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Four New Formulas for Schubert Polynomials

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We state several new combinatorial formulas for the Schubert polynomials. They are generalizations of well-known expressions for the Schur polynomials: (1) the Demazure character formula; (2) the realization as the generating function of semi-standard tableaux of a given shape; and (3), (4) the Weyl character formula. Our formulas appear surprising from a combinatorial point of view because their derivation and proof involve a new geometric model, the configuration varieties.

The results we state here are a special case of formulas for a broad class of Schur-type polynomials, the (flagged) Schur polynomials of strictly separated diagrams = [?], [?]. These include skew Schur and key polynomials [?], and the Schur polynomials of northwest diagrams [?], [?], [?], [?].

1 Schubert polynomials

The Schubert polynomials $\mathcal{S}(w)$ of permutations $w \in \Sigma_n$ are polynomials in variables $x_1, \dots, x_n$. They were originally considered as representatives of Schubert classes in the Borel picture of the cohomology of the flag variety $GL(n)/B$, though we will give a completely different geometric interpretation in the later sections of this note.

They are constructed in terms of the following *divided difference operators* [?], [?], [?]. First, the operator ∂_i is defined by

$$\partial_i f\,(x_1, \dots, x_n) = 3D\frac{f(x_1, \dots, x_i, x_{i+1}, \dots, x_n) - = f(x_1, \dots, x_{i+1}, x_i, \dots, x_n)}{x_i - x_{i+1}}.$$

Then for a reduced decomposition of a permutation $u = 3Ds_{i_1}s_{i_2}\cdots$, the operator $\partial_u = 3D\partial_{i_1}\partial_{i_2}\cdots$ is independent of the reduced decomposition chosen. Also, take $\partial_e = 3D\mathrm{id}$.

Now we may define the Schubert polynomials as follows. Let $w_0 = 3Dn, n-1, \dots, 2, 1$ be the longest permutation, and take $u = 3Dw^{-1}w_0$, so that $wu =$

$3Dw_0$. Then

$$\mathcal{S}(w) = 3D\partial_u(x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1}).$$

We have deg $\mathcal{S}(w) = 3D\ell(w)$.

Example. For the permutation $w = 3D24153 \in \Sigma_5$, by inverting first ascents we get $ws_1s_3s_2s_1s_4s_3 = 3Dw_0$, so

$$\mathcal{S}(w) = 3D\partial_1\partial_3\partial_2\partial_1\partial_4\partial_3(x_1^4x_2^3x_3^2x_4) = 3Dx_1x_2\,(x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4).$$

2 Orthodontia on a Rothe diagram

Consider the *Rothe diagram* of a permutation $w \in \Sigma_n$,

$$D = 3DD(w) = 3D\{(i,j) \in \mathbf{N}\times\mathbf{N} \mid i < w^{-1}(j),\ \ j < w(i)\}.$$

Its elements are called *squares*. We shall often think of D as a sequence $(C_1, C_2, \ldots, C_r)$ of columns $C_j \subset \mathbf{N}$, by projecting squares (i,j) to their first coordinate. We omit any empty columns from the sequence.

In the sequel, our main interest is in Rothe diagrams, but our analysis will involve more general diagrams $D \subset \mathbf{N}\times\mathbf{N}$ of squares in the plane. In fact, to any such diagram one can associate a "Schur polynomial", which is the character of its flagged Schur module (see Section 7). The formulas we state will apply not only to the Schubert polynomials (associated to Rothe diagrams), but to the Schur polynomials of any "strongly separated" diagram. (See Section 9.)

Now, let D be a Rothe diagram. For our formulas, we will require a sequence of permutations $w_1, w_2, \ldots, w_r$ which is compatible with D in the sense that $w_j\{1,2,\ldots,|C_j|\} = 3DC_j$ for all j. We also demand that the sequence be monotone in the weak order: that is, for some $u_1, u_2, \ldots, u_r$, we must have $w_1 = 3Du_1$, $w_2 = 3Du_1u_2$, $\ldots$, $w_r = 3Du_1\cdots u_r$, with $\ell(w_j) = 3D\ell(u_1) + \ell(u_2) + \cdots + \ell(u_j)$. (It is enough to require $w_r = 3D\ell(u_1) + \cdots + \ell(u_r)$.)

This can be done by means of the following algorithm. Given a column $C \subset \{1,\ldots,n\}$, a *missing tooth* of C is an integer i such that $i \notin C$, but $i' \in C$ for some $i' > i$. The only C without any missing teeth are $\{1,2,3,\ldots,j\}$. Given a diagram $D = 3D(C_1,\ldots,C_r)$, let (i_0, j_0) denote a *special* missing tooth in D which is in the leftmost column possible, and as high as possible in this column subject to the condition that $(i_0+1, j_0) \in D$.

Now perform *orthodontia* on D to get a new diagram D' with fewer missing teeth, by switching rows i_0 and i_0+1 in the columns weakly right of (i_0, j_0). That is, change D to

$$\begin{aligned} D' \ = 3D \quad & \{(i,j) \mid (i,j) \in D \text{ and } j < j_0\} \\ & \cup\{(s_{i_0}i, j) \mid (i,j) \in D \text{ and } j \geq j_0\}. \end{aligned}$$

Next, locate the special missing tooth (i_1, j_1) of D', and perform this procedure again on D' to get D'' and (i_2, j_2), and so on until we reach a diagram with no missing teeth. Notice that $j_0 \leq j_1 \leq \cdots$.

Finally, define the *orthodontic sequence* of $D = 3DD(w)$ to be $w_1, w_2, \ldots$, where $w_j = 3D\prod_{k\,:\,j_k \leq j} s_{i_k}$, the product being taken over all k such that (i_k, j_k) is weakly left of column j. It is easily seen that this sequence has the desired properties.

Example. For the same $w = 3D24153$, we have

$$D = 3DD(w) = 3D \begin{array}{cccc} 1 & \square & & \\ 2 & \square & & \square \\ 3 & & & \\ 4 & & & \square \end{array} = 3D \begin{array}{cc} \square & \circ \\ \square & \square \\ & \\ & \square \end{array}$$

$$D' = 3D \begin{array}{ccc} 1 & \square & \square \\ 2 & \square & \\ 3 & & \circ \\ 4 & & \square \end{array} \qquad D'' = 3D \begin{array}{ccc} 1 & \square & \square \\ 2 & \square & \circ \\ 3 & & \square \end{array} \qquad D''' = 3D \begin{array}{ccc} 1 & \square & \square \\ 2 & \square & \square \end{array}$$

so that the special missing teeth (as indicated by $\circ$) are $(i_0, j_0) = 3D(1,2)$, $(i_1, j_1) = 3D(3,2)$, $(i_2, j_2) = 3D(2,2)$, and $w_1 = 3De$, $\quad w_2 = 3Ds_{i_0}s_{i_1}s_{i_2} = 3Ds_1s_3s_2$.

Note that $w_r = 3Dw_2 = 3Ds_1s_3s_2$ is a reduced subword of the first-ascent sequence $s_1s_3s_2s_1s_4s_3$ which raises w to the maximal permutation w_0, as in the previous section. This is always the case, and we can give an algorithm for extracting this subword.

3 Demazure character formula

The definition of $\mathcal{S}(w)$ involves *descending* induction (lowering the degree), but we give the following *ascending* algorithm.

Let $D(w) = 3D(C_1, \ldots, C_r)$ (omitting empty columns), and let $c_j = 3D|\,C_j|$. Take a monotone compatible sequence $w_1, \ldots, w_r$ for $D(w)$, such as the orthodontic sequence defined above, and let $u_j = 3Dw_{j-1}^{-1}w_j$, so that $w_1 = 3Du_1$, $w_2 = 3Du_1u_2, \cdots$. Furthermore, let $\lambda_i = 3Dx_1x_2\cdots x_i$.

Define the *Demazure operators* (isobaric divided differences) $\pi_i = 3D\partial_i x_i$ and $\pi_u = 3D\pi_{i_1}\pi_{i_2}\cdots$, for $u = 3Ds_{i_1}s_{i_2}\cdots$ a reduced decomposition. (See [?].) These are analogous to the ∂ operators, but do not change the degree of a homogeneous polynomial.

Finally, let $\mathcal{S}_0(w) = 3D1$ and

$$\mathcal{S}_k(w) = 3D\pi_{u_k}(\lambda_{c_k}\mathcal{S}_{k-1}(w)).$$

Theorem 1

$$\mathcal{S}_r(w) = 3D\mathcal{S}(w).$$

Example. For our permutation $w = 3D24153$, we have $c_1 = 3Dc_2 = 3D2$, $u_1 = 3De$, $u_2 = 3Ds_1 s_3 s_2$, and we may verify that

$$\mathcal{S}(w) = 3Dx_1 x_2 \, \pi_1 \pi_3 \pi_2 (x_1 x_2).$$

Note that this makes the factorization evident.

4 Young tableaux

The work of Lascoux-Schutzenberger [?] and Littlemann [?] allows us to "quantize" our Demazure formula, realizing the terms of the polynomial by certain tableaux endowed with a crystal graph structure. Reiner and Shimozono have shown that our construction gives the same non-commutative Schubert polynomials as those in [?]. In fact, A. Lascoux has informed me that some of the contents of this section were known to him, and motivated [?], though not explicitly stated there. Our tableaux are different, however, from the "balanced tableaux" of Fomin, Greene, Reiner, and Shimozono.

Recall that a *column-strict filling* of a diagram D (with entries in $\{1, \dots, n\}$) is a map t filling the squares of D with numbers from 1 to n, strictly increasing down each column. The *content* of a filling is a monomial $x^t = 3D\prod_{(i,j)\in D} x_{t(i,j)}$, so that the exponent of x_i is the number of times i appears in the filling. We will define a set of fillings $\mathcal{T}$ of the Rothe diagram $D(w)$ which satisfy

$$\mathcal{S}(w) = 3D\sum_{t\in\mathcal{T}} x^t.$$

The set $\mathcal{T}$ will be defined recursively, $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_r = 3D\mathcal{T}$, so that

$$\mathcal{S}_k(w) = 3D\sum_{t\in\mathcal{T}_k} x^t.$$

We will need the *root operators* first defined in [?]. These are operators f_i which take a filling t of a diagram D either to another filling of D or to the empty set $\emptyset$. To define them we first encode a filling t in terms of its *reading word*: that is, the sequence of its entries starting at the upper left corner, and reading down the columns one after another: $t(1,1), t(2,1), t(3,1), \dots, t(1,2), t(2,2), \dots$.

The lowering operator f_i either takes a word t to the empty word $\emptyset$, or it changes one of the i entries to $i+1$, according to the following rule. First, we ignore all the entries in t except those containing i or $i+1$; if an i is followed by an $i+1$ (ignoring non i or $i+1$ entries in between), then henceforth we ignore that pair of entries; we look again for an i followed (up to ignored entries) by an $i+1$, and henceforth ignore this pair; and iterate until we obtain a subword of the form $i+1, i+1, \dots, i+1, i, i, \dots, i$. If there are *no* i entries in this word, then $f_i(t) = 3D\emptyset$, the empty word. If there are some i entries, then the *leftmost* is changed to $i+1$.

For example, we apply f_2 to the word

$$\begin{array}{llcccccccccccc}
t & = 3D & 1 & 2 & 2 & 1 & 3 & 2 & 1 & 4 & 2 & 2 & 3 & 3 \\
 & & . & 2 & 2 & . & 3 & 2 & . & . & 2 & 2 & 3 & 3 \\
 & & . & 2 & . & . & . & 2 & . & . & 2 & . & . & 3 \\
 & & . & \mathbf{2} & . & . & . & \mathbf{2} & . & . & . & . & . & . \\
f_2(t) & = 3D & 1 & \mathbf{3} & 2 & 1 & 3 & \mathbf{2} & 1 & 4 & 2 & 2 & 3 & 3 = \\
f_2^2(t) & = 3D & 1 & \mathbf{3} & 2 & 1 & 3 & \mathbf{3} & 1 & 4 & 2 & 2 & 3 & 3 = \\
f_2^3(t) & = 3D & \emptyset & & & & & & & & & & &
\end{array}$$

Decoding the image word back into a filling of the same diagram D, we have defined our operators.

Moreover, consider the column $\phi_m = 3D\{1, 2, \ldots, m\}$ and its minimal column-strict filling t_m (ith level filled with i). For a filling t of any diagram $D = 3D(C_1, \ldots, C_r)$, define in the obvious way the composite filling $t_m \sqcup t$ of the juxtaposed diagram $\phi_i \sqcup D = 3D(\phi_i, C_1, \ldots, C_r)$. In terms of words, this means concatenating the words $(1, 2, \ldots, m)$ and t.

Now we can define our sets of tableau. Let our notation be as in the Demazure character formula, $D(w) = 3D(C_1, \ldots, C_r)$, etc, and take a reduced decomposition $u_k = 3Ds_{i_1} \cdots s_{i_l}$. Define $\mathcal{T}_0 = 3D\{\emptyset\}$, and

$$\mathcal{T}_k = 3D\langle f_{i_1}\rangle \cdots \langle f_{i_l}\rangle (t_{c_k} \sqcup \mathcal{T}_{k-1}),$$

where $\langle f_i \rangle$ means the set of powers $\{\mathrm{id}, f_i, f_i^2, \ldots\}$.

Theorem 2 *The Schubert polynomial $\mathcal{S}(w)$ is the generating function for the tableaux $\mathcal{T} = 3D\mathcal{T}_r$:*

$$\mathcal{S}(w) = 3D \sum_{t \in \mathcal{T}} x^t.$$

Furthermore, the crystal graph structure of $\mathcal{T}$ reflects the splitting of $\mathcal{S}(w)$ into key polynomials:

$$\mathcal{S}(w) = 3D \sum_{t \in \mathit{Yam}(\mathcal{T})} \kappa_{w_t(x^t)},$$

where $\mathit{Yam}(\mathcal{T})$ is the set of Yamanouchi words in $\mathcal{T}$, and the w_t are some permutations.

Example. As above, when $c_1 = 3Dc_2 = 3D2$, $u_1 = 3De$, $u_2 = 3Ds_1 s_3 s_2$, the set of tableaux (words) grows as follows:

$$\mathcal{T}_0 = 3D\{\emptyset\} \overset{t_2 \sqcup}{\to} \{12\} \overset{\langle f_2\rangle}{\to} \{12, 13\} \overset{\langle f_3\rangle}{\to} \{12, 13, 14\} \overset{\langle f_1\rangle}{\to} \mathcal{T}_1 = 3D\{12, 13, 14, 23, = 24\}$$

$$\overset{t_2 \sqcup}{\to} \quad \mathcal{T} = 3D\mathcal{T}_2 = 3D\{1212, 1213, 1214, 1223, 1224\}.$$

This clearly gives us the Schubert polynomial as generating function, and furthermore we see the crystal graph (with vertices the tableaux in $\mathcal{T}$ and edges all pairs of the form $(t, f_i t)$):

$$\begin{array}{ccccc} 1223 & \stackrel{1}{\leftarrow} & 1213 & & \\ 3\downarrow & & \downarrow 3 & & 1212 \\ 1224 & \stackrel{1}{\leftarrow} & 1214 & & \end{array}$$

The highest-weight elements in each component are the Yamanouchi words $\mathrm{Yam}(\mathcal{T}) = 3D\{1213, 1212\}$, and by looking at the corresponding lowest elements, we may deduce $\mathcal{S}(w) = 3D\kappa_{x_1 x_2^2 x_4} + \kappa_{x_1^2 x_2^2} = 3D\kappa_{1201} + \kappa_{2200}$. Lascoux and Schutzenberger [?] have obtained another characterization of such lowest-weight tableaux.

5 Weyl character formula I

Our next character formula mixes the rational terms of the Weyl character formula with the chains of Weyl group elements in the Standard Monomial Theory of Lakshmibai-Seshadri-Musili. For computations, this formula is very inefficient: a related expression with much fewer terms, more directly generalizing the Weyl formula, is given in the following section. The main advantage of the current expression is that one can use it to obtain character formulas for certain analogous polynomials associated with other root systems (though unfortunately these analogous polynomials do not seem to include the Schubert polynomials of other root systems).

Suppose we have any permutation with choice of reduced decomposition, $u = 3Ds_{i_1} \cdots s_{i_l}$, and a sequence of zeroes and ones $\epsilon = 3D(\epsilon_1, \epsilon_2, \cdots)$, $\epsilon_j \in= \{0, 1\}$. Denote

$$u^\epsilon = 3Ds_{i_1}^{\epsilon_1} \cdots s_{i_l}^{\epsilon_l},$$

a subword of u, not necessarily reduced.

Consider as before a monotone sequence of permutations $w_1, w_2, \ldots, w_r$ compatible with the Rothe diagram $D = 3DD(w)$, and choose a reduced decomposition for $u_j = 3Dw_{j-1}^{-1} w_j$. This gives a choice of reduced decomposition for each $w_j = 3Du_1 = \cdots u_j$, and in particular for $w_r = 3Ds_{i_1} \cdots s_{i_l}$. Let $v_k = 3Ds_{i_1} \cdots s_{i_k}$ for $k \leq l$. Recall that $\lambda_{c_j} = 3Dx_1 x_2 \cdots x_{c_j}$, the fundumental weight associated to the length c_j of the jth column of D (not counting empty columns).

Theorem 3

$$\mathcal{S}(w) = 3D\sum_\epsilon \frac{\prod_{j=3D1}^r w_j^\epsilon(\lambda_{c_j})}{\prod_{k=3D1}^l (1 - v_k^\epsilon(x_{i_k}^{-1} x_{i_k+1}))},$$

where the summation is over all 2^l sequences of zeroes and ones $\epsilon = 3D(\epsilon_1, \ldots, \epsilon_l)$.

This follows straight-forwardly from our Demazure formula, though it also has a geometric interpretation in terms of Bott-Samelson varieties (see below). **Example.** For the same $w = 3D24153$, we have the reduced decomposition $w_r = 3Dw_2 = 3Ds_1 s_3 s_2$, $l = 3D3$, so the Schubert polynomial is the following sum of 8 terms corresponding to $\epsilon = 3D(000), (001), (010), (011), \ldots$:

$$\begin{aligned} S(w) \quad = 3D \quad & \frac{x_1^2 x_2^2}{(1-x_1^{-1}x_2)(1-x_3^{-1}x_4)(1-x_2^{-1}x_3)} + \frac{x_1^2 x_2 x_3}{(1-x_1^{-1}x_2)(1-x_3^{-1}x_4)(1-x_3^{-1}x_2)} \\ & + \frac{x_1^2 x_2^2}{(1-x_1^{-1}x_2)(1-x_4^{-1}x_3)(1-x_2^{-1}x_4)} + \frac{x_1^2 x_2 x_4}{(1-x_1^{-1}x_2)(1-x_4^{-1}x_3)(1-x_4^{-1}x_2)} \\ & + \frac{x_1^2 x_2^2}{(1-x_2^{-1}x_1)(1-x_3^{-1}x_4)(1-x_1^{-1}x_3)} + \frac{x_1 x_2^2 x_3}{(1-x_2^{-1}x_1)(1-x_3^{-1}x_4)(1-x_3^{-1}x_1)} \\ & + \frac{x_1^2 x_2^2}{(1-x_2^{-1}x_1)(1-x_4^{-1}x_3)(1-x_1^{-1}x_4)} + \frac{x_1 x_2^2 x_4}{(1-x_2^{-1}x_1)(1-x_4^{-1}x_3)(1-x_4^{-1}x_1)}. \end{aligned}$$

6 Weyl Character Formula II

Finally, we state a result directly generalizing the Weyl character formula (Jacobi bialternant), reducing to it in case $\mathcal{S}(w)$ is a Schur polynomial.

The formula involves certain extensions of the Rothe diagram $D = 3DD(w)$. Define the Young diagram $\Phi = 3D\{(i,j) \mid 1 \le i \le j \le n-1\}$. Let the *flagged diagram* $\Phi \sqcup D$ be the concatenation of the two diagrams placed horizontally next to each other: that is, the columns of $\Phi \sqcup D$ are those of Φ followed by those of D.

Now, given $\Phi \sqcup D = 3D(C_1, \ldots, C_r)$, define the *blowup* of the flagged diagram $\widehat{\Phi \sqcup D} = 3D(C_1, \ldots, C_r, C'_1, C''_2, \ldots)$, where the extra columns are the intersections $\tilde{C} = 3DC_{i_1} \cap C_{i_2} \cap \cdots \subset \mathbf{N}$, for all lists $C_{i_1}, C_{i_2}, \ldots$ of columns of $\Phi \sqcup D$; but if an intersection $C_{i_1} \cap C_{i_2} \cap \cdots = 3DC_k$ is already a column of $\Phi \sqcup D$, then we do not append it.

Now let $\widetilde{D} = 3D\widehat{\Phi \sqcup D}$. Define a *standard tabloid* t of $\widetilde{D}$ to be a column-strict filling such that if C, C' are columns of $\widetilde{D}$ with C horizontally contained in C', then the numbers filling C all appear in the filling of C'. In symbols, $t : \widetilde{D} \to \{1, \ldots, n\}$, $t(i,j) < t(i+1,j)$ for all i, j, and $C \subset C' \Rightarrow t(C) \subset t(C')$.

For $1 \le i \ne j \le n$ and a tabloid t of $\widetilde{D}$, we define certain integers: $d_{ij}(t)$ is the number of connected components of the following graph. The vertices are columns C of $\widetilde{D}$ such that $i \in t(C)$, $j \notin t(C)$; the edges are (C, C') such that $C \subset C'$ or $C' \subset C$.

Finally, since there are inclusions of diagrams $D, \Phi \subset \widetilde{D} = 3D\widehat{\Phi \sqcup D}$, we have the *restrictions* of a tabloid t for $\widetilde{D}$ to D and Φ, which we denote $t|D$ and $t|\Phi$.

Theorem 4

$$\mathcal{S}(w) = 3D \sum_t \frac{x^{(t|D)}}{\prod_{i<j} (1 - x_i^{-1} x_j)^{d_{ij}(t)-1} \; (1 - x_j^{-1} x_i)^{d_{ji}(t)}},$$

where t runs over the standard tabloids for $\Phi \widehat{\sqcup} D$ such that $(t|\Phi)(i,j) = 3Di$ for all $(i,j) \in \Phi$.

Example. For the same $w = 3D24153$,

$$D = 3DD(w) = 3D \begin{array}{c|cc} 1 & \square & \\ 2 & \square & \square \\ 3 & & \\ 4 & & \square \end{array} \qquad \Phi \sqcup D = 3D \begin{array}{cccccc} \square & \square & \square & \square & \square & \\ & \square & \square & \square & \square & \square \\ & & \square & \square & & \\ & & & \square & & \square \end{array}$$

$$\Phi \widehat{\sqcup} D = 3D \begin{array}{c|ccccccc} 1 & \square & \square & \square & \square & \square & & \\ 2 & & \square & \square & \square & \square & \square & \square \\ 3 & & & \square & \square & & & \\ 4 & & & & \square & & \square & \end{array} .$$

There are six standard tabloids of the type occurring in the theorem. Their restrictions to the last three columns of $\Phi \widehat{\sqcup} D$ are:

$$\begin{array}{ccc} 1 & & \\ 2 & 1 & 1 \\ & & \\ & 2 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 1 & 1 \\ & & \\ & 3 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 1 & 1 \\ & & \\ & 4 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 1 & 2 \\ & & \\ & 2 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 2 & 2 \\ & & \\ & 3 & \end{array}, \quad \begin{array}{ccc} 1 & & \\ 2 & 2 & 2 \\ & & \\ & 4 & \end{array}.$$

The integers $d_{ij}(t)$ are 0, 1, or 2, and we obtain

$$\begin{aligned} S(w) \quad = 3D \quad & \frac{x_1^2 x_2^2}{(1-x_1^{-1}x_2)(1-x_2^{-1}x_3)(1-x_2^{-1}x_4)} + \frac{x_1^2 x_2 x_3}{(1-x_1^{-1}x_2)(1-x_3^{-1}x_4)(1-x_3^{-1}x_2)} \\ & + \frac{x_1^2 x_2 x_4}{(1-x_1^{-1}x_2)(1-x_4^{-1}x_2)(1-x_4^{-1}x_3)} + \frac{x_1^2 x_2^2}{(1-x_1^{-1}x_3)(1-x_1^{-1}x_4)(1-x_2^{-1}x_1)} \\ & + \frac{x_1 x_2^2 x_3}{(1-x_2^{-1}x_1)(1-x_3^{-1}x_4)(1-x_3^{-1}x_1)} + \frac{x_1 x_2^2 x_4}{(1-x_2^{-1}x_1)(1-x_4^{-1}x_1)(1-x_4^{-1}x_3)}. \end{aligned}$$

Note that some, but not all, of the above six terms are among the eight terms of the previous example. As in the case of the original Weyl character formula, it is not clear a priori why either of these rational functions should simplify to a polynomial (with positive integer coefficients).

This formula has been implemented by the author in Mathematica, = available on request.

7 Schur modules

The above formulas arise naturally from a Borel-Weil theory which relates Schubert polynomials with certain algebraic varieties similar to the Schubert varieties of $GL(n)$. (See [?].) The starting point of our theory is the result of Kraskiewicz and Pragacz which realizes Schubert polynomials as characters of "flagged Schur modules".

We shall write $G = 3DGL(n, \mathbf{C})$, $B = 3D$ the subgroup of upper triangular matrices, $V = 3D\mathbf{C}^n$ the defining representation, and V_i the subspace of V spanned by the first i coordinate vectors.

Let the diagram $D \subset \mathbf{N} \times \mathbf{N}$ be any set of squares (i, j) in the plane. Let Σ_D be the symmetric group permuting the squares of D, $\mathrm{Col}(D) \subset \Sigma_D$ the subgroup permuting the squares within each column, and $\mathrm{Row}(D)$ similarly for rows. Define (almost) idempotents α_D, β_D in the group algebra $\mathbf{C}[\Sigma_D]$ by

$$\alpha_D = 3D \sum_{\sigma \in \mathrm{Row}\, D} \sigma, \qquad \beta_D = 3D \sum_{\sigma \in \mathrm{Col}\, D} \mathrm{sgn}(\sigma)\sigma,$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation.

Σ_D acts on the right of the tensor product $V^{\otimes D}$ by permuting factors, and G and B act on the left by the diagonal action. These two actions commute. Define the *flagged Schur module* to be the B-stable subspace

$$S_D^B = 3D(\bigotimes_{(i,j) \in D} V_i \ ) \ \alpha_D \beta_D \subset V^{\otimes D}.$$

Theorem 5 *(Kraskiewicz-Pragacz) The Schubert polynomial for a permutation is the character of the flagged Schur module of its Rothe diagram:*

$$\mathcal{S}(w) = 3D \, \mathrm{tr}(\, \mathrm{diag}(x_1, \ldots, x_n) \mid S^B_{D(w)}).$$

Here the constant n is taken so that $D(w)$ has at most n rows.

Reiner, Shimozono, and the author have given a proof of this theorem using configuration varieties.

Example. For $w = 3D24153$, if we number the squares of $D(w)$ as

$$D = 3DD(w) = 3D \begin{array}{cc} \square & \\ \square & \square \\ & \\ & \square \end{array} = 3D \begin{array}{cc} 1 & \\ 2 & 3 \\ & \\ & 4 \end{array},$$

then we have (in cycle notation for $\Sigma_D \cong \Sigma_4$),

$$\begin{aligned} \alpha_D \beta_D &= 3D \quad (1 + (23))\ (1 - (12))(1 - (34)) \\ &= 3D \quad 1 + (23) - (12) - (34) - (132) - (234) + (12)(34) + (1324). \end{aligned}$$

Take $n = 3D5$, so that $V^{\otimes D} \cong (\mathbf{C}^5)^{\otimes 4}$, a 20-dimensional space with coordinate vectors $e_{i_1 i_2 i_3 i_4} = 3De_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}$, $i_1, \ldots, i_4 \in \{1, \ldots, 5\}$. Σ_D acts by $e_{i_1 i_2 i_3 i_4} \cdot \sigma = 3De_{i_{\sigma(1)} i_{\sigma(2)} i_{\sigma(3)} i_{\sigma(4)}}$, and $GL(5)$ acts diagonally on the tensor factors. By definition, S_D^B is spanned by vectors of the form

$$v_t = 3De_{t(1),t(2),t(3),t(4)} \, \alpha_D \beta_D,$$

for all fillings $t : D \to \{1, \ldots, 5\}$ with $t(1) \leq 1$, $t(2), t(3) \leq 2$, $t(4) \leq 4$, and it should follow from our theory that we obtain a basis if we take only the 6 fillings $t \in \mathcal{T}$, the set of tableaux defined in our second character formula. For instance, if $t = 3D(1213)$,

$$\begin{aligned} v_t &= 3D \quad e_{1213} + e_{1123} - e_{2113} - e_{1231} - e_{1123} - e_{1132} + e_{2131} + e_{1132} \\ &= 3D \quad e_{1213} - e_{2113} - e_{1231} + e_{2131}. \end{aligned}$$

8 Configuration varieties

Now we translate our algebra into geometry, realizing a flagged Schur module S_D^B as the space of sections of a line bundle over a (possibly singular) algebraic variety $\mathcal{F}_D^B$. Because the singularities are sufficiently tame, we can obtain non-trivial transformations of our problem by considering desingularizations of $\mathcal{F}_D^B$ and applying known character formulas for spaces of sections over smooth varieties. The first three formulas come from a Bott-Samelson resolution of $\mathcal{F}_D^B$ [?], the last from an analog of Zelevinsky's resolution [?].

Let $D \subset \mathbf{N} \times \mathbf{N}$ be a diagram with all its squares in rows $i = 3D1, \ldots, n$. Let $C_1, \ldots, C_r \subset \{1, \ldots, n\}$ be the columns of D, and for each $C = 3DC_j$, let $V_C = 3D\mathrm{Span}\,\{e_i \mid i \in C\} \subset \mathbf{C}^n$, a coordinate subspace of dimension $c_j = 3D|C_j|$. Consider the $= r$-tuple $(V_{C_1}, \ldots, V_{C_r})$ as a point in the product of Grassmannians $\mathrm{Gr}(D) = 3D\,\mathrm{Gr}(c_1, \mathbf{C}^n) \times \cdots \times \mathrm{Gr}(c_r, \mathbf{C}^n)$, and define the *flagged configuration variety*

$$\mathcal{F}_D^B = 3D\overline{B \cdot (V_{C_1}, \ldots, V_{C_r})} \subset \mathrm{Gr}(D),$$

the closure of the B-orbit of the above point. This is an irreducible projective variety, and the Schubert varieties of $GL(n)/B$ and $GL(n)/P$ are clearly special cases. $\mathcal{F}_D^B$ has a natural line bundle $\mathcal{L}_D$ defined by restricting the Plucker bundle $\mathcal{O}(1, \ldots, 1)$ over the product of Grassmannians $\mathrm{Gr}(D)$. These varieties are very tractable in the case of a Rothe diagram, and we may state the following Borel-Weil-Bott theorem.

Theorem 6 *(Magyar-van der Kallen) Let $D = 3DD(w)$ a Rothe diagram. Then $\mathcal{F}_D^B$ has rational singularities and is projectively normal with respect to $\mathcal{L}_D$. Furthermore, the space of global sections*

$$H^0(\mathcal{F}_D^B, \mathcal{L}_D) \cong (S_D^B)^*$$

as B-modules, and $H^i(\mathcal{F}_D^B, \mathcal{L}_D) = 3D0$ for all $i > 0$.

Now let $w_1, \ldots, w_r$ be the orthodontic sequence of $D = 3DD(w)$ and $w_r = 3Ds_{i_1} \cdots s_{i_l}$ the associated reduced decomposition. Thus the initial subwords are reduced decompositions $w_j = 3Ds_{i_1} \cdots s_{i_{l(j)}}$, where $l(j) = 3Dl(w_j)$. Consider the associated Bott-Samelson variety [?]

$$Z = 3DP_{i_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{i_l}/B,$$

where P_i denotes the maximal parabolic of $G = 3DGL(n)$ such that $G/P_i \cong \mathrm{Gr}(i, \mathbf{C}^n)$. Define the multiplication map

$$\begin{array}{cccc} \phi: & Z & \to & \mathrm{Gr}(D) \cong G/P_{c_1} \times G/P_{c_2} \times \cdots \times G/P_{c_r} = \\ & (p_1, \ldots, p_l) & \mapsto & (p_1 p_2 \cdots p_{l(1)},\ p_1 p_2 \cdots p_{l(2)},\ \ldots,\ p_1, \cdots p_{l(r)}). \end{array}$$

Theorem 7 *The map ϕ maps Z birationally onto $\mathcal{F}_D^B$, and so is a resolution of singularities. Furthermore, for all i,*

$$H^i(Z, \phi^* \mathcal{L}_D) \cong H^i(\mathcal{F}_D^B, \mathcal{L}_D).$$

From this, the computations of Demazure [?] on the Bott-Samelson variety directly imply our formula (1), and (2) follows by the theory of root operators. Formula (3) results from applying the Atiyah-Bott-Lefschetz fixed-point formula to Z.

Theorem 8 *Let $D = 3DD(w)$ and $\widetilde{D} = 3D\widehat{\Phi \sqcup D}$ the blowup diagram of section ??. Then the configuration variety $\mathcal{F}_{\widetilde{D}}^B$ is a smooth variety, and is a resolution of singularities of $\mathcal{F}_D^B$ via the natural projection map map $\psi : \mathrm{Gr}(\widetilde{D}) \to \mathrm{Gr}(D)$. Furthermore, for all i,*

$$H^i(\mathcal{F}_{\widetilde{D}}^B, \psi^* \mathcal{L}_D) \cong H^i(\mathcal{F}_D^B, \mathcal{L}_D).$$

Formula (4) now results from Atiyah-Bott-Lefschetz applied to $\mathcal{F}_{\widetilde{D}}^B$.

Example. In our case $w = 3D24153$, we may take $n = 3D4$, so that we have $\mathrm{Gr}(D) = 3D\,\mathrm{Gr}(2, \mathbf{C}^4) \times \mathrm{Gr}(2, \mathbf{C}^4)$, which we may think of as the variety of pairs of lines in $\mathbf{P}^3$. The B-orbit of the special point (V_{12}, V_{24}) is precisely the pairs of the form (V_{12}, W), where $W = 3D\langle v_1, v_2 =\rangle$, $v_1 \in V_{12}$, and v_1, v_2 are linearly independent. Thus

$$\mathcal{F}_D^B \cong \{W \in \mathrm{Gr}(2, \mathbf{C}^4) \mid \dim(V_{12} \cap W) \geq 1\},$$

the variety of lines in $\mathbf{P}^3$ which intersect the coordinate axis. (We can give such a description of $\mathcal{F}_D^B$ as configurations with intersection conditions for any Rothe diagram.) This is the singular Schubert variety in $\mathrm{Gr}(2, \mathbf{C}^4)$, the resolutions mentioned are the original Bott-Samelson and Zelevinsky resolutions, and there exist regular maps $Z \to \mathcal{F}_{\widetilde{D}}^B \to \mathcal{F}_D^B$.

$\mathbf{P}^3 plane\}$.

9 Generalizations

The above results can be used to compute the characters of Schur modules more general than those associated to Rothe diagrams. In fact, let us replace $D(w)$ by any diagram which satisfies the following *strictly separated* condition. For two sets $S, S' \subset \mathbf{N}$, we say $S < S'$ if $s < s'$ for all $s \in S$, $s' \in S'$. Now, a diagram $D = 3D(C_1, \ldots, C_r)$ with columns $C_j \subset \{1, \ldots, n\}$, is strictly separtated if, for any two columns C, C' of D, we have

$$(C \setminus C') < (C' \setminus C) \quad \text{or} \quad (C \setminus C') > (C' \setminus C) \ ,$$

where $C \setminus C'$ denotes the complement of C' in C. See [?], [?]. Replace the Schubert polynomial by the character of the flagged Schur module S_D^B. Then our Theorems 1, 2, 3, 6, and 7 remain valid: the orthodontic algorithm gives a sequence of Weyl group elements compatible with D, the first three formulas compute the character of S_D^B, and the associated configuration variety $\mathcal{F}_D^B$ satisfies the Borel-Weil Theorem and possesses a Bott-Samelson resolution.

Suppose that D satisfies an even stronger property, the *northwest condition*:

$$(i, j), (i', j') \in D \Rightarrow (\min(i, i'), \min(j, j')) \in D \ = .$$

Then Theorems 4 and 8 are valid as well: the fourth character formula is true for S_D^B, and the variety has a Zelevinsky resolution smaller than the Bott-Samelson one.

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