Left ternary trees and non-separable rooted planar maps

Alberto Del Lungo*, Francesco Del Ristoro*, Jean-Guy Penaud†

Abstract

In this paper, we illustrate a bijective proof of the enumerative formula regarding non-separable rooted planar maps $\mathcal{NS}$, by means of a class $\mathcal{L}$ of certain ternary trees (called left trees). Our first step consists in determining the left trees' combinatorial enumeration according to the number of their internal nodes. We then establish a bijection between the left trees having n internal nodes and the non-separable rooted planar maps with $n+1$ edges. We wish to point out that in the bijection, $\mathcal{L}$ and $\mathcal{NS}$ have many corresponding parameters to each other.

keywords: non-separable rooted planar maps, left trees, bijective proof, overturning operation.

1 Introduction

A *Planar Map* is the embedding of a connected graph in the surface of a sphere, which divides the surface up into some simply-connected regions called *faces*. A map is called *separable* if its edge-set can be partitioned into two disjoint non-null subsets S and T so that there is just one vertex v incident with both a member of S and a menmber of T. We call v a *cut-vertex* of the map. A map is *rooted* when one of its vertices is chosen as the *root vertex*; an edge that is incident to the root vertex is called a *root edge*. This is represented graphically by an arrow on the root edge that points to the root vertex. The face on the root edge's right is the *exterior face*. A planar map is represented by a stereographic projection of the sphere mapped from a point interior to a face to the point of infinity. Figure 1 (a) illustrates a non-separable rooted planar map. We denote the set of the all non-separable rooted planar maps having at least 2 edges by $\mathcal{NS}$.

Brown [2] and Tutte [10] enumerated non-separable rooted planar maps, by means of recursive decomposition and the quadratic method used for solving the occurring equation. See [3, 6] for a thorough treatment of planar map enumeration. The number of nonseparable rooted planar maps with $(n+1)$ edges is:

$$|\mathcal{NS}_{n+1}| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}. \tag{1.1}$$

Cori [5], proposed the problem of determining a bijective proof of the enumerative formula, during the conference entitled "Formal Power Series and Algebraic Combinatorics", held in Florence in 1993.

In this paper, we describe a bijective proof of the enumerative formula for non-separable rooted planar maps. We obtained the proof by means of a certain ternary trees called *left trees* (see figure 1 (b)). The main steps of our proof are the following:

*Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, Firenze, Italy,[dellungo,fdr]@dsi2.ing.unifi.it
†LaBRI, Université de Bordeaux I, 33405 Talence Cedex, France,penaud@labri.u-bordeaux.fr

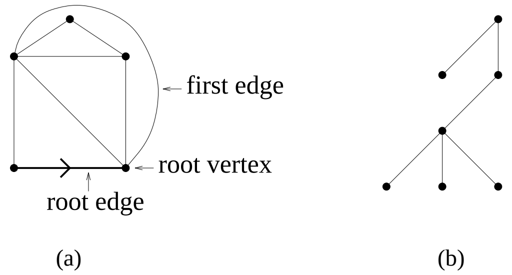

Figure 1: A nonseparable rooted planar map and a left tree

1. the combinatorial enumeration of left trees having n internal nodes;
2. the determination of a bijection between the left trees having n internal nodes and the non-separable rooted planar maps having $n+1$ edges.

The first step consists in making a bijection between the non-left trees having n internal nodes and the class of ternary tree pairs (T_1, T_2) such that the sum of T_1's and T_2's internal nodes is n. The enumerative left trees' formula follows from this.
The second step consists in using a variant of Brown's [2, 7] combinatorial decomposition method for nonseparable rooted planar maps. We determine a combinatorial decomposition for left trees which is analogous to the above mentioned map decomposition. The bijection is a result of the decompositions. We wish to point out that Jacquard and Schaeffer [8] discovered another bijection between our class of left trees and non-separable rooted planar maps.

2 The combinatorial proof

After some preliminary remarks, in section 2.1 we go on to describe left tree combinatorial enumeration, while in section 2.2 we treat the bijection between left trees and non-separable rooted planar maps.
In the non-separable rooted planar map enumerative formula (1.1) we find the term $\binom{3n}{n}/(2n+1)$, that represents the cardinality of the set of ternary trees $\mathcal{T}$ having n nodes. A *ternary tree* is a finite set of nodes which is either empty, or contains a root and three disjoint ternary trees, called the root's left, middle and right subtrees (see [9]). We can write Brown and Tutte's result in the following way:

$$|\mathcal{NS}_{n+1}| = \frac{2}{n+1}|\mathcal{T}_n|.$$

This suggests the possible existence of a ternary tree subclass enumerated by $|\mathcal{NS}_{n+1}|$. This subclass exists and is the set of *left trees*. For each node x of a ternary tree T, there is a path $\gamma(x)$ from T's root to x. The path $\gamma(x)$ is made up of three steps, called *left*, *middle* and *right* (see figure 2 (a)). The *abscissa* of x is the difference between the number of $\gamma(x)$'s left and right steps. We denote x's abscissa by $\alpha(x)$. A left tree is a ternary tree whose nodes have a positive or zero abscissa ($\alpha(x) \geq 0$, see figure 2 (b)). We denote the left trees' set by $\mathcal{L}$ and use the ECO method [1] for determining a recursive construction of them. We proceed as follows: let us consider the set $\mathcal{L}_n$ of the left trees having n nodes. We define an operator ϑ able to construct each left tree $T \in \mathcal{L}_{n+1}$ from another left tree $T' \in \mathcal{L}_n$ and every $T \in \mathcal{L}_{n+1}$ is obtained from only one $T' \in \mathcal{L}_n$ (see [1] for ϑ's definition). Therefore, we have a recursive construction of the left trees that can be

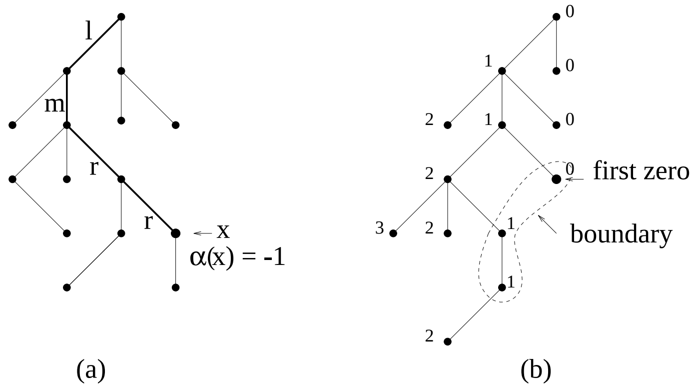

Figure 2: (a) a ternary tree; (b) a left tree

described by means of a rewriting rule. We implement this rule, by means of *Maple*, and obtain that the cardinality of $\mathcal{L}_n$ is the same as $\mathcal{NS}_{n+1}$, for $n \leq 30$. Consequently, we can conjecture that the number of left trees havig n nodes is given by formula (1.1).
As proof , we introduce the class $(\mathcal{T} \times \mathcal{T})_n$ of ternary tree pairs (T_1, T_2) such that the sum of T_1's and T_2's nodes is n. It is easy to give a bijective proof of the following result:

$$|(\mathcal{T} \times \mathcal{T})_n| = \frac{n-1}{(n+1)(2n+1)} \binom{3n}{n},$$

and so:

$$|\mathcal{T}_n| - |(\mathcal{T} \times \mathcal{T})_n| = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}.$$

Therefore, if we determine a bijection between the class $(\mathcal{T} \setminus \mathcal{L})_n$ of non-left trees and the class $(\mathcal{T} \times \mathcal{T})_n$, we prove our conjecture:

$$|\mathcal{L}_n| = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}.$$

Moreover, if we can determine a bijection between $\mathcal{NS}_{n+1}$ and $\mathcal{L}_n$, we get a bijective proof of the enumerative formula for non-separable rooted planar maps.

2.1 The bijection between $\mathcal{T} \setminus \mathcal{L}$ and $\mathcal{T} \times \mathcal{T}$

Let $T \in \mathcal{T} \setminus \mathcal{L}$. There is at least one node whose abscissa is 0 and which has a right son. Let p be the first of these node in the post-order traversal, and let q be p's right son ($\alpha(p) = 0$ and $\alpha(q) = -1$). By deleting the edge (p, q), we obtain two ternary trees T_1 and T_2 such that:

- T_1 has a distinct node p having abscissa 0 and no right son, whose subsequent nodes in the post-order all have a positive or 0 abscissa;
- T_2 is a generic ternary tree (see figure 3).

The class of ternary trees such that:

- there is a marked node p;

- $\alpha(p) = 0$;

- $\alpha(q) \geq 0$, for each node q subsequent to p in the post-order;

is denoted by $\widetilde{\mathcal{T}}$. Therefore, if there is a bijection between $\widetilde{\mathcal{T}}_n$ and $\mathcal{T}_n$, there is also a bijection

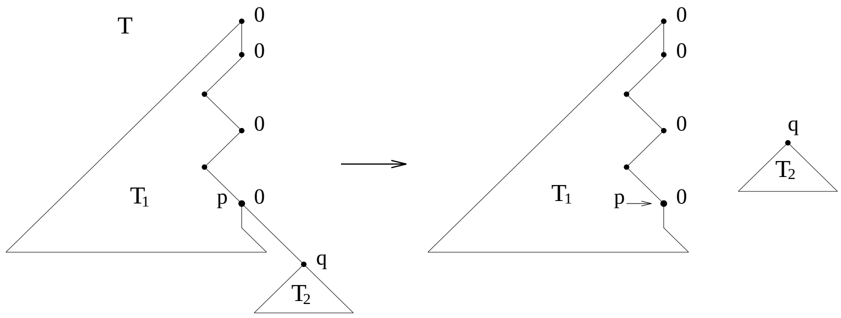

Figure 3: The decomposition of $\mathcal{T} \setminus \mathcal{L}$

between $(\mathcal{T} \setminus \mathcal{L})_n$ and $(\mathcal{T} \times \mathcal{T})_n$. We determine the bijection between marked trees and ternary trees by means of an operation on ternary trees that we call *overturning*, denoted by ρ. Given a ternary tree T and one of its leaves b_1, the overturning operation naturally consists in taking T for the leaf b_1 and rotating T counterclockwise until we obtain a new ternary tree having b_1 as its root (see figure 6).
We now wish to introduce some notations that allow us to define of the overturning operation formally. By adding $2n+1$ external nodes to a ternary tree $T \in \mathcal{T}_n$, we get a *complete ternary tree* whose internal nodes have exactly 3 sons. By adding an external node (called *over root*) to the complete tree so that the original root becomes the new one's middle son, we obtain a *complete planted ternary tree* $\hat{T}$ (see figure 4). We denote this class having n internal nodes by $\hat{\mathcal{T}}_n$. Let

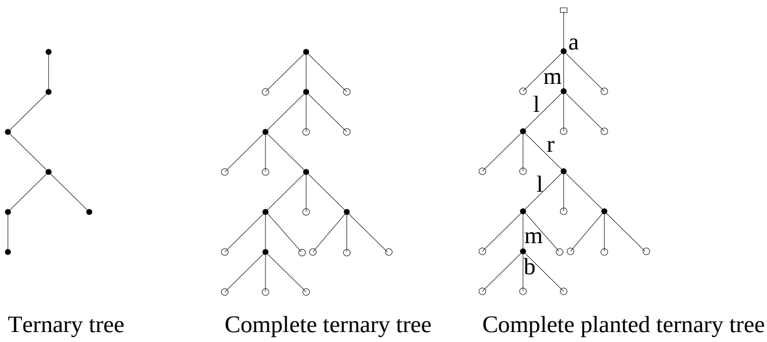

Figure 4: A ternary tree and its corresponding complete and complete planted ternary trees

$T \in \mathcal{T}_n$ and $\hat{X}$ be the set of $\hat{T}$'s nodes. We introduce the following functions: $f_i : \hat{X} \to \hat{X}$, with $i = 0, 1, 2, 3$, such that, if $x \in \hat{X}$,

- $f_0(x)$,$f_1(x)$ and $f_2(x)$, are the left, middle and right son of x, respectively;

- $f_3(x)$ is x's father.

These functions can represent the tree T . We can now define the overturning operation.

Definition 1 *Let $T \in \mathcal{T}_n$. Let a_1 and b_1 be the over-root and a leaf of $\hat{T}$, respectively. We denote T's path from a_1 to b_1 by γ, and γ's nodes by $\hat{X}_\gamma$. The* overturning *on T according to b_1 defines four new partial functions $f_j^{'}$, with $j = 0, 1, 2, 3$, such that:*

- *for each $x \in \hat{X} \setminus \hat{X}_\gamma$, $f_j^{'}(x) = f_j(x)$, for $j = 0, 1, 2, 3$,*
- *if x, y and z are three consecutive nodes of γ such that $x = f_3(y)$ and $z = f_i(y)$, with $0 \leq i \leq 2$, then $f_j^{'}(y) = f_{(j+i+1) mod\, 4}(y)$ (see figure 5),*
- *$f_3^{'}(a_1) = f_1(a_1) = a$ and $f_1^{'}(b_1) = f_3(b_1) = b$, where a is T's root and b is b_1's father.*

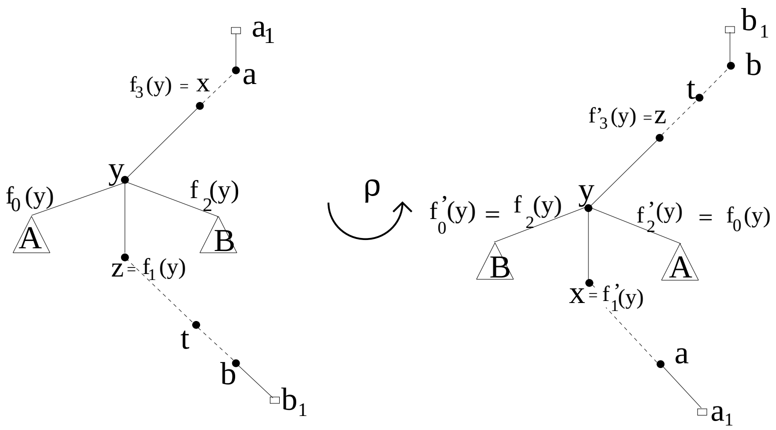

Figure 5: Overturning on node y

We can immediately deduce that:

Proposition 2.1 *The four new functions $f_j^{'}$, with $j = 0, 1, 2, 3$, obtained by performing the overtuning on T according to b_1, define a ternary tree having over-root b_1. We denote this tree by $\rho(\hat{T}, b_1)$. By performing the overturning on $\rho(\hat{T}, b_1)$ according to a_1 , we obtain the original tree $\hat{T}$ (see figures 6 and 7).*

Overturning properties

We now describe some properties of overturning. For brevity's sake we do not illustrate the proofs. The reader can use the trees in figures 7 to check the properties.

Let $T \in \mathcal{T}_n$ and a_1 and b_1 be the over-root and a leaf of $\hat{T}$. We denote $\hat{T}$'s path from a_1 to b_1 by γ and $\rho(\hat{T}, b_1)$'s path from b_1 to a_1 by $\gamma^{'}$.

Proposition 2.2 *If x is a node of the path γ, x's abscissa $\alpha_{\rho(\hat{T},b_1)}(x)$ in the transformed tree $\rho(\hat{T}, b_1)$ is such that:*

$$\alpha_{\rho(\hat{T},b_1)}(x) = \alpha_T(x) + \delta - \alpha_T(b_1),$$

where $\delta = 1, 0, -1$ whenever x's subsequent step in γ is a left, middle or right step, respectively.

Proposition 2.3 *By denoting the set of $\hat{T}$'s nodes that precede (follow) γ's nodes according to post-order by P (F), we have:*

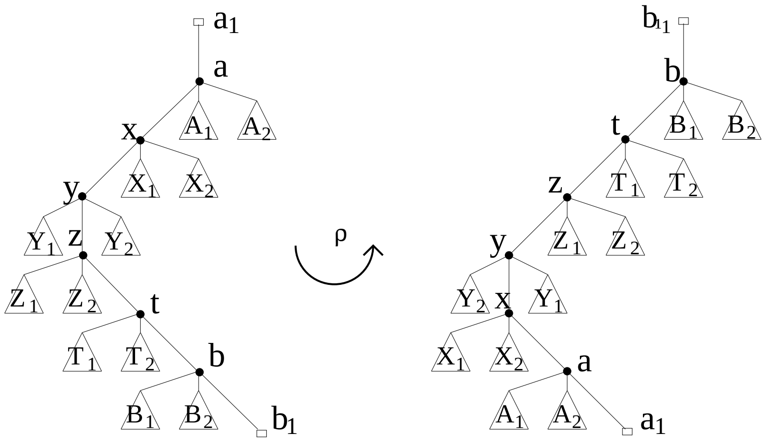

Figure 6: Ternary trees $\hat{T}$ and $\rho(\hat{T}, b_1)$

i) if $x \in P$, *then* $\alpha_{\rho(\hat{T},b_1)}(x) = \alpha_T(x) - 2 - \alpha_T(b_1)$,

ii) if $x \in F$, *then* $\alpha_{\rho(\hat{T},b_1)}(x) = \alpha_T(x) + 2 - \alpha_T(b_1)$.

We can now determine the bijection between the class $\widetilde{\mathcal{T}}_n$ of marked trees and the class $\mathcal{T}_n$ of ternary trees.

Theorem 2.4 *The overturning* ρ *is a bijection between the class* $\widetilde{\mathcal{T}}_n$ *and the class* $\mathcal{T}_n$.

Proof. Let $T \in \widetilde{\mathcal{T}}_n$ and let a_1 and b be $\hat{T}$'s over-root and $\hat{T}$'s marked node. If b_1 is b's right son, we can deduce from Proposition 2.1 that we can obtain a ternary tree $\rho(\hat{T}, b_1)$ by overturning $\hat{T}$ according b_1. Moreover, from Propositions 2.2 and 2.3, it follows that a_1 becomes $\rho(\hat{T}, b_1)$'s first external node whose abscissa is $+1$ according to pre-order traversal.
Vice versa, let $T_1 \in T_n$ and let b and a_1 be $\hat{T}_1$'s root and its first external node whose abscissa is $+1$ according to pre-order. By overturning $\hat{T}$ according to a_1, we obtain a ternary tree $\rho(\hat{T}_1, a_1)$. By marking the image of b in $\rho(\hat{T}_1, a_1)$, we get a marked ternary tree of $\widetilde{\mathcal{T}}_n$. □

From this Theorem and our preliminary remarks, it follows that:

Corollary 2.5 *There is a bijection between the class* $(\mathcal{T} \setminus \mathcal{L})_n$ *of non-left ternary trees having* n *vertices and the class* $(\mathcal{T} \times \mathcal{T})_n$ *of ternary tree pairs having a total of* n *vertices.*

Moreover, by using Propositions 2.2 and 2.3, we can prove a very nice property:

Corollary 2.6 *Let* $T \in \widetilde{\mathcal{T}}_n$ *and let* $o(T)$ $(e(T))$ *be the number of* T*'s nodes having an odd (even) abscissa. The ternary tree* $\rho(T)$ *is such that:*

$$e(\rho(T)) = o(T) + 1, \qquad o(\rho(T)) = e(T) - 1.$$

2.2 The bijection between $\mathcal{NS}$ and $\mathcal{L}$

In this subsection, we determine a bijection between $\mathcal{NS}$ and $\mathcal{L}$ by making a combinatorial decomposition of both .

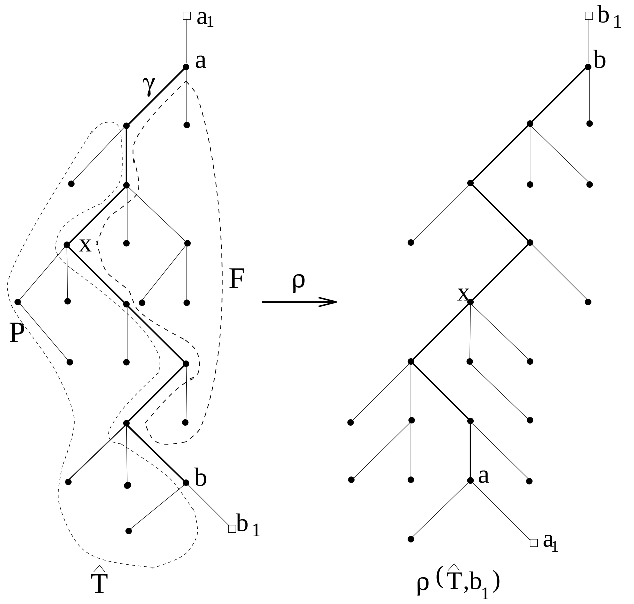

Figure 7: Ternary trees $\hat{T}$ and $\rho(\hat{T}, b_1)$

2.2.1 $\mathcal{NS}$'s decomposition

The following decomposition is a variation of Brown's method [2, 7]. There is only one map in $\mathcal{NS}$ having two edges and we denote it by D (see figure 8). The *first edge* of $\mathcal{NS}$'s map is the first

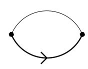

Figure 8: A rooted digon

edge encountered counterclockwise around the root vertex from the root edge (see figure 1 (a)). We divide $\mathcal{NS}$ into two subsets:

- $\mathcal{NS}^{(I)}$ contains the maps M in $\mathcal{NS}$ such that we obtain a map M_1 of $\mathcal{NS}$ if we delete M's first edge;
- $\mathcal{NS}^{(II)}$ contains all other maps in $\mathcal{NS}$ (including D).

Since we now want to determine a combinatorial decomposition for $\mathcal{NS}^{(I)}$ and $\mathcal{NS}^{(II)}$, we introduce the relative parameters involved. Let M be a non-separable map, we denote the number of its edges minus 1 by $e(M)$, the number of vertices minus 2 by $v(M)$, the number of its faces minus 1 by $f(M)$, the number of its incidental external face edges minus 1 by $de(M)$ (degree of M's external face) and the number of its incident root vertex edges minus 1 by $dr(M)$ (degree of M's root vertex).
For instance, the map M in figure 1 (a) is such that $e(M) = 7$, $v(M) = 3$, $f(M) = 4$, $de(M) = 2$ and $dr(M) = 3$.
Let $M \in \mathcal{NS}^{(I)}$. The deletion of M's first edge gives $M_1 \in \mathcal{NS}$. Let γ be the non-root vertex of

M's first edge (γ is a vertex of M's external face). We need to mark γ on the M_1's external face to reconstruct M from M_1 (see figure 9). Therefore, if we denote the class of $\mathcal{NS}$'s maps whose

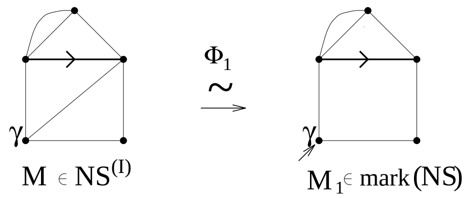

Figure 9: $\mathcal{NS}^{(I)}$'s decomposition

external face's vertex is marked by $mark(\mathcal{NS})$, we obtain:

Lemma 2.7

$$\phi_1 : \mathcal{NS}^{(I)} \tilde{\to} mark(\mathcal{NS}) \; : M \to M_1$$

is a bijection such that $e(M) = e(M_1) + 1$, $v(M) = v(M_1)$, $f(M) = f(M_1) + 1$, $de(M) \leq de(M_1)$ *and* $dr(M) = dr(M_1) - 1$.

Let $M \in \mathcal{NS}^{(II)}$. By deleting M's first edge, we obtain a map M' having almost a cut-vertex. The cut-vertices belong to the face on the first edge's left. Let the cut-vertices on this face be $v_1, v_2, \ldots, v_k$, so that they can be encountered around the face starting out from its root vertex, and let $\beta = v_1$ (see figure 10). We divide M in to vertices β and γ and obtain the maps shown in

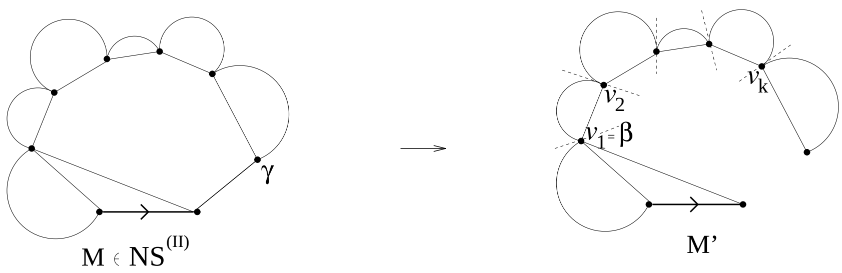

Figure 10: A map of $\mathcal{NS}^{(II)}$

figure 11. We then take γ' on β' and add a root edge between γ'' and β'' whose root vertex γ''. We

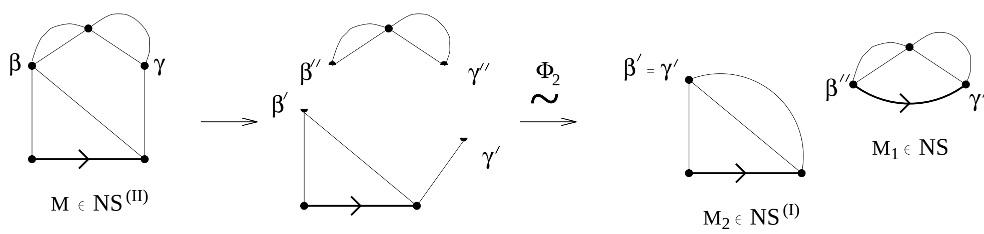

Figure 11: $\mathcal{NS}^{(II)}$'s decomposition

obtain maps $M_1 \in \mathcal{NS}$, and $M_2 \in \mathcal{NS}^{(I)}$ and by deleting M_2's first edge, we get a map $M' \in \mathcal{NS}$. It follows that:

Lemma 2.8

$$\phi_2 : \mathcal{NS}^{(II)} \xrightarrow{\sim} \mathcal{NS} \times \mathcal{NS}^{(I)} \ : M \to (M_1, M_2)$$

is a bijection such that $e(M) = e(M_1) + e(M_2)$, $v(M) = v(M_1) + v(M_2) + 1$, $f(M) = f(M_1) + f(M_2) - 1$, $de(M) = de(M_1) + de(M_2)$ *and* $dr(M) = dr(M_2)$.

In figure 16 an example of this decomposition is showed.

2.2.2 $\mathcal{L}$'s decomposition

We have to determine an analogous decomposition for $\mathcal{L}$. We start out dividing $\mathcal{L}$ into two classes: $\mathcal{L}^{(I)}$ and $\mathcal{L}^{(II)}$. A left tree's *first zero* is the first node whose abscissa is zero according to the post-order. The *boundary* of a tree $T \in \mathcal{L}$ contains the first zero and T's nodes having abscissa 1 such that if we add a right son a to one of these nodes, we obtain a tree $T' \in \mathcal{L}$ in which a is the first zero of T' (see figure 2(b)).

- $\mathcal{L}^{(I)}$ contains $\mathcal{L}$'s trees T whose first zero is a leaf;
- $\mathcal{L}^{(II)}$ contains all of $\mathcal{L}$'s other trees (including the left tree having only one node).

We now introduce some parameters involved in the decomposition. Let T be a left ternary tree, denote the number of its nodes by $n(T)$, the number of its nodes whose abscissa is an odd (even) number by $o(T)$ ($e(T)$), the number of nodes whose abscissa is zero by $z(T)$ and the number of nodes contained in T's boundary by $db(T)$ (degree of T's boundary).
For instance, the left tree T in figure 2 (b) is such that $n(T) = 13$, $o(T) = 5$, $e(T) = 8$, $df(T) = 3$ and $z(T) = 4$.
Let $T \in \mathcal{L}^{(I)}$. T's first zero is a leaf and so its deletion produces a tree $T_1 \in \mathcal{L}$. We need to mark the father of T's first zero to reconstruct T from T_1 (see figure 12). Since this node belongs to

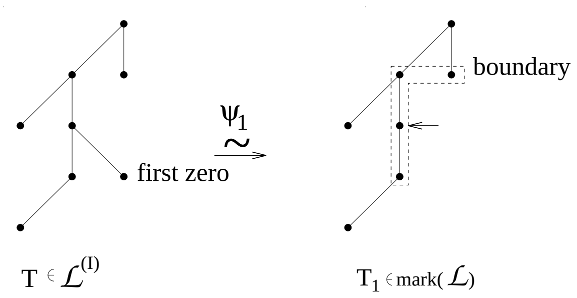

Figure 12: $\mathcal{L}^{(I)}$'s decomposition

T_1's boundary, if we denote the class of the left trees having a marked node on the boundary by $mark(\mathcal{L})$, we obtain:

Lemma 2.9

$$\psi_1 : \mathcal{L}^{(I)} \xrightarrow{\sim} mark(\mathcal{L}^{(I)}) \ : T \to T_1$$

is a bijection such that $n(T) = n(T_1) + 1$, $o(T) = o(T_1)$, $e(T) = e(T_1) + 1$, $db(T)) \leq db(T_1)$ *and* $z(T) = z(T_1) + 1$.

This Lemma is analogous to Lemma 2.7 and so we can deduce the following similarities between the map and tree parameters:

edges $e(M)$	$\longleftrightarrow$	nodes $n(T)$,
vertices $v(M)$	$\longleftrightarrow$	nodes having an odd abscissa $o(T)$,
faces $f(M)$	$\longleftrightarrow$	nodes having an even abscissa $e(T)$,
external face degree $de(M)$	$\longleftrightarrow$	boundary degree $db(T)$,
root vertex degree $dr(M)$	$\longleftrightarrow$	nodes having 0 abscissa $z(T)$.

Let $T \in \mathcal{L}^{(II)}$. T's first zero is the first node having abscissa zero according to the post-order and it is not a leaf. Consequently, it does not have a middle and right son, and its left son is the root of a subtree $T_1 \in \mathcal{L}$. If we cut the left branch of T's first zero, we obtain two subtrees: $T_1 \in \mathcal{L}$ and $T_2 \in \mathcal{L}^{(I)}$ (T_2's first zero is a leaf; see figure 13). It follows that:

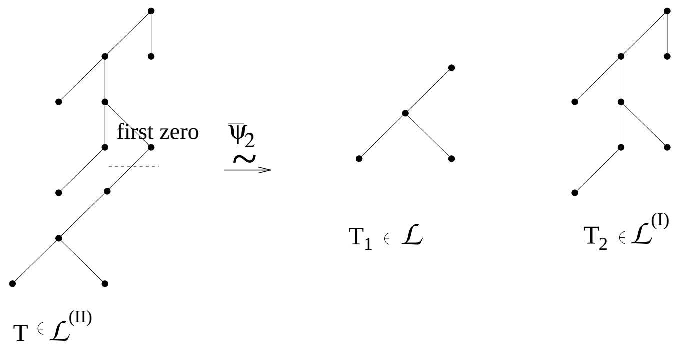

Figure 13: $\mathcal{L}^{(II)}$'s first decomposition

Lemma 2.10

$$\overline{\psi}_2 : \mathcal{L}^{(II)} \xrightarrow{\sim} \mathcal{L} \times \mathcal{L}^{(I)} : T \rightarrow (T_1, T_2)$$

is a bijection such that $n(T) = n(T_1) + n(T_2)$, $o(T) = e(T_1) + o(T_2)$, $e(T) = o(T_1) + e(T_2)$, $db(T) = z(T_1) + db(T_2)$ *and* $z(T) = z(T_2)$.

This Lemma is not analogous to Lemma 2.8 because the followinf three three map and tree parameters do not correspond:

$$v(M) \longleftrightarrow o(T), \qquad f(M) \longleftrightarrow e(T), \qquad de(M) \longleftrightarrow db(T).$$

For the trees:

$$o(T) = e(T_1) + o(T_2), \qquad e(T) = o(T_1) + e(T_2), \qquad db(T) = z(T_1) + db(T_2),$$

while for the maps:

$$v(M) = v(M_1) + v(M_2) + 1, \;\; f(M) = f(M_1) + f(M_2) - 1, \;\; de(M) = de(M_1) + de(M_2).$$

Therefore, if we determine a bijective transformation $\sigma : \mathcal{L}_n \xrightarrow{\sim} \mathcal{L}_n : T_1 \rightarrow \overline{T}_1$ such that:

$$e(T_1) = o(\overline{T}_1) + 1, \qquad o(T_1) = e(\overline{T}_1) - 1, \qquad z(T_1) = db(\overline{T}_1),$$

the decomposition for $\mathcal{L}^{(II)}$ is analogous to $\mathcal{NS}^{(II)}$'s, and so we also obtain a bijection between their maps and trees. Let us now define this bijective transformation σ on $\mathcal{L}$ by means of the overturning operation ρ (defined in section 2.1). Let $\widetilde{\mathcal{L}}$ be the class of left trees whose first zero is marked. $\widetilde{\mathcal{L}}$ is a subclass of $\widetilde{\mathcal{T}}$ and it obviously bijects $\mathcal{L}$. If we perform overturning ρ on a tree of $\widetilde{\mathcal{L}}$ according to the external right son of the marked node, we obtain a ternary tree. In the following Proposition we prove that this ternary tree is a left tree; that is, we show that the overturning ρ is a bijective transformation on $\mathcal{L}$.

Proposition 2.11

$$\rho : \widetilde{\mathcal{L}}_n \xrightarrow{\sim} \mathcal{L}_n \ : \overline{T}_1 \to T_1$$

is a bijection such that: $e(\overline{T}_1) = o(T_1) + 1, \qquad o(\overline{T}_1) = e(T_1) - 1, \qquad db(\overline{T}_1) = z(T_1).$

Proof. Let $\overline{T}_1 \in \widetilde{\mathcal{L}}_n$. Let b_1 be the external right son of $\overline{T}_1$'s marked node. We denote $\rho(\overline{T}_1, b_1)$ by T_1. Theorem 2.4 establishes that overturning ρ is a bijection between $\widetilde{\mathcal{T}}_n$ and $\mathcal{T}_n$. Therefore, if we prove that T_1 is a left tree, ρ is a bijection between $\widetilde{\mathcal{L}}_n$ and $\mathcal{L}_n$. We denote:

- the path from $\overline{T}_1$'s root to $\overline{T}_1$'s marked node (first zero) by γ;
- the set of $\overline{T}_1$'s nodes that precede (follow) γ's nodes according to the post-order by P (F) (see figure 14).

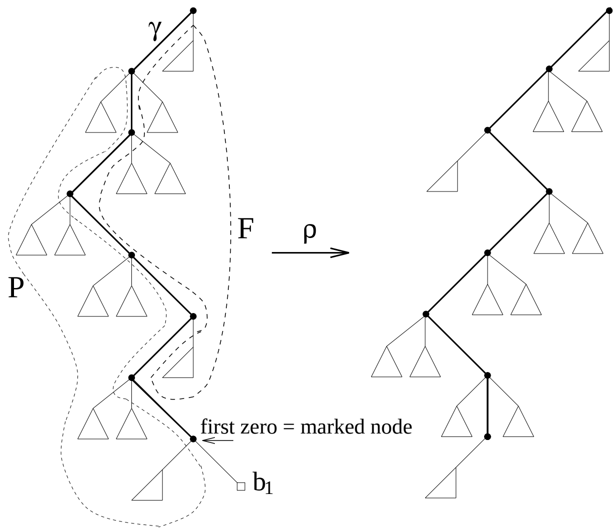

Figure 14: Overturning ρ on $\overline{T}_1$ according to b_1

We have to prove that $\alpha(x) \geq 0$ for each node x of T_1. Since $\alpha(b_1) = -1$, from Proposition 2.2 and 2.3, it follows that the abscissa of the nodes belonging to the transformed path $\rho(\gamma)$ and to set $\rho(F)$ of T_1 is non-negative. The marked node of $\overline{T}_1$ is the first zero; that is, the first node having abscissa zero according to post-order. Since P's node precedes the first zero, $\alpha(x) \geq 1$ for each node x of P. Therefore, from Proposition 2.3, the abscissas of the nodes belonging to the transformed set $\rho(P)$ of T_1 are non-negative. Consequently, $T_1 \in \mathcal{L}$.
Let us now examine the relationship between $\overline{T}_1$'s and T_1's parameters. The relationship $e(\overline{T}_1) = o(T_1) + 1$ and $o(\overline{T}_1) = e(T_1) - 1$ follow from Corollary 2.6. The last relation is obtained by proving that the image of $\overline{T}_1$'s boundary is the set of T_1's nodes having abscissa zero.

Let x be a node of $\overline{T}_1$'s boundary. If x is the first zero, then $\rho(x)$ is T_1's root and so $\alpha(\rho(x)) = 0$. If x is not the first zero, from the boundary definition it follows that $x \in P$ and $\alpha(x) = 1$. Therefore, from Proposition 2.3, we get $\alpha(\rho(x)) = 0$.
Vice versa, let $x^{'} \in T_1$ and $\alpha(x') = 0$. There is a node $x \in \overline{T}_1$ such that $\rho(x) = x'$.

- If $x \in \gamma$ and x is not the first zero, from Proposition 2.2 it follows that $\alpha(x) = 0$ and γ's step $(x, \bar{x})$ is a right step. Thus, $\alpha(\bar{x}) = -1$ and this is impossible because $\overline{T}_1$ is a left tree. Therefore, if $x \in \gamma$, then x is the first zero and so x belongs to $\overline{T}_1$'s boundary.

- If $x \notin \gamma$, from Proposition 2.3 we can deduce that $x \in P$ and $\alpha(x) = 1$; this in turn, means that x belongs to $\overline{T}_1$'s boundary.

Consequently, we obtain the third relation. □

From this Proposition, it follows that the bijective transformation σ on $\mathcal{L}$ is ρ^{-1}. The results in section 2.1 establish that this inverse function is obtained by performing the overturning on the left tree according to its first external node b_1 having abscissa $+1$ according to the prefix order (see Theorem 2.4's proof). Therefore, with $T \in \mathcal{L}^{(II)}$, the steps of T's decomposition are:

1. cut the left branch of T's first zero, which gives us two subtrees: $T_1 \in \mathcal{L}$, and $T_2 \in \mathcal{L}^{(I)}$;

2. determine T_1's first external node b_1 having abscissa $+1$ according to the pre-order;

3. perform the overturning ρ on T_1 according to b_1 (see figure 15).

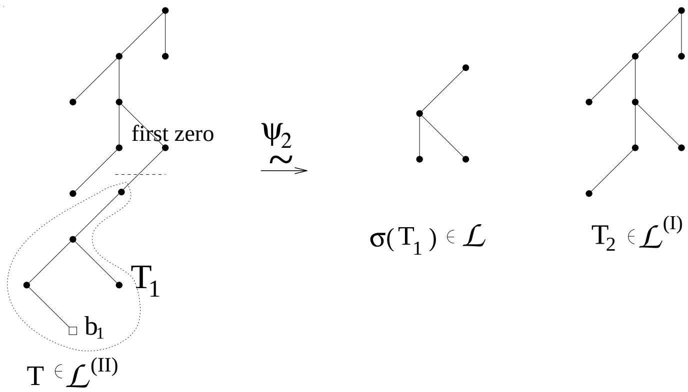

Figure 15: $\mathcal{L}^{(II)}$'s decomposition

Consequently, the lemma on $\mathcal{L}^{(II)}$ analogous to Lemma 2.8 is:

Lemma 2.12

$$\psi_2 : \mathcal{L}^{(II)} \xrightarrow{\sim} \mathcal{L} \times \mathcal{L}^{(I)} \; : T \to (\overline{T}_1, T_2)$$

is a bijection such that $\overline{T}_1 = \sigma(T_1)$ *and the relationship among the parameters are:* $n(T) = n(\overline{T}_1) + n(T_2)$, $o(T) = o(\overline{T}_1) + o(T_2) + 1$, $e(T) = e(\overline{T}_1) + e(T_2) - 1$, $db(T) = db(\overline{T}_1) + db(T_2)$ *and* $z(T) = z(T_2)$.

We have a combinatorial decomposition ψ for $\mathcal{L}$ and an analogous decomposition ϕ for $\mathcal{NS}$. Therefore, our final result is:

Theorem 2.13

$$\psi^{-1}\phi : \mathcal{NS} \xrightarrow{\sim} \mathcal{L} : M \to T$$

is a bijection such that $e(M) = n(T)$, $v(M) = o(T)$, $f(M) = e(T)$, $de(M) = db(T)$ *and* $dr(M) = z(T)$. *Moreover, in this bijection,* $\mathcal{NS}^{(i)}$ *corresponds to* $\mathcal{L}^{(i)}$, *for* $i = I, II$.

A non-separable rooted planar map and the correspondent left tree obtained by bijection $\psi^{-1}\phi$ are shown in figure 16.

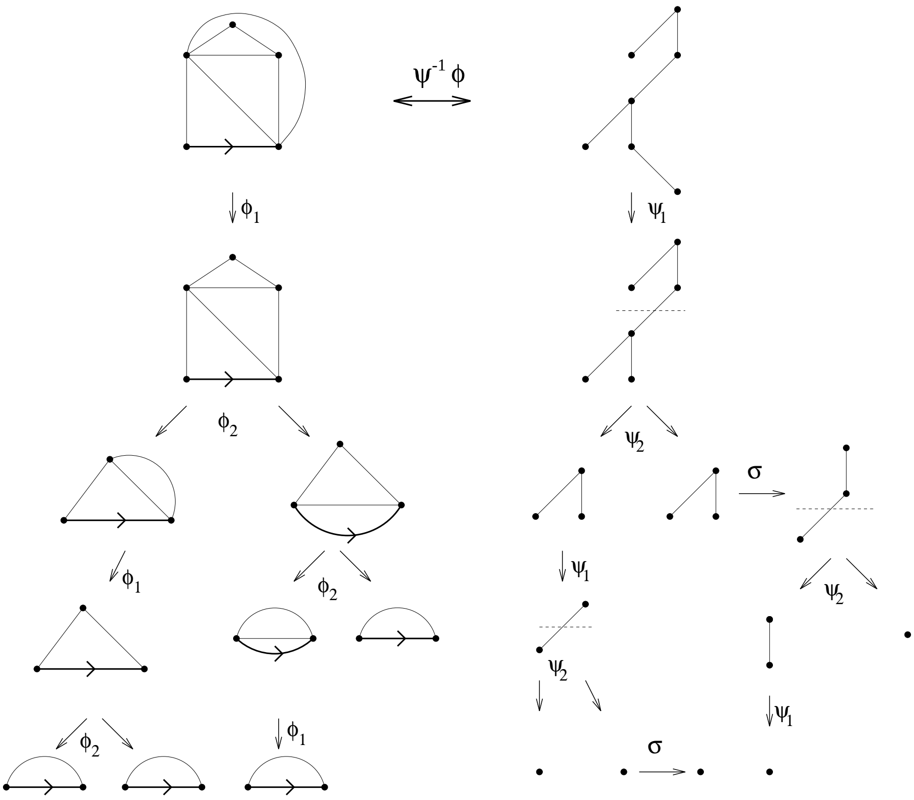

Figure 16: The bijection between $\mathcal{NS}$ and $\mathcal{L}$

Remark 1 In [7], Goulden and West describe a bijection between non-separable planar maps and two-stack-sortable permutations that is obtained by Brown's decomposition and an analogous one for two-stack-sortable permutations. Our map decompositions are analogous both to Brown's and to two-stack-sortable permutation decompositions. Consequently, we have a bijection between left trees and two-stack-sortable permutations.

3 Acknowledgments

The authors would like to thank R. Pinzani for his helpful opinion on this work and R. Cori for suggesting the problem.

References

[1] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, Towards a methodology for tree enumeration, *Proc. of the* 7^{th} *FPSAC*, Marne-la-Vallée (1995) 53-65.

[2] W. G. Brown, Enumeration of nonseparable planar maps, *Canad. J. Math.* 15 (1963) 526-545.

[3] R. Cori, Une code pour les graphes planaires et ses apllications, *Astérisque, Société Math. de France* 27 (1975).

[4] R. Cori and B. Vauquelin, Planar maps are well labeled trees, *Canad J. Math.* 33 (5) (1981) 1023-1042.

[5] R. Cori, Bijective Census of Rooted Planar Maps A survey, *Proc. of the* 6^{th} *FPSAC*, Firenze (1993) 131-141.

[6] I. P. Goulden and D. N. Jackson, *Combinatorial Enumeration*, Wiley, New York, 1993.

[7] I. P. Goulden and J. West, Raney paths and a combinatorial relationship between rooted nonseparable planar maps and two-stack-sortable permutations, to appear in *J. Combinatorial Theory (A)*.

[8] B. Jacquard and G. Schaeffer, Three families of trees and nonseparable planar maps, *private comunication.*

[9] D. E. Knuth, *The Art of Computer Programming*, Vol. 1, (Addison-Wesley, Reading, MA, 2nd ed., 1973).

[10] W. T. Tutte, A census of planar maps, *Canad. J. Math.* 15 (1963) 249-271.