac{3}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{161}{16}\,\zeta(3)\,\zeta(7) + \frac{79}{16}\,\zeta(5)^2 + \frac{7}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{3}{2}\,\zeta(2)^2\,\zeta(3)^2 - \frac{361}{231}\,\zeta(2)^5 \qquad (177)\\
\zeta(5,\,2,\,1,\,1,\,1) &= \frac{61}{32}\,\zeta(8,\,2) - \frac{3}{2}\,\zeta(2)\,\zeta(6,\,2) + \frac{259}{16}\,\zeta(3)\,\zeta(7) + \frac{195}{16}\,\zeta(5)^2 - 5\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \zeta(2)^2\,\zeta(3)^2 - \frac{2843}{1848}\,\zeta(2)^5 \qquad (178)\\
\zeta(5,\,1,\,4) &= -\frac{7}{2}\,\zeta(8,\,2) + \frac{5}{2}\,\zeta(2)\,\zeta(6,\,2) + 14\,\zeta(3)\,\zeta(7) + \frac{9}{2}\,\zeta(5)^2 - 10\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{1}{5}\,\zeta(2)^2\,\zeta(3)^2 - \frac{1027}{5775}\,\zeta(2)^5 \qquad (179)\\
\zeta(5,\,1,\,3,\,1) &= \frac{7}{32}\,\zeta(8,\,2) - \frac{5}{2}\,\zeta(2)\,\zeta(6,\,2) - \frac{129}{16}\,\zeta(3)\,\zeta(7) + \frac{1}{16}\,\zeta(5)^2 + \frac{19}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{443}{550}\,\zeta(2)^5 \qquad (180)\\
\zeta(5,\,1,\,2,\,2) &= -3\,\zeta(8,\,2) + \frac{9}{2}\,\zeta(2)\,\zeta(6,\,2) + \frac{467}{16}\,\zeta(3)\,\zeta(7) + \frac{67}{8}\,\zeta(5)^2 - \frac{33}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{33}{20}\,\zeta(2)^2\,\zeta(3)^2 - \frac{184}{525}\,\zeta(2)^5 \qquad (181)\\
\zeta(5,\,1,\,2,\,1,\,1) &= -\frac{273}{64}\,\zeta(8,\,2) + \frac{15}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{555}{32}\,\zeta(3)\,\zeta(7) - \frac{15}{32}\,\zeta(5)^2 - 11\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{9}{20}\,\zeta(2)^2\,\zeta(3)^2 + \frac{2029}{6600}\,\zeta(2)^5 \qquad (182)
\end{aligned}$$

$$\begin{aligned}
\zeta(5,\,1,\,1,\,3) &= \frac{161}{32}\,\zeta(8,\,2) - \frac{9}{2}\,\zeta(2)\,\zeta(6,\,2) - \frac{525}{16}\,\zeta(3)\,\zeta(7) - \frac{253}{16}\,\zeta(5)^2 + 22\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{1}{10}\,\zeta(2)^2\,\zeta(3)^2 + \frac{10921}{11550}\,\zeta(2)^5 \qquad (183) \\
\zeta(5,\,1,\,1,\,2,\,1) &= \frac{119}{64}\,\zeta(8,\,2) - 3\,\zeta(2)\,\zeta(6,\,2) + \frac{239}{32}\,\zeta(3)\,\zeta(7) + \frac{313}{32}\,\zeta(5)^2 + 3\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{19}{20}\,\zeta(2)^2\,\zeta(3)^2 - \frac{84113}{46200}\,\zeta(2)^5 \qquad (184) \\
\zeta(5,\,1,\,1,\,1,\,2) &= \frac{93}{64}\,\zeta(8,\,2) + \frac{3}{4}\,\zeta(2)\,\zeta(6,\,2) - \frac{215}{32}\,\zeta(3)\,\zeta(7) - \frac{141}{32}\,\zeta(5)^2 - 5\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{3}{2}\,\zeta(2)^2\,\zeta(3)^2 + \frac{11023}{7700}\,\zeta(2)^5 \qquad (185) \\
\zeta(5,\,1,\,1,\,1,\,1,\,1) &= -8\,\zeta(3)\,\zeta(7) - 4\,\zeta(5)^2 + 3\,\zeta(2)\,\zeta(3)\,\zeta(5) + \frac{9}{20}\,\zeta(2)^2\,\zeta(3)^2 + \frac{139}{275}\,\zeta(2)^5 \qquad (186) \\
\zeta(4,\,4,\,2) &= -\frac{15}{2}\,\zeta(8,\,2) + 10\,\zeta(2)\,\zeta(6,\,2) + 49\,\zeta(3)\,\zeta(7) + 15\,\zeta(5)^2 - 40\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{996}{1925}\,\zeta(2)^5 \qquad (187) \\
\zeta(4,\,4,\,1,\,1) &= 7\,\zeta(8,\,2) - \frac{15}{4}\,\zeta(2)\,\zeta(6,\,2) - \frac{275}{16}\,\zeta(3)\,\zeta(7) - 11\,\zeta(5)^2 + 12\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{7}{10}\,\zeta(2)^2\,\zeta(3)^2 + \frac{10429}{11550}\,\zeta(2)^5 \qquad (188) \\
\zeta(4,\,3,\,3) &= \frac{7}{2}\,\zeta(8,\,2) + 10\,\zeta(2)\,\zeta(6,\,2) - 7\,\zeta(3)\,\zeta(7) - \frac{59}{2}\,\zeta(5)^2 - 20\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{12932}{1925}\,\zeta(2)^5 \qquad (189) \\
\zeta(4,\,3,\,2,\,1) &= -\frac{75}{16}\,\zeta(8,\,2) - \frac{13}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{131}{4}\,\zeta(3)\,\zeta(7) + \frac{259}{8}\,\zeta(5)^2 - \frac{25}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{7}{5}\,\zeta(2)^2\,\zeta(3)^2 - \frac{10403}{2310}\,\zeta(2)^5 \qquad (190) \\
\zeta(4,\,3,\,1,\,2) &= \frac{893}{64}\,\zeta(8,\,2) - \frac{21}{2}\,\zeta(2)\,\zeta(6,\,2) - \frac{2545}{32}\,\zeta(3)\,\zeta(7) - \frac{1217}{32}\,\zeta(5)^2 + \frac{97}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{3}{4}\,\zeta(2)^2\,\zeta(3)^2 + \frac{67457}{23100}\,\zeta(2)^5 \qquad (191) \\
\zeta(4,\,3,\,1,\,1,\,1) &= -\frac{273}{64}\,\zeta(8,\,2) + \frac{15}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{555}{32}\,\zeta(3)\,\zeta(7) - \frac{15}{32}\,\zeta(5)^2 - 11\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{9}{20}\,\zeta(2)^2\,\zeta(3)^2 + \frac{2029}{6600}\,\zeta(2)^5 \qquad (192) \\
\zeta(4,\,2,\,3,\,1) &= -\frac{21}{8}\,\zeta(8,\,2) + 10\,\zeta(2)\,\zeta(6,\,2) + \frac{991}{16}\,\zeta(3)\,\zeta(7) + \frac{43}{4}\,\zeta(5)^2 - \frac{91}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{21}{10}\,\zeta(2)^2\,\zeta(3)^2 + \frac{6994}{5775}\,\zeta(2)^5 \qquad (193) \\
\zeta(4,\,2,\,2,\,2) &= -\frac{141}{32}\,\zeta(8,\,2) + \frac{9}{2}\,\zeta(2)\,\zeta(6,\,2) + \frac{455}{16}\,\zeta(3)\,\zeta(7) + \frac{229}{16}\,\zeta(5)^2 - 24\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{9}{10}\,\zeta(2)^2\,\zeta(3)^2 - \frac{1369}{3850}\,\zeta(2)^5 \qquad (194) \\
\zeta(4,\,2,\,2,\,1,\,1) &= \frac{285}{32}\,\zeta(8,\,2) - 9\,\zeta(2)\,\zeta(6,\,2) - \frac{1169}{16}\,\zeta(3)\,\zeta(7) - \frac{349}{16}\,\zeta(5)^2 + 40\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{14}{5}\,\zeta(2)^2\,\zeta(3)^2 + \frac{73957}{46200}\,\zeta(2)^5 \qquad (195) \\
\zeta(4,\,2,\,1,\,3) &= -\frac{629}{64}\,\zeta(8,\,2) + \frac{27}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{1803}{32}\,\zeta(3)\,\zeta(7) + \frac{1025}{32}\,\zeta(5)^2 - 38\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2 - \frac{4901}{2100}\,\zeta(2)^5 \qquad (196) \\
\zeta(4,\,2,\,1,\,2,\,1) &= -\frac{107}{32}\,\zeta(8,\,2) + \frac{27}{4}\,\zeta(2)\,\zeta(6,\,2) - \frac{39}{2}\,\zeta(3)\,\zeta(7) - \frac{207}{16}\,\zeta(5)^2 - 6\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{21}{10}\,\zeta(2)^2\,\zeta(3)^2 + \frac{6361}{1848}\,\zeta(2)^5 \qquad (197) \\
\zeta(4,\,2,\,1,\,1,\,2) &= -\frac{359}{64}\,\zeta(8,\,2) + \frac{3}{4}\,\zeta(2)\,\zeta(6,\,2) + \frac{847}{32}\,\zeta(3)\,\zeta(7) + \frac{271}{32}\,\zeta(5)^2 + \frac{5}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{17}{5}\,\zeta(2)^2\,\zeta(3)^2 - \frac{12667}{4620}\,\zeta(2)^5 \qquad (198)
\end{aligned}$$

$$
\begin{aligned}
\zeta(4,2,1,1,1,1) &= \frac{141}{64}\,\zeta(8,2) - \frac{7}{4}\,\zeta(2)\,\zeta(6,2) + \frac{371}{32}\,\zeta(3)\,\zeta(7) + \frac{251}{32}\,\zeta(5)^2 - \frac{5}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{9}{10}\,\zeta(2)^2\,\zeta(3)^2 - \frac{8823}{7700}\,\zeta(2)^5 \qquad (199) \\
\zeta(4,1,3,2) &= \frac{187}{64}\,\zeta(8,2) - \frac{427}{32}\,\zeta(3)\,\zeta(7) - \frac{583}{32}\,\zeta(5)^2 + 13\,\zeta(2)\,\zeta(3)\,\zeta(5) - \frac{9}{4}\,\zeta(2)^2\,\zeta(3)^2 \\
&+ \frac{11441}{7700}\,\zeta(2)^5 \qquad (200) \\
\zeta(4,1,3,1,1) &= \frac{7}{16}\,\zeta(8,2) - \frac{17}{8}\,\zeta(3)\,\zeta(7) + \frac{33}{8}\,\zeta(5)^2 - 5\,\zeta(2)\,\zeta(3)\,\zeta(5) + \frac{3}{2}\,\zeta(2)^2\,\zeta(3)^2 \\
&+ \frac{9689}{46200}\,\zeta(2)^5 \qquad (201) \\
\zeta(4,1,2,3) &= \frac{293}{64}\,\zeta(8,2) - \frac{45}{4}\,\zeta(2)\,\zeta(6,2) - \frac{1383}{32}\,\zeta(3)\,\zeta(7) + \frac{31}{32}\,\zeta(5)^2 + \frac{71}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{9}{5}\,\zeta(2)^2\,\zeta(3)^2 - \frac{54029}{23100}\,\zeta(2)^5 \qquad (202) \\
\zeta(4,1,2,2,1) &= -\frac{7}{8}\,\zeta(8,2) + \frac{9}{2}\,\zeta(2)\,\zeta(6,2) + \frac{141}{4}\,\zeta(3)\,\zeta(7) - \frac{41}{8}\,\zeta(5)^2 - \frac{21}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{24}{5}\,\zeta(2)^2\,\zeta(3)^2 + \frac{11321}{46200}\,\zeta(2)^5 \qquad (203) \\
\zeta(4,1,2,1,2) &= \frac{141}{32}\,\zeta(8,2) - \frac{21}{2}\,\zeta(2)\,\zeta(6,2) - \frac{321}{16}\,\zeta(3)\,\zeta(7) + \frac{219}{16}\,\zeta(5)^2 + 19\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{9}{4}\,\zeta(2)^2\,\zeta(3)^2 - \frac{6052}{1925}\,\zeta(2)^5 \qquad (204) \\
\zeta(4,1,2,1,1,1) &= -\frac{245}{32}\,\zeta(8,2) + \frac{25}{4}\,\zeta(2)\,\zeta(6,2) + \frac{341}{16}\,\zeta(3)\,\zeta(7) + \frac{113}{16}\,\zeta(5)^2 - \frac{31}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{3}{20}\,\zeta(2)^2\,\zeta(3)^2 - \frac{83}{825}\,\zeta(2)^5 \qquad (205) \\
\zeta(4,1,1,3,1) &= -\frac{7}{32}\,\zeta(8,2) - \frac{15}{4}\,\zeta(2)\,\zeta(6,2) - \frac{83}{4}\,\zeta(3)\,\zeta(7) - \frac{33}{16}\,\zeta(5)^2 + \frac{35}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{13}{20}\,\zeta(2)^2\,\zeta(3)^2 - \frac{41977}{46200}\,\zeta(2)^5 \qquad (206) \\
\zeta(4,1,1,2,2) &= -\frac{229}{64}\,\zeta(8,2) + \frac{33}{4}\,\zeta(2)\,\zeta(6,2) + \frac{457}{32}\,\zeta(3)\,\zeta(7) + \frac{29}{32}\,\zeta(5)^2 - 26\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{27}{20}\,\zeta(2)^2\,\zeta(3)^2 + \frac{2831}{1155}\,\zeta(2)^5 \qquad (207) \\
\zeta(4,1,1,2,1,1) &= 7\,\zeta(8,2) - \frac{15}{4}\,\zeta(2)\,\zeta(6,2) - \frac{275}{16}\,\zeta(3)\,\zeta(7) - 11\,\zeta(5)^2 + 12\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{7}{10}\,\zeta(2)^2\,\zeta(3)^2 + \frac{10429}{11550}\,\zeta(2)^5 \qquad (208) \\
\zeta(4,1,1,1,3) &= \frac{35}{16}\,\zeta(8,2) - \frac{3}{4}\,\zeta(2)\,\zeta(6,2) - \frac{49}{8}\,\zeta(3)\,\zeta(7) - \frac{75}{8}\,\zeta(5)^2 + \frac{15}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{7}{5}\,\zeta(2)^2\,\zeta(3)^2 + \frac{2921}{4620}\,\zeta(2)^5 \qquad (209) \\
\zeta(4,1,1,1,2,1) &= -\frac{147}{32}\,\zeta(8,2) - \frac{7}{4}\,\zeta(2)\,\zeta(6,2) + \frac{393}{16}\,\zeta(3)\,\zeta(7) + \frac{315}{16}\,\zeta(5)^2 - 4\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{9}{20}\,\zeta(2)^2\,\zeta(3)^2 - \frac{6539}{1925}\,\zeta(2)^5 \qquad (210) \\
\zeta(4,1,1,1,1,2) &= \frac{275}{64}\,\zeta(8,2) + \zeta(2)\,\zeta(6,2) - \frac{697}{32}\,\zeta(3)\,\zeta(7) - \frac{459}{32}\,\zeta(5)^2 + \zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{23}{20}\,\zeta(2)^2\,\zeta(3)^2 + \frac{901}{308}\,\zeta(2)^5 \qquad (211) \\
\zeta(4,1,1,1,1,1,1) &= -4\,\zeta(3)\,\zeta(7) - 2\,\zeta(5)^2 + \zeta(2)\,\zeta(3)\,\zeta(5) + \frac{1}{5}\,\zeta(2)^2\,\zeta(3)^2 + \frac{666}{1925}\,\zeta(2)^5 \qquad (212) \\
\zeta(3,3,3,1) &= \frac{7}{2}\,\zeta(8,2) - \frac{27}{4}\,\zeta(2)\,\zeta(6,2) - 7\,\zeta(3)\,\zeta(7) + \frac{43}{8}\,\zeta(5)^2 + \frac{27}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&- \frac{317}{154}\,\zeta(2)^5 \qquad (213) \\
\zeta(3,3,2,2) &= -\frac{23}{64}\,\zeta(8,2) - \frac{27}{4}\,\zeta(2)\,\zeta(6,2) + \frac{965}{32}\,\zeta(3)\,\zeta(7) + \frac{1079}{32}\,\zeta(5)^2 - 9\,\zeta(2)\,\zeta(3)\,\zeta(5) \\
&+ \frac{27}{20}\,\zeta(2)^2\,\zeta(3)^2 - \frac{37999}{7700}\,\zeta(2)^5 \qquad (214)
\end{aligned}
$$

$$\begin{aligned}
\zeta(3,\,3,\,2,\,1,\,1) &= -\frac{107}{32}\,\zeta(8,\,2)+\frac{27}{4}\,\zeta(2)\,\zeta(6,\,2)-\frac{39}{2}\,\zeta(3)\,\zeta(7)-\frac{207}{16}\,\zeta(5)^2-6\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ \frac{21}{10}\,\zeta(2)^2\,\zeta(3)^2+\frac{6361}{1848}\,\zeta(2)^5 \qquad (215)\\
\zeta(3,\,3,\,1,\,2,\,1) &= \frac{189}{32}\,\zeta(8,\,2)-\frac{27}{2}\,\zeta(2)\,\zeta(6,\,2)-\frac{189}{16}\,\zeta(3)\,\zeta(7)+\frac{207}{16}\,\zeta(5)^2+27\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- \frac{69793}{15400}\,\zeta(2)^5 \qquad (216)\\
\zeta(3,\,3,\,1,\,1,\,2) &= \frac{85}{32}\,\zeta(8,\,2)+\frac{9}{2}\,\zeta(2)\,\zeta(6,\,2)-\frac{243}{8}\,\zeta(3)\,\zeta(7)-\frac{489}{16}\,\zeta(5)^2+\frac{15}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- \frac{9}{20}\,\zeta(2)^2\,\zeta(3)^2+\frac{8938}{1925}\,\zeta(2)^5 \qquad (217)\\
\zeta(3,\,3,\,1,\,1,\,1,\,1) &= -\frac{119}{64}\,\zeta(8,\,2)+\zeta(2)\,\zeta(6,\,2)+\frac{561}{32}\,\zeta(3)\,\zeta(7)+\frac{151}{32}\,\zeta(5)^2-\frac{15}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- \frac{7}{10}\,\zeta(2)^2\,\zeta(3)^2-\frac{15787}{23100}\,\zeta(2)^5 \qquad (218)\\
\zeta(3,\,2,\,3,\,1,\,1) &= -\frac{7}{8}\,\zeta(8,\,2)+\frac{9}{2}\,\zeta(2)\,\zeta(6,\,2)+\frac{141}{4}\,\zeta(3)\,\zeta(7)-\frac{41}{8}\,\zeta(5)^2-\frac{21}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- \frac{24}{5}\,\zeta(2)^2\,\zeta(3)^2+\frac{11321}{46200}\,\zeta(2)^5 \qquad (219)\\
\zeta(3,\,2,\,2,\,2,\,1) &= -\frac{141}{16}\,\zeta(8,\,2)+\frac{153}{2}\,\zeta(3)\,\zeta(7)+\frac{471}{8}\,\zeta(5)^2-45\,\zeta(2)\,\zeta(3)\,\zeta(5)+\frac{27}{10}\,\zeta(2)^2\,\zeta(3)^2\\
&- \frac{94989}{15400}\,\zeta(2)^5 \qquad (220)\\
\zeta(3,\,2,\,2,\,1,\,2) &= -\frac{9}{64}\,\zeta(8,\,2)+\frac{9}{2}\,\zeta(2)\,\zeta(6,\,2)+\frac{909}{32}\,\zeta(3)\,\zeta(7)+\frac{129}{32}\,\zeta(5)^2-\frac{51}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ \frac{8559}{7700}\,\zeta(2)^5 \qquad (221)\\
\zeta(3,\,2,\,2,\,1,\,1,\,1) &= \frac{477}{64}\,\zeta(8,\,2)-\frac{9}{2}\,\zeta(2)\,\zeta(6,\,2)-\frac{2417}{32}\,\zeta(3)\,\zeta(7)-\frac{1101}{32}\,\zeta(5)^2+35\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ \frac{13}{5}\,\zeta(2)^2\,\zeta(3)^2+\frac{29871}{7700}\,\zeta(2)^5 \qquad (222)\\
\zeta(3,\,2,\,1,\,3,\,1) &= \frac{75}{32}\,\zeta(8,\,2)+\frac{27}{4}\,\zeta(2)\,\zeta(6,\,2)+\frac{279}{16}\,\zeta(3)\,\zeta(7)-\frac{75}{16}\,\zeta(5)^2-27\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ \frac{3}{20}\,\zeta(2)^2\,\zeta(3)^2+\frac{5903}{1848}\,\zeta(2)^5 \qquad (223)\\
\zeta(3,\,2,\,1,\,2,\,2) &= \frac{27}{4}\,\zeta(8,\,2)-\frac{27}{4}\,\zeta(2)\,\zeta(6,\,2)-\frac{1089}{16}\,\zeta(3)\,\zeta(7)-\frac{273}{8}\,\zeta(5)^2+\frac{99}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- \frac{9}{10}\,\zeta(2)^2\,\zeta(3)^2+\frac{547}{308}\,\zeta(2)^5 \qquad (224)\\
\zeta(3,\,2,\,1,\,2,\,1,\,1) &= -\frac{75}{16}\,\zeta(8,\,2)-\frac{13}{4}\,\zeta(2)\,\zeta(6,\,2)+\frac{131}{4}\,\zeta(3)\,\zeta(7)+\frac{259}{8}\,\zeta(5)^2-\frac{25}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ \frac{7}{5}\,\zeta(2)^2\,\zeta(3)^2-\frac{10403}{2310}\,\zeta(2)^5 \qquad (225)\\
\zeta(3,\,2,\,1,\,1,\,2,\,1) &= \frac{281}{64}\,\zeta(8,\,2)+10\,\zeta(2)\,\zeta(6,\,2)-\frac{1963}{32}\,\zeta(3)\,\zeta(7)-\frac{1645}{32}\,\zeta(5)^2+\frac{13}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ \frac{13}{10}\,\zeta(2)^2\,\zeta(3)^2+\frac{19339}{2100}\,\zeta(2)^5 \qquad (226)\\
\zeta(3,\,2,\,1,\,1,\,1,\,2) &= -11\,\zeta(8,\,2)+\zeta(2)\,\zeta(6,\,2)+66\,\zeta(3)\,\zeta(7)+\frac{275}{8}\,\zeta(5)^2-15\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- \frac{31}{10}\,\zeta(2)^2\,\zeta(3)^2-\frac{11839}{1925}\,\zeta(2)^5 \qquad (227)\\
\zeta(3,\,2,\,1,\,1,\,1,\,1,\,1) &= \frac{9}{4}\,\zeta(8,\,2)-\zeta(2)\,\zeta(6,\,2)+\frac{9}{2}\,\zeta(3)\,\zeta(7)+\frac{9}{4}\,\zeta(5)^2-\frac{2}{5}\,\zeta(2)^2\,\zeta(3)^2-\frac{3016}{5775}\,\zeta(2)^5 \qquad (228)\\
\zeta(3,\,1,\,3,\,1,\,2) &= -\frac{75}{64}\,\zeta(8,\,2)+\frac{99}{32}\,\zeta(3)\,\zeta(7)+\frac{75}{32}\,\zeta(5)^2-\frac{3}{20}\,\zeta(2)^2\,\zeta(3)^2-\frac{10891}{23100}\,\zeta(2)^5 \qquad (229)\\
\zeta(3,\,1,\,3,\,1,\,1,\,1) &= \frac{7}{32}\,\zeta(8,\,2)-\frac{5}{2}\,\zeta(2)\,\zeta(6,\,2)-\frac{129}{16}\,\zeta(3)\,\zeta(7)+\frac{1}{16}\,\zeta(5)^2+\frac{19}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- \frac{443}{550}\,\zeta(2)^5 \qquad (230)\\
\zeta(3,\,1,\,2,\,2,\,2) &= \frac{141}{32}\,\zeta(8,\,2)-\frac{153}{4}\,\zeta(3)\,\zeta(7)-\frac{471}{16}\,\zeta(5)^2+\frac{45}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)-\frac{27}{20}\,\zeta(2)^2\,\zeta(3)^2
\end{aligned}$$

$$
\begin{aligned}
&+ && \frac{5937}{1925}\,\zeta(2)^5 && (231)\\
\zeta(3,1,2,2,1,1) &= && -\frac{21}{8}\,\zeta(8,2)+10\,\zeta(2)\,\zeta(6,2)+\frac{991}{16}\,\zeta(3)\,\zeta(7)+\frac{43}{4}\,\zeta(5)^2-\frac{91}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- && \frac{21}{10}\,\zeta(2)^2\,\zeta(3)^2+\frac{6994}{5775}\,\zeta(2)^5 && (232)\\
\zeta(3,1,2,1,2,1) &= && \frac{7}{2}\,\zeta(8,2)-\frac{27}{4}\,\zeta(2)\,\zeta(6,2)-7\,\zeta(3)\,\zeta(7)+\frac{43}{8}\,\zeta(5)^2+\frac{27}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- && \frac{317}{154}\,\zeta(2)^5 && (233)\\
\zeta(3,1,2,1,1,2) &= && \frac{359}{64}\,\zeta(8,2)-\frac{25}{2}\,\zeta(2)\,\zeta(6,2)-\frac{629}{32}\,\zeta(3)\,\zeta(7)+\frac{197}{32}\,\zeta(5)^2+29\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ && \frac{1}{4}\,\zeta(2)^2\,\zeta(3)^2-\frac{82261}{23100}\,\zeta(2)^5 && (234)\\
\zeta(3,1,2,1,1,1,1) &= && -\frac{35}{4}\,\zeta(8,2)+\frac{5}{2}\,\zeta(2)\,\zeta(6,2)+\frac{29}{2}\,\zeta(3)\,\zeta(7)+\frac{37}{4}\,\zeta(5)^2-4\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ && \frac{1}{5}\,\zeta(2)^2\,\zeta(3)^2-\frac{3219}{1925}\,\zeta(2)^5 && (235)\\
\zeta(3,1,1,2,2,1) &= && \frac{7}{8}\,\zeta(8,2)-\frac{11}{2}\,\zeta(2)\,\zeta(6,2)-\frac{189}{4}\,\zeta(3)\,\zeta(7)-21\,\zeta(5)^2+\frac{85}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- && \frac{6}{5}\,\zeta(2)^2\,\zeta(3)^2-\frac{167}{825}\,\zeta(2)^5 && (236)\\
\zeta(3,1,1,2,1,2) &= && -\frac{73}{16}\,\zeta(8,2)+10\,\zeta(2)\,\zeta(6,2)+\frac{107}{8}\,\zeta(3)\,\zeta(7)-\frac{21}{8}\,\zeta(5)^2-\frac{49}{2}\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ && \frac{1}{2}\,\zeta(2)^2\,\zeta(3)^2+\frac{503}{175}\,\zeta(2)^5 && (237)\\
\zeta(3,1,1,2,1,1,1) &= && \frac{21}{2}\,\zeta(8,2)-10\,\zeta(3)\,\zeta(7)-\frac{27}{2}\,\zeta(5)^2-5\,\zeta(2)\,\zeta(3)\,\zeta(5)-\frac{2}{5}\,\zeta(2)^2\,\zeta(3)^2\\
&+ && \frac{18418}{5775}\,\zeta(2)^5 && (238)\\
\zeta(3,1,1,1,2,2) &= && -\frac{173}{64}\,\zeta(8,2)+\frac{5}{2}\,\zeta(2)\,\zeta(6,2)+\frac{1191}{32}\,\zeta(3)\,\zeta(7)+\frac{677}{32}\,\zeta(5)^2-27\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ && \frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2-\frac{1433}{1100}\,\zeta(2)^5 && (239)\\
\zeta(3,1,1,1,2,1,1) &= && -7\,\zeta(8,2)-\frac{5}{2}\,\zeta(2)\,\zeta(6,2)-3\,\zeta(3)\,\zeta(7)+9\,\zeta(5)^2+15\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- && \frac{1173}{385}\,\zeta(2)^5 && (240)\\
\zeta(3,1,1,1,1,2,1) &= && \frac{7}{4}\,\zeta(8,2)+\zeta(2)\,\zeta(6,2)+\frac{35}{2}\,\zeta(3)\,\zeta(7)+\frac{21}{4}\,\zeta(5)^2-13\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&- && \frac{2}{5}\,\zeta(2)^2\,\zeta(3)^2+\frac{212}{1925}\,\zeta(2)^5 && (241)\\
\zeta(3,1,1,1,1,1,2) &= && \frac{9}{4}\,\zeta(8,2)-\frac{41}{2}\,\zeta(3)\,\zeta(7)-\frac{43}{4}\,\zeta(5)^2+6\,\zeta(2)\,\zeta(3)\,\zeta(5)+\frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2\\
&+ && \frac{3352}{1925}\,\zeta(2)^5 && (242)\\
\zeta(3,1,1,1,1,1,1,1) &= && -\zeta(3)\,\zeta(7)-\frac{1}{2}\,\zeta(5)^2+\frac{8}{55}\,\zeta(2)^5 && (243)\\
\zeta(2,2,2,2,1,1) &= && -\frac{141}{32}\,\zeta(8,2)+\frac{9}{2}\,\zeta(2)\,\zeta(6,2)+\frac{455}{16}\,\zeta(3)\,\zeta(7)+\frac{229}{16}\,\zeta(5)^2-24\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ && \frac{9}{10}\,\zeta(2)^2\,\zeta(3)^2-\frac{1369}{3850}\,\zeta(2)^5 && (244)\\
\zeta(2,2,2,1,2,1) &= && -\frac{23}{64}\,\zeta(8,2)-\frac{27}{4}\,\zeta(2)\,\zeta(6,2)+\frac{965}{32}\,\zeta(3)\,\zeta(7)+\frac{1079}{32}\,\zeta(5)^2-9\,\zeta(2)\,\zeta(3)\,\zeta(5)\\
&+ && \frac{27}{20}\,\zeta(2)^2\,\zeta(3)^2-\frac{37999}{7700}\,\zeta(2)^5 && (245)\\
\zeta(2,2,2,1,1,1,1) &= && 3\,\zeta(8,2)+\zeta(2)\,\zeta(6,2)-28\,\zeta(3)\,\zeta(7)-18\,\zeta(5)^2+8\,\zeta(2)\,\zeta(3)\,\zeta(5)+\frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2\\
&+ && \frac{14}{5}\,\zeta(2)^5 && (246)\\
\zeta(2,2,1,2,1,1,1) &= && \frac{19}{4}\,\zeta(8,2)-\frac{25}{2}\,\zeta(2)\,\zeta(6,2)-\frac{63}{2}\,\zeta(3)\,\zeta(7)-\frac{3}{4}\,\zeta(5)^2+35\,\zeta(2)\,\zeta(3)\,\zeta(5)
\end{aligned}
$$

$$+\ \frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2-\frac{5683}{1925}\,\zeta(2)^5 \tag{247}$$

$$\zeta(2,\,2,\,1,\,1,\,2,\,1,\,1) = -\frac{15}{2}\,\zeta(8,\,2)+10\,\zeta(2)\,\zeta(6,\,2)+49\,\zeta(3)\,\zeta(7)+15\,\zeta(5)^2-40\,\zeta(2)\,\zeta(3)\,\zeta(5)$$

$$+\ \frac{996}{1925}\,\zeta(2)^5 \tag{248}$$

$$\zeta(2,\,2,\,1,\,1,\,1,\,2,\,1) = \frac{13}{2}\,\zeta(8,\,2)+\frac{5}{2}\,\zeta(2)\,\zeta(6,\,2)-60\,\zeta(3)\,\zeta(7)-\frac{81}{2}\,\zeta(5)^2+20\,\zeta(2)\,\zeta(3)\,\zeta(5)$$

$$+\ \frac{4}{5}\,\zeta(2)^2\,\zeta(3)^2+\frac{329}{55}\,\zeta(2)^5 \tag{249}$$

$$\zeta(2,\,2,\,1,\,1,\,1,\,1,\,1,\,1) = \zeta(8,\,2) \tag{250}$$

$$\zeta(2,\,1,\,2,\,1,\,2,\,1,\,1) = \frac{7}{2}\,\zeta(8,\,2)+10\,\zeta(2)\,\zeta(6,\,2)-7\,\zeta(3)\,\zeta(7)-\frac{59}{2}\,\zeta(5)^2-20\,\zeta(2)\,\zeta(3)\,\zeta(5)$$

$$+\ \frac{12932}{1925}\,\zeta(2)^5 \tag{251}$$

$$\zeta(2,\,1,\,2,\,1,\,1,\,1,\,1,\,1) = -\frac{7}{2}\,\zeta(8,\,2)+7\,\zeta(3)\,\zeta(7)+4\,\zeta(5)^2-\frac{408}{385}\,\zeta(2)^5 \tag{252}$$

$$\zeta(2,\,1,\,1,\,2,\,1,\,1,\,1,\,1) = \frac{7}{2}\,\zeta(8,\,2)-7\,\zeta(3)\,\zeta(7)-5\,\zeta(5)^2+\frac{2216}{1925}\,\zeta(2)^5 \tag{253}$$

$$\zeta(2,\,1,\,1,\,1,\,1,\,1,\,1,\,1,\,1) = \frac{32}{385}\,\zeta(2)^5 \tag{254}$$

References

[1] V.I. Arnold. The Vassiliev theory of discriminants and knots. In *First European Congress of Mathematics*, volume 1, pages 3–29. Birkhauser, 1994.

[2] D. Borwein, J.M. Borwein, and R. Girgensohn. Explicit evaluation of Euler sums. In *Proc. Edin. Math. Soc.*, volume 38, pages 277–294, 1995.

[3] J.M. Borwein, D.M. Bradley, and D.J. Broadhurst. Evaluation of k-fold Euler/Zagier sums : a compendium of results for arbitrary k.

[4] J.M. Borwein and R. Girgensohn. Evaluation of triple Euler sums. *Electronic J. Combinatorics*, 3, 1996.

[5] Ph. Flajolet, G. Labelle L. Laforest, and B. Salvy. Hypergeometrics and the cost structure of quadtree. *Ramdom Structures and Algorithms*, 7:117–143, 1995.

[6] Ph. Flajolet and B. Salvy. Euler sums and contour integral representations. Technical report, Rapport technique, INRIA Rocquencourt, 1996.

[7] M. Fliess. Fonctionnelles causales non linéaires et indéterminées non commutatives. *Bull. Soc. Math. France*, 109:3–40, 1981.

[8] I.M. Guelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, and J.Y.Thibon. Noncommutative symmetric functions. *Adv. Math.*, 112:218–348, 1995.

[9] M. Hoffman. Multiple harmonic series. *Pacific Journal of Mathematics*, 152(2):275–290, 1995.

[10] M. Hoffman. The algebra of multiple harmonic series. *Journal of Algebra*, August 1997.

[11] C. Markett. Triple sums and the Riemann zeta function. *Journal of Number Theory*, 48:113–132, 1994.

[12] Hoang Ngoc Minh. Polylogarithms & evaluation transform. In *"IMACS Symposium"*, Lille, June 1993.

[13] Hoang Ngoc Minh. Intégration symbolique des équations différentielles linéaires. In *FPSAC'96*, Minneapolis, June 1996.

[14] Hoang Ngoc Minh. Summations of polylogarithms via evaluation transform. *Mathematics and Computers in Simulation*, 1336, 1996.

[15] N. Nielsen. Recherches sur des généralisations d'une fonction de Legendre et d'Abel. *Annali di Matematica*, 9:219–235, 1904.

[16] N. Nielsen. Recherches sur le carré de la dérivée logarithmique de la fonction gamma et sur quelques fonctions analogues. *Annali di Matematica*, 9:190–210, 1904.

[17] M. Petitot. *Algèbre non commutative en Scratchpad. Application au problème de la réalisation minimale analytique.* Thèse de Doctorat, Université Lille I, Janvier 1992.

[18] D.E. Radford. A natural ring basis for the shuffle algebra and an application to group schemes. *Journal of Algebra*, 58:432–454, 1979.

[19] C. Reutenauer. *Free Lie Algebras*, volume New Series-7 of *London Mathematical Society Monographs*. Oxford Science Publications, 1993.

[20] D. Zagier. Values of zeta functions and their applications. In *First European Congress of Mathematics*, volume 2, pages 179–512. Birkhauser, 1994.