Doubly alternating Baxter permutations are Catalan

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Abstract

The Baxter permutations who are alternating and whose inverse is also alternating are shown to be enumerated by the Catalan numbers. A bijection to complete binary trees is also given.

Résumé

Nous montrons que les permutations de Baxter alternantes dont l'inverse est également une permutation de Baxter alternante sont énumérées par les nombres de Catalan. De plus, nous donnons une bijection avec les arbres binaires complets.

1 Introduction

Baxter [Ba] studied a particular class of permutations by studying fixed points of the composite of commuting functions. These permutations can be defined by forbidden subsequences [Gi, Gu, Kn, SS, We]. In fact, the **Baxter permutations** are exactly the permutations π of length n which verify the two following conditions:

for all $1 \leq i < j < k < l \leq n$,

$$\text{if } \pi_i + 1 = \pi_l \text{ and } \pi_j > \pi_l \text{ then } \pi_k > \pi_l,$$
$$\text{if } \pi_l + 1 = \pi_i \text{ and } \pi_k > \pi_i \text{ then } \pi_j > \pi_i.$$

For example, 2413 and 3142 are the only permutations on four elements which are not Baxter permutations.

Chung, Graham, Hoggatt and Kleiman [CGHK] enumerated analytically the Baxter permutations of length n by the formula $\sum_{r=0}^{n-1}[\binom{n+1}{r}.\binom{n+1}{r+1}.\binom{n+1}{r+2}]/[\binom{n+1}{1}.\binom{n+1}{2}]$. Later, Mallows [Ma] found a more precise formula whereas Viennot [Vi] (see also [DG1]) gave a combinatorial proof of the formula obtained by Chung and al..

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A permutation is said to be **alternating** if the permutation starts with a rise and then descents and rises come in turn. More precisely, an alternating permutation of length n is such that $\pi(2i-1) < \pi(2i) > \pi(2i+1)$ for $1 \leq i \leq \lfloor n/2 \rfloor$ that is to say its rise (respectively descent) set denoted $Rise(\pi)$ (respectively $Des(\pi)$) is equal to odd (respectively even) index.
Cori, Dulucq and Viennot [CDV] established a one-to-one correspondence which proved that alternating Baxter permutations of length $2n$ and $2n+1$ are enumerated by $c_n.c_n$ and $c_{n+1}.c_n$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

More recently, Dulucq and Guibert [DG2] found a new bijection which unifies [Vi] and [CDV] and gave a combinatorial interpretation of Mallows's formula [Ma].

Considering the definition, it is clear that if π is a Baxter permutation of length n then

- its **mirror** π^* defined by $\pi^*(i) = \pi(n+1-i) = i$ for $1 \leq i \leq n$,
- its **complement** π^c defined by $\pi^c(i) = n+1-\pi(i) = i$ for $1 \leq i \leq n$, are also
- its **inverse** π^{-1} defined by $\pi^{-1}(i) = j \Longleftrightarrow \pi(j) = i$ for $1 \leq i \leq n$

Baxter permutations.

However, one can find an alternating Baxter permutation whose inverse is not alternating. A permutation is said to be **doubly alternating** if it is alternating and its inverse is alternating.

We will use the remark

Remark 1 *It does not make any difference if we define alternating to start with a rise or with a descent.*

Proof Indeed, the mirror permutation and the complement permutation of an alternating Baxter permutation are also alternating Baxter permutations. □

Example 2 *The permutation* 5 7 6 15 13 14 9 11 10 12 8 17 16 18 1 3 2 4 *is a doubly alternating Baxter permutation of length* 18.

The purpose of this note is to establish the following.

Theorem 3 (Main Theorem) *The number of doubly alternating Baxter permutations of length $2n+\epsilon$ where $\epsilon = 0$ or 1 denoted $d_{2n+\epsilon}$ is the Catalan number*

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Note that we have $d_0 = 1$ by considering the empty permutation.

Now we consider another way to regard permutations: the **essential set** of a permutation with its **rank function**, introduced by Fulton [Fu] and further studied by Eriksson and Linusson [EL1, EL2].

First, let every permutation π of length n be represented by its dotted permutation matrix, regarded as an $n \times n$-collection of squares in the plane, where square $(i, \pi(i))$ in row i and column $\pi(i)$ has a dot for all $1 \leq i \leq n$.

We then get the diagram of the permutation by shading the square containing the dot, the squares in that row east of the dot, and the squares in each column south of the dot.

We call a white (nonshaded) square a **white corner** if the squares directly east and south of it are shaded.

The essential set of a permutation π is defined to be the set of white corners of the diagram of π. For every white corner of π, its rank is defined by the number of dots northwest of the square.

Eriksson and Linusson have characterized the essential sets of Baxter permutations.

Proposition 4 *([EL2]) A permutation is Baxter if and only if its essential set has at most one white corner in each column and row.*

Clearly, the essential set of a permutation has a white corner in row i if and only if the permutation has a descent in position i and the essential set has a white corner in column i if and only if the inverse permutation has a descent in position i. Hence we get the following corollary which is the key to the proof of Theorem 3.

Corollary 5 *A permutation is a doubly alternating Baxter permutations if and only if its essential set has exactly one square in each even row that is not the last row and in each even column that is not the last column.*

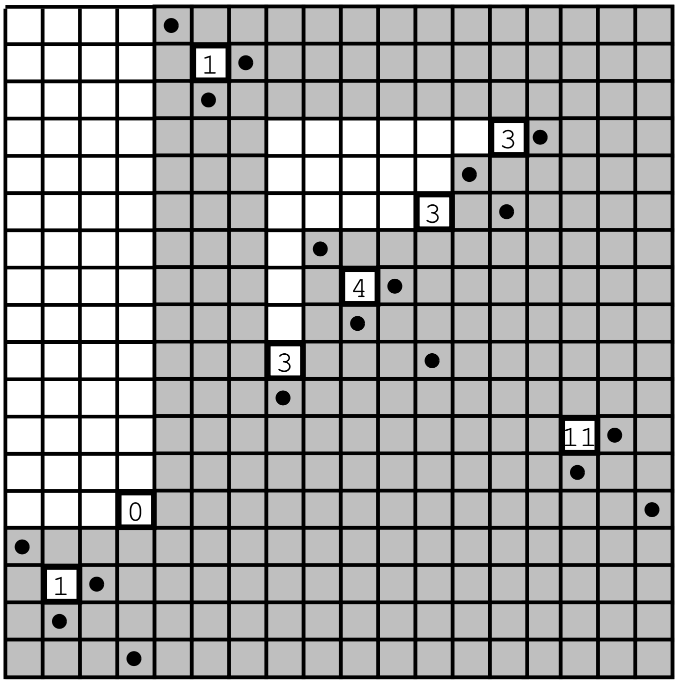

Figure 1: The essential set (and the ranks of the white corners) of the doubly alternating Baxter permutation 5 7 6 15 13 14 9 11 10 12 8 17 16 18 1 3 2 4.

Example 6 *Figure 1 shows the essential set of the doubly alternating Baxter permutation* 5 7 6 15 13 14 9 11 10 12 8 17 16 18 1 3 2 4*.*

2 Proof

Note that the construction of the essential set works as well for a dot configuration that is not of full rank. First a technical Lemma about a rectangular part R of the matrix of a doubly alternating Baxter permutation.

Lemma 7 *Let R be a rectangle with $2n+1$ rows and $s \geq 2n+1$ columns. Assume that R has one dot $\pi(j)$ on each row j such that π is alternating,that is R has exactly one white corner in the rows $\{1,3,5,\ldots,2n-1\}$ and none in the other rows. Assume also that if R has a dot in an odd column (different from $2n+1$) then there is also a white corner in that column, and that there are no white corners in the even columns.*

Then $\pi(j) \leq 2n$ for each $1 \leq j \leq 2n$ and $\pi(2n+1) = 2n+1$.

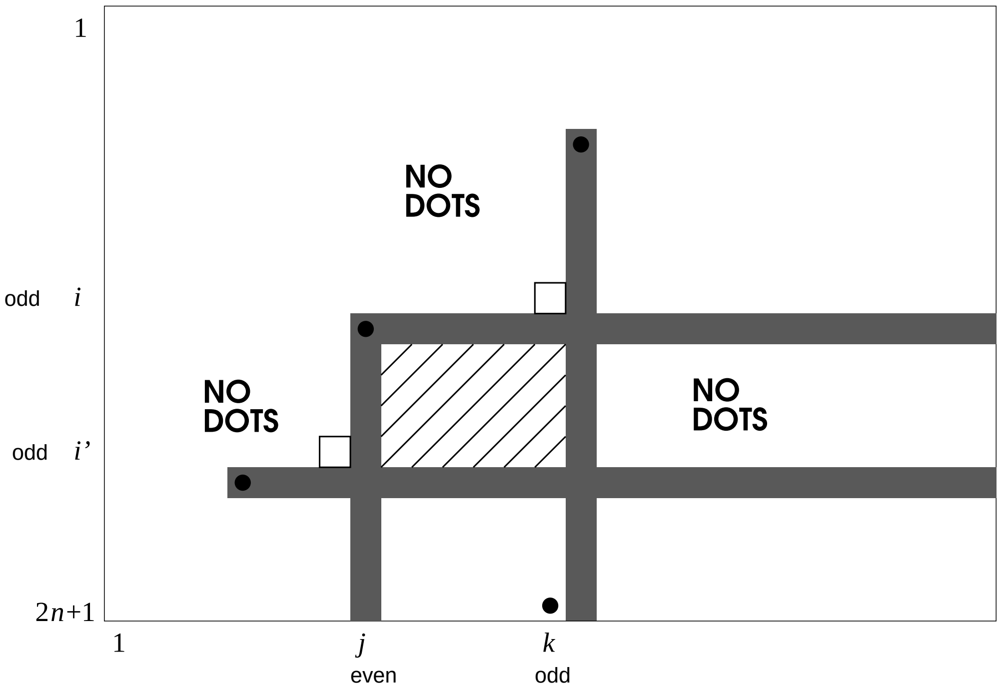

Figure 2: The slightly striped rectangle has the same properties as the entire rectangle in Lemma 7.

Proof The proof will be done by induction on n. It is trivial for $n = 0$. For the reasoning below it will help to look at Figure 2. Let $k := \pi(2n+1)$. Since we do not have a white corner on row $2n+1$, we know that R has dots in all the columns $1,\ldots,k-1$, therefore it suffices to show $k = 2n+1$.

Assume that $k < 2n+1$. If k was even then there could not be a white corner in column $k-1$ so k is odd and we have a white corner (i,k) with i odd (i odd implies directly that $k > 1$). We have $j := \pi(i+1) < k$, because (i,k) is a white corner. There can be no dots in the rectangle $[i+1, 2n+1] \times [k+1, s]$, since a dot there would imply a white square and hence a white corner in column k below row $i+1$ contradicting

Baxter. There can also be no dots in $[1,i]\times[j+1,k]$, since otherwise there would be a second white corner on row i, a contradiction. The latter implies that j has to be even.

Now, let $(i',j-1)$ be the white corner in row $j-1$, where i' is odd, $2n+1>i'>i$. There are no dots in $[i+2,i']\times[1,j-1]$, since such would cause a second white corner in row $j-1$. Hence the rectangle $R':=[i+2,i']\times[j+1,k-1]$, striped in Figure 2, contains a dot in every row. It fulfills also all the other conditions in the lemma. By induction we get that $\pi(i')=j+i'-i-1\leq k-1$ is the last of the dots in R'. Since $j+i'-i-1$ is odd, we should have a white corner in that column, which is impossible, a contradiction. $\square$

Corollary 8 *If π is a Baxter permutation of length $2n+1$ with $Des(\pi)=Des(\pi^{-1})=\{1,3,5,\ldots,2n-1\}$, then $\pi(2n+1)=2n+1$. In particular, we have $d_{2n}=d_{2n+1}$.*

Proof Follows directly from Lemma 7. $\square$

Lemma 9 *If π is a Baxter permutation of length $2n$ with $Des(\pi)=Des(\pi^{-1})=\{2,4,6,\ldots,2n-2\}$, then*

$$\pi(1)=2k+1 \implies \pi(2n-2k)=2n.$$

Proof Let $(i,2k)$ be the white corner in column $2k$, i even. This gives that there are no dots in $[1,i]\times[1,2k]$. So the rectangle $[2,i]\times[2k+2,2n]$ fulfills the conditions of Lemma 7 which means that $\pi(i)=2k+1+i-1$, an even number. There are no white squares above the dot in row i and hence it must be in the last column. This means $2k+i=2n$ and the lemma is proved. $\square$

Proof of Theorem 3 (Main Theorem) We will count the number of Baxter permutations of length $2n$ with $Des(\pi)=Des(\pi^{-1})=\{2,4,6,\ldots,2n-2\}$. From the conditions on π^{-1} it follows that $\pi(1)$ is odd, say $2k+1$. From Lemma 9 we get $\pi(2n-2k)=2n$ which implies that there are dots neither in $[1,2n-2k]\times[1,2k]$ nor in $[2n-2k+1,2n]\times[2k+1,2n]$. We have that $[2,2n-2k-1]\times[2k+2,2n-1]$ is isomorphic to any matrix of a Baxter permutation of length $2n-2k-2$ with descents in odd rows and columns; and we have that $[2n-2k+1,2n]\times[1,2k]$ is isomorphic to any matrix of a Baxter permutation of length $2k$ with descents in even rows and columns. Using remark 1, we can study permutations both with descents in all odd or descents in all even positions, and we get the recursion

$$d_{2n}=\sum_{k=0}^{n-1} d_{2n-2k-2}\cdot d_{2k}.$$

By Corollary 8 this is the well-known recursion for the Catalan numbers $c_n=\sum_{k=0}^{n-1} c_{n-k-1}\cdot c_k$. $\square$

3 Example of bijection

Now we exhibit a bijection for doubly alternating Baxter permutations which can be regarded as a specialization of the correspondence of Dulucq and Guibert [DG2].

Lemma 10 *There is a bijection between doubly alternating Baxter permutations of length $2n$ and complete binary trees with $2n+1$ nodes.*

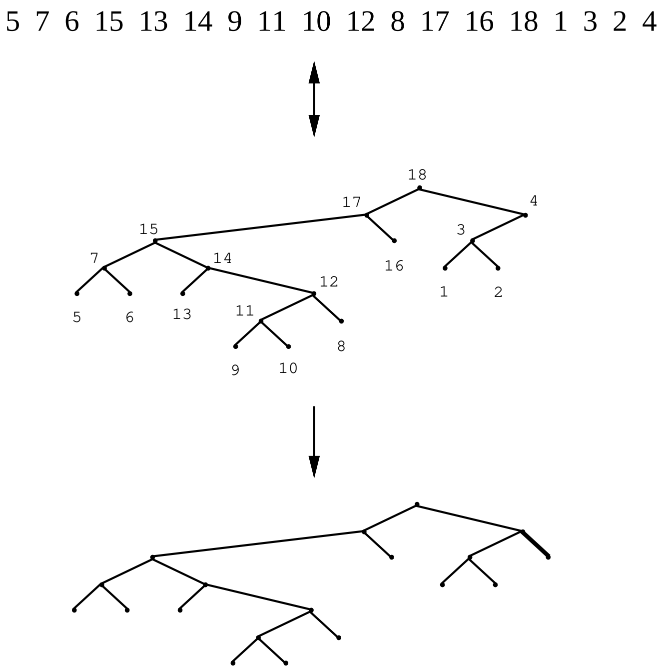

Figure 3: The mapping of the bijection: from a doubly alternating Baxter permutation to a complete binary tree.

The mapping and its inverse can be defined in the following way.

- The mapping (an example is given by Figure 3) consists in first building the decreasing complete binary tree of the doubly alternating Baxter permutation and then forgetting its labels.
  We briefly recall the correspondence between alternating permutations and decreasing complete binary trees:
  $$decr(w) = \lhd decr(u), x, decr(v) \rhd$$
  where $w = uxv$ and $x = \max\{w_i : w = w_1 w_2 \dots w_p\}$.

- The inverse mapping (an example is given by Figure 4) consists in labeling a complete binary tree with $2n+1$ nodes, in decreasing order. The algorithm given in Figure 5 is called with the complete binary tree where its rightmost leaf is omitted (the tree is said to be right-truncated) and with the set of consecutive labels $\{1, 2, \dots, 2n\}$.

Thus, we recursively obtain a decreasing labeling of the initial complete binary tree. The doubly alternating Baxter permutation is obtained by projecting in infix order this labeling.

The mapping and its inverse are well defined by considering section 2.

Thus, we obtain the following results.

Corollary 11 *The number of doubly alternating Baxter permutations of length* $2n+\epsilon$ *where* $\epsilon = 0$ *or* 1 *such that* $\pi(2k) = 2n+\epsilon$ *where* $1 \leq k \leq n$ *is*

$$c_{k-1} \cdot c_{n-k} \textit{ where } c_i = \frac{1}{i+1}\binom{2i}{i}.$$

PROOF We immediatly deduce this result from the recursive decomposition (also given in section 2). □

Corollary 12 *The number of doubly alternating Baxter permutations of length* $2n+\epsilon$ *where* $\epsilon = 0$ *or* 1 *having* k *left-to-right maxima is the ballot number*

$$\binom{2n-k}{n-1} - \binom{2n-k}{n}.$$

PROOF We deduce this result from the enumeration of the complete binary tree according to the number of nodes and the length of the left main branch, called ballot numbers or Delannoy numbers [Er]. □

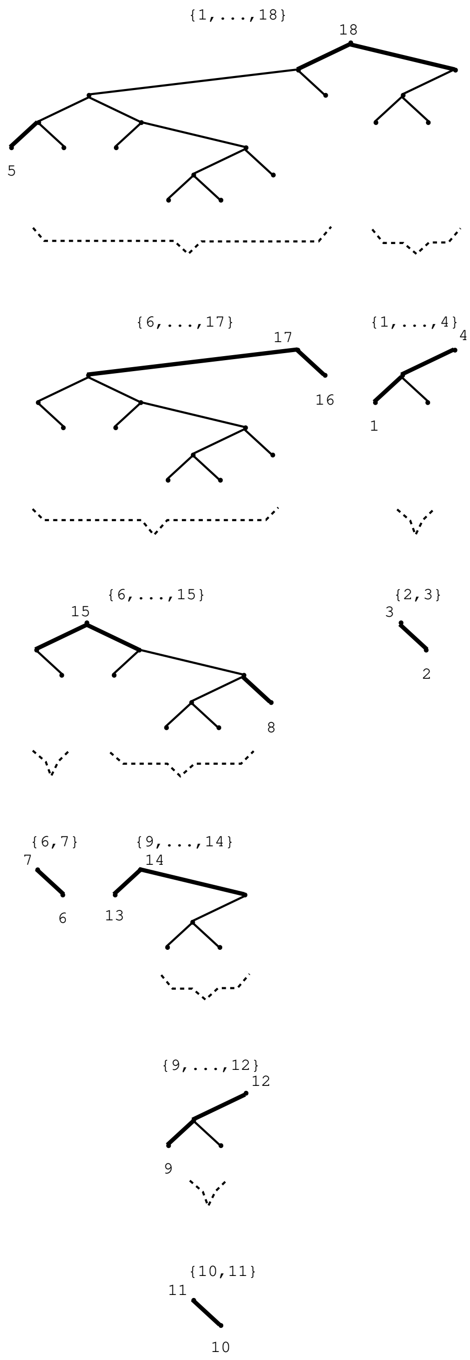

Figure 4: The inverse mapping of the bijection: labeling a complete binary tree in decreasing order.

Algorithm for a right-truncated tree with the set of labels $\{2i+1, \ldots, 2j\}$.

- Label $2j$ the root.
- Let $2k-1$ be the number of nodes of the left subtree.
- Label $2(j-k)+1$ the leftmost leaf of the tree.
- Call the algorithm for the left-truncated tree (given in Figure 6) with the left subtree without its leftmost leaf, and the labels $\{2(j-k)+2, \ldots, 2j-1\}$ as input.
- Call the algorithm for the right-truncated tree with the right subtree and labels $\{2i+1, \ldots, 2(j-k)\}$ as input.

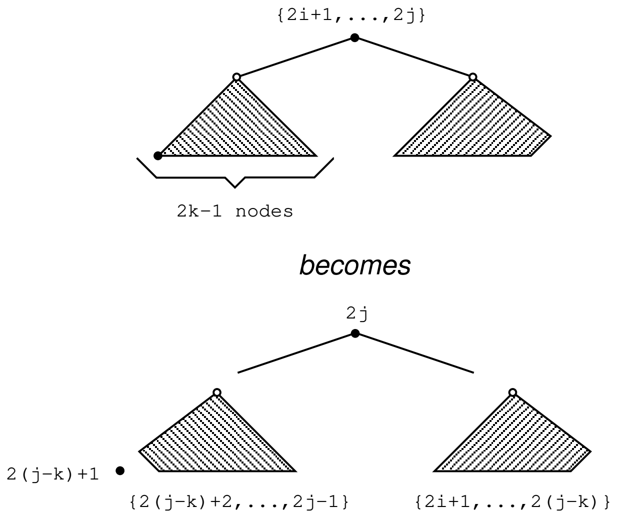

Figure 5: Algorithm for a right-truncated (complete binary) tree with the set of consecutive labels $\{2i+1, \ldots, 2j\}$.

Algorithm for a left-truncated tree with the set of labels $\{2i+2, \ldots, 2j+1\}$.

- Label $2j+1$ the root.
- Let $2k-1$ be the number of nodes of the right subtree.
- Label $2(j-k+1)$ the rightmost leaf of the tree.
- Call the algorithm for the left-truncated tree with the left subtree and labels $\{2i+2, \ldots, 2(j-k)+1\}$ as input.
- Call the algorithm for the right-truncated tree with the right subtree without its rightmost leaf, and the labels $\{2(j-k+1)+1, \ldots, 2j\}$ as input.

Figure 6: Algorithm for a left-truncated (complete binary) tree with the set of consecutive labels $\{2i+2, \ldots, 2j+1\}$.

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