Coproducts and the **cd**-index

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Dedicated to Gian-Carlo Rota on his 2^6th birthday.

Abstract

The linear span of isomorphism classes of posets, $\mathcal{P}$, has a Newtonian coalgebra structure. We observe that the **ab**-index is a Newtonian coalgebra map from the vector space $\mathcal{P}$ to the algebra of polynomials in the non-commutative variables **a** and **b**. This enables us to obtain explicit formulas showing how the **cd**-index of the face lattice of a convex polytope changes when taking the pyramid and the prism of the polytope. As a corollary we have new recursion formulas for the **cd**-index of the Boolean algebra and the cubical lattice. Moreover, these operations also have interpretations for certain classes of permutations, including simsun and signed simsun permutations. Lastly, we prove an identity for the shelling components of the simplex.

Résumé

L'espace vectoriel $\mathcal{P}$ engendré par les classes d'isomorphismes des ensembles partiellement ordonnés a une structure d'une coalgèbre newtonienne. Nous observons que l'index **ab** est un homomorphisme de coalgèbre newtonienne de l'espace vectoriel $\mathcal{P}$ à l'algèbre des polynômes en les variables noncommutatives **a** et **b**. Cette observation nous permet d'obtenir des formules explicites montrant comment l'index **cd** du treillis des faces d'un polytope convexe change quand on prend la pyramide et le prisme du polytope. Comme corollaire nous avons des nouvelles formules récursives pour l'index **cd** de l'algèbre de Boole et du treillis cubique. De plus, ces opérations ont aussi des interprétations pour certaines classes de permutations, comportent les permutations de "simsun" et leur variante signée. Finalement, nous prouvons une identité pour les composantes d'effeuillage du simplexe.

1 Introduction

The **cd**-index is an efficient way to encode the flag f-vector (equivalently the flag h-vector) of an Eulerian poset. It also gives an explicit basis for the generalized Dehn-Sommerville equations, also known as the Bayer-Billera relations [1]. An important example of an Eulerian poset is the face lattice of a convex polytope.

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In this paper we study how the **cd**-index of the face lattice of a convex polytope changes after applying each of the following geometric operations to the convex polytope itself: taking the pyramid, taking the prism, truncating at a vertex, and pasting two polytopes together at a common facet. All four of these operations act on the face lattice of the polytope. The change in the **cd**-index from the pasting operation follows from a result of Stanley [17, Lemma 2.1]. Similarly the change from truncating at a vertex follows from the same result of Stanley and the pyramid and prism operations.

To understand how the **cd**-index changes under the prism and pyramid operations, we consider $\mathcal{P}$, the linear span of isomorphism classes of graded posets. This vector space is an algebra under the star product $*$ of posets, first described by Stanley [17]. More importantly, $\mathcal{P}$ has a coalgebra structure. The pair formed by the star product $*$ and the coproduct Δ do not form a bialgebra, but instead a Newtonian coalgebra, a concept introduced by Joni and Rota [12]. The main observation we make is that the **cd**-index is a Newtonian coalgebra map from the vector space $\mathcal{E}$ spanned by all isomorphism classes of Eulerian posets to the algebra $\mathcal{F}$ of polynomials in the non-commutative variables **c** and **d**. We thus obtain that the prism operation corresponds to a certain derivation D on **cd**-polynomials, and the pyramid operation corresponds to a second derivation G. Hence given the **cd**-index of a polytope, we may easily compute the **cd**-index of the prism and the pyramid of the polytope with the help of these two derivations. Using these two derivations, we obtain new explicit recursion formulas for the **cd**-index of the Boolean algebra B_n and the cubical lattice C_n.

There is a relation between the **cd**-index of the Boolean algebra B_n and certain classes of permutations. For instance, the **cd**-index of B_n is a refined enumeration of André permutations [14]. Similarly, it is also a refined enumeration of simsun permutations, first defined by Simion and Sundaram [19, 20]. Another known example of a poset–permutations pair is the cubical lattice and signed André permutations [7, 14]. This motivates us to ask the following question. Given an Eulerian poset P, is it possible to find a canonical class of permutations which correspond to the **cd**-index of the poset P? We show that given a poset–permutations pair (P, T), we can construct a class of permutations corresponding to the pyramid of P. A similar signed result holds for the prism of P. The simsun permutations may be built up by repeated use of this correspondence. Also, we define signed simsun permutations, which correspond to the cubical lattice C_n.

In [17] Stanley studies the shelling components of a simplex and their **cd**-indexes, given by a sum of $\check{\Phi}_i^n$'s. Using our techniques we obtain a recursion formula for $\check{\Phi}_i^n$. As a corollary to this recursion we prove a version of Stanley's conjecture [17, Conjecture 3.1] concerning the correspondence between simsun permutations and the $\check{\Phi}_i^n$'s.

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2 Newtonian coalgebras

Let k be a field of characteristic 0. Let V be a vector space over the field k. A product on the vector space V is a linear map $\mu : V \otimes V \longrightarrow V$. The product μ is associative if $\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$. Similarly, a coproduct on the vector space V is a linear map $\Delta : V \longrightarrow V \otimes V$. The coproduct Δ is coassociative if $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$.

Definition 2.1 *Let V be a vector space with an associative product μ and a coassociative coproduct Δ. We call the triplet (V, μ, Δ) a* Newtonian coalgebra *if it satisfies the identity*

$$\Delta \circ \mu = (1 \otimes \mu) \circ (\Delta \otimes 1) + (\mu \otimes 1) \circ (1 \otimes \Delta).$$

Recall the Sweedler notation of a coproduct Δ. That is, we write $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$. Then the Newtonian condition may be written

$$\Delta(x \cdot y) = \sum_x x_{(1)} \otimes (x_{(2)} \cdot y) + \sum_y (x \cdot y_{(1)}) \otimes y_{(2)}.$$

Observe that this identity is a generalization of the product rule for a derivative. In fact, for any element $v \in V$, the linear map $x \longmapsto D_v(x) = \sum_x x_{(1)} \cdot v \cdot x_{(2)}$ is a derivative on the algebra (V, μ). That is, $D_v(x \cdot y) = D_v(x) \cdot y + x \cdot D_v(y)$, or $D_v \circ \mu = \mu \circ (D_v \otimes 1 + 1 \otimes D_v)$.

The definition of Newtonian coalgebra originated from Joni and Rota [12] under the name infinitesimal coalgebra. Our definition is from [6]. The first in–depth study of a Newtonian coalgebra was by Hirschhorn and Raphael [11], who studied the coalgebra on $k[x]$ where the coproduct is given by $\Delta(x^n) = \sum_{i+j=n-1} x^i \otimes x^j$. In this section we will introduce two important examples of Newtonian coalgebras, which we denote by $\mathcal{A}$ and $\mathcal{P}$. These two examples appear in [6].

Let $\mathcal{A} = k\langle \mathbf{a}, \mathbf{b} \rangle$ be the polynomial algebra in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$. Let the product on $\mathcal{A}$ be the ordinary multiplication. Define the coproduct Δ on a monomial $v_1 \cdot v_2 \cdots v_n$ by

$$\Delta(v_1 \cdot v_2 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n.$$

It is easy to see that this is a Newtonian coalgebra. The Newtonian coalgebra $\mathcal{A}$ is naturally graded, that is, we may write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$, where $\mathcal{A}_n$ is spanned by monomials of degree n. Then $\dim(\mathcal{A}_n) = 2^n$ and we have $\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ and $\Delta(\mathcal{A}_n) \subseteq \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$.

We will now consider graded posets P whose minimal element differs from its maximal element. Hence the rank of such a poset is at least 1. (See [16] for terminology on posets.) If two posets are isomorphic we say that they have the same *type*. We denote the type of a poset P by $\overline{P}$. Let $\mathcal{P}$ be the vector space over the field k spanned by all types of posets.

We define a coproduct on the vector space $\mathcal{P}$ by

$$\Delta(\overline{P}) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \overline{[\hat{0}, x]} \otimes \overline{[x, \hat{1}]},$$

and extend this definition by linearity. Observe that this coproduct differs from the ordinary coproduct that is defined on the reduced incidence Hopf algebra of posets; see [4, 12, 15].

Let P and Q be two graded posets. We define their star product, $R = P * Q$, by letting R be the set $(P - \{\hat{1}\}) \cup (Q - \{\hat{0}\})$ and defining the order relation on R by $x \leq_R y$ if (i) $x, y \in P$ and $x \leq_P y$, (ii) $x, y \in Q$ and $x \leq_Q y$, or (iii) $x \in P$ and $y \in Q$. This product was first mentioned in [17]. Observe that the rank of the poset $P * Q$ is given by $\rho(P) + \rho(Q) - 1$. The product $*$ extends naturally to a product on $\mathcal{P}$.

Proposition 2.2 (Ehrenborg and Hetyei) *The triplet $(\mathcal{P}, *, \Delta)$ is a Newtonian coalgebra.*

This Newtonian algebra has a natural grading, $\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n$, where $\mathcal{P}_n$ is the linear span of types of graded posets of rank $n+1$. Then we have $\mathcal{P}_i * \mathcal{P}_j \subseteq \mathcal{P}_{i+j}$ and $\Delta(\mathcal{P}_n) \subseteq \bigoplus_{i+j=n-1} \mathcal{P}_i \otimes \mathcal{P}_j$.

There are two other products on posets that we consider. First, there is the Cartesian product of posets, which we denote by $P \times Q$. Secondly, define the diamond product by $P \diamond Q = (P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$. The diamond product corresponds to the Cartesian product of convex polytopes, that is, $\mathcal{L}(V \times W) = \mathcal{L}(V) \diamond \mathcal{L}(W)$, where V and W are two convex polytopes and $\mathcal{L}(V)$ denotes the face lattice of V. Both of these products on posets extend naturally to the linear space $\mathcal{P}$, and we have that $\mathcal{P}_i \times \mathcal{P}_j \subseteq \mathcal{P}_{i+j+1}$ and $\mathcal{P}_i \diamond \mathcal{P}_j \subseteq \mathcal{P}_{i+j}$.

3 The cd-index of Eulerian posets

To each graded poset P we will assign a non-commutative polynomial in the variables $\mathbf{a}$ and $\mathbf{b}$ called the $\mathbf{ab}$-index. Let P be a graded poset of rank $n+1$. To every chain $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ of the poset P we associate a *weight* $w_P(c) = w(c) = z_1 \cdots z_n$, where

$$z_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

Observe that the chain $\{\hat{0} < \hat{1}\}$ receives the weight $(\mathbf{a} - \mathbf{b})^n$ and a maximal chain has weight $\mathbf{b}^n$. Note also that the degree of the weight $w(c)$ is n. Define the $\mathbf{ab}$-*index* of the poset P to be the sum

$$\Psi(P) = \sum_c w(c),$$

where c ranges over all chains $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ in the poset P.

By linearity we may extend the map Ψ to a linear map $\Psi : \mathcal{P} \longrightarrow \mathcal{A}$.

Proposition 3.1 *The linear map $\Psi : \mathcal{P} \longrightarrow \mathcal{A}$ is a Newtonian coalgebra map. That is, $\Psi \circ \mu = \mu \circ (\Psi \otimes \Psi)$ and $\Delta \circ \Psi = (\Psi \otimes \Psi) \circ \Delta$.*

The first identity is equivalent to $\Psi(P * Q) = \Psi(P) \cdot \Psi(Q)$, for two posets P and Q. This is due to Stanley; see [17, Lemma 1.1].

Recall that a poset P is *Eulerian* if the Möbius function μ on any interval $[x, y]$ in P is given by $\mu(x, y) = (-1)^{\rho(x,y)}$. Let $\mathcal{E}$ be the subspace of $\mathcal{P}$ spanned by all types of Eulerian posets. It is easy to see that $\mathcal{E}$ is closed under the product $*$ and the coproduct Δ. Hence $\mathcal{E}$ forms a Newtonian subalgebra of $\mathcal{P}$. Observe that $\mathcal{E}$ is also closed under the Cartesian product and the diamond product.

Fine observed that the $\mathbf{ab}$-index of an Eulerian poset may be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$; see [2]. When the $\mathbf{ab}$-index can be written as a polynomial in $\mathbf{c}$ and $\mathbf{d}$, we call this polynomial the $\mathbf{cd}$-*index*. See Stanley [17] for an elementary proof of this fact.

Let $\mathcal{F}$ be the subalgebra of $\mathcal{A}$ spanned by the elements $\mathbf{c}$ and $\mathbf{d}$. $\mathcal{F}$ is closed under the coproduct Δ, since $\Delta(\mathbf{c}) = \Delta(\mathbf{a}+\mathbf{b}) = 1\otimes 1 + 1\otimes 1 = 2\cdot 1\otimes 1$ and $\Delta(\mathbf{d}) = \Delta(\mathbf{ab}+\mathbf{ba}) = \mathbf{a}\otimes 1 + 1\otimes \mathbf{b} + \mathbf{b}\otimes 1 + 1\otimes \mathbf{a} = \mathbf{c}\otimes 1 + 1\otimes \mathbf{c}$. The Newtonian coalgebra $\mathcal{F}$ inherits the grading from $\mathcal{A}$. That is, $\mathcal{F} = \bigoplus_{n\geq 0}\mathcal{F}_n$, where $\mathcal{F}_n \subseteq \mathcal{A}_n$. It is easy to see that $\dim(\mathcal{F}_n) = f_{n+1}$, where f_n is the nth Fibonacci number. (Recall f_n is defined recursively by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$.)

The important observation to make here is that the linear map $\Psi : \mathcal{P} \longrightarrow \mathcal{A}$ restricts to a linear map from the Newtonian coalgebra $\mathcal{E}$ to the Newtonian coalgebra $\mathcal{F}$. Note that there exist posets which are not Eulerian, but whose $\mathbf{ab}$-index may be expressed in terms of $\mathbf{c}$ and $\mathbf{d}$.

Let V be a convex polytope. Then the face lattice of V, $\mathcal{L}(V)$, is an Eulerian poset. Hence we may compute the $\mathbf{cd}$-index of $\mathcal{L}(V)$, that is, $\Psi(\mathcal{L}(V))$. For the remainder of this paper we will write $\Psi(V)$ instead of the more cumbersome $\Psi(\mathcal{L}(V))$.

4 The pyramid and the prism of a polytope

There are two well-known operations defined on convex polytopes: the pyramid and the prism. From a convex polytope V we may construct the pyramid of V, $\mathrm{Pyr}(V)$, and the prism of V, $\mathrm{Prism}(V)$. See [22] for a formal treatment of these operations. Let B_n be the Boolean algebra of rank n, that is, the face lattice of the simplex of dimension $n-1$. Also let C_n be the cubical lattice of rank $n+1$, namely the face lattice of an n-dimensional cube.

Proposition 4.1 *Let V be a convex polytope. Then the face lattice of the pyramid of V and the face lattice of the prism of V are given by $\mathcal{L}(\mathrm{Pyr}(V)) = \mathcal{L}(V)\times B_1$ and $\mathcal{L}(\mathrm{Prism}(V)) = \mathcal{L}(V)\diamond B_2$.*

Two natural questions occur now. Given the $\mathbf{cd}$-index $\Psi(V)$, are we able to compute $\Psi(\mathrm{Pyr}(V)) = \Psi(\mathcal{L}(V)\times B_1)$ and $\Psi(\mathrm{Prism}(V)) = \Psi(\mathcal{L}(V)\diamond B_2)$?

Proposition 4.2 *Let P be a graded poset. Then we have that*

$$\begin{aligned}
\Psi(P\times B_1) &= \frac{1}{2}\left[\Psi(P)\cdot\mathbf{c} + \mathbf{c}\cdot\Psi(P) + \sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}} \Psi([\hat{0},x])\cdot\mathbf{d}\cdot\Psi([x,\hat{1}])\right],\\
\Psi(P\diamond B_2) &= \Psi(P)\cdot\mathbf{c} + \sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}} \Psi([\hat{0},x])\cdot\mathbf{d}\cdot\Psi([x,\hat{1}]).
\end{aligned}$$

Since $C_{n+1} = C_n \diamond B_2$, Proposition 4.2 gives a recursion formula for the $\mathbf{cd}$-index of the cubical lattice C_n which was first developed by Purtill [14]. The second part of Proposition 4.2 may be generalized in the following manner. Let A_r be a graded poset of rank 2 which has r atoms (and hence r coatoms). Note that $A_2 = B_2$. Let $\overline{\mathbf{c}}_r = \mathbf{a} + (r-1)\cdot\mathbf{b}$ and $\overline{\mathbf{d}}_r = \mathbf{ab} + (r-1)\cdot\mathbf{ba}$.

Proposition 4.3 *Let P be a graded poset. Then we have that*

$$\Psi(P \diamond A_r) = \Psi(P)\cdot\overline{\mathbf{c}}_r + \sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}} \Psi([\hat{0},x])\cdot\overline{\mathbf{d}}_r\cdot\Psi([x,\hat{1}]).$$

This proposition generalizes the recursion for the $\mathbf{r}$-$\mathbf{cd}$-index given in [7].

Define a linear operator $D : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D(w) = \sum_w w_{(1)}\cdot\mathbf{d}\cdot w_{(2)}.$$

Recall that D is a derivation. We could have defined D directly as a derivation on $\mathcal{A}$ such that $D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab} + \mathbf{ba} = \mathbf{d}$. Note that D is also a derivation on $\mathcal{F}$ since $D(\mathbf{c}) = 2\cdot\mathbf{d}$ and $D(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$.

Combining Proposition 4.2 with the fact that Ψ is a Newtonian coalgebra map we obtain

Theorem 4.4 *Let P be a graded poset. Then we have*

$$\begin{aligned}\Psi(P\times B_1) &= \frac{1}{2}[\Psi(P)\cdot\mathbf{c} + \mathbf{c}\cdot\Psi(P) + D(\Psi(P))],\\ \Psi(P\diamond B_2) &= \Psi(P)\cdot\mathbf{c} + D(\Psi(P)).\end{aligned}$$

Similarly, let V be a convex polytope. Then we obtain

$$\begin{aligned}\Psi(\mathrm{Pyr}(V)) &= \frac{1}{2}[\Psi(V)\cdot\mathbf{c} + \mathbf{c}\cdot\Psi(V) + D(\Psi(V))],\\ \Psi(\mathrm{Prism}(V)) &= \Psi(V)\cdot\mathbf{c} + D(\Psi(V)).\end{aligned}$$

This theorem gives us a new recursion formula for the $\mathbf{cd}$-index of the cubical lattice C_n. Directly we have

$$\Psi(C_{n+1}) = \Psi(C_n)\cdot\mathbf{c} + D(\Psi(C_n)).$$

This is a different recursion formula than Purtill obtained in [14].

Example 4.5 *Let the convex polytope V be a 3-cube with a vertex cut off. The polytope V has 10 vertices and 7 facets. Hence the $\mathbf{cd}$-index of V is $\Psi(V) = \mathbf{c}^3 + (10-2)\mathbf{dc} + (7-2)\mathbf{cd} = \mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}$. We have*

$$\begin{aligned}\Delta(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) &= 7\cdot\mathbf{c}^2\otimes 1 + 15\cdot\mathbf{c}\otimes\mathbf{c} + 10\cdot 1\otimes\mathbf{c}^2 + 16\cdot\mathbf{d}\otimes 1 + 10\cdot 1\otimes\mathbf{d}.\\ D(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) &= 7\cdot\mathbf{c}^2\mathbf{d} + 15\cdot\mathbf{cdc} + 10\cdot\mathbf{dc}^2 + 26\cdot\mathbf{d}^2.\end{aligned}$$

Hence the $\mathbf{cd}$-index of the prism of V is equal to

$$\Psi(\mathrm{Prism}(V)) = \mathbf{c}^4 + 7\cdot\mathbf{c}^2\mathbf{d} + 20\cdot\mathbf{cdc} + 18\cdot\mathbf{dc}^2 + 26\cdot\mathbf{d}^2.$$

5 The coproduct Γ

On the algebra $\mathcal{F}$ define two Newtonian coproducts, Γ and Γ', by

$$\begin{array}{rclcrcl} \Gamma(\mathbf{c}) & = & 1 \otimes 1, & \qquad & \Gamma'(\mathbf{c}) & = & 1 \otimes 1, \\ \Gamma(\mathbf{d}) & = & \mathbf{c} \otimes 1, & & \Gamma'(\mathbf{d}) & = & 1 \otimes \mathbf{c}. \end{array}$$

Observe that $\Delta = \Gamma + \Gamma'$. It is interesting to note that neither Γ or Γ' can be extended nicely to a Newtonian coproduct on $\mathcal{A}$. Define G, a linear operator from $\mathcal{F}$ to itself, by

$$G(w) = \sum_w w^{\Gamma}_{(1)} \cdot \mathbf{d} \cdot w^{\Gamma}_{(2)},$$

where the Sweedler notation applies to the coproduct Γ. Similarly, let

$$G'(w) = \sum_w w^{\Gamma'}_{(1)} \cdot \mathbf{d} \cdot w^{\Gamma'}_{(2)}.$$

The linear maps G and G' are derivations on $\mathcal{F}$ such that $G(\mathbf{c}) = G'(\mathbf{c}) = \mathbf{d}$, $G(\mathbf{d}) = \mathbf{cd}$, and $G'(\mathbf{d}) = \mathbf{dc}$. Also we have that $D = G + G'$. More importantly, we have the following lemma.

Lemma 5.1 *For all* $\mathbf{cd}$*-monomials* w *we have* $w \cdot \mathbf{c} + G(w) = \mathbf{c} \cdot w + G'(w)$.

Theorem 5.2 *Let* P *be an Eulerian poset. Then we have*

$$\Psi(P \times B_1) = \Psi(P) \cdot \mathbf{c} + G(\Psi(P)).$$

Similarly, let V *be a convex polytope. Then we obtain*

$$\Psi(\mathrm{Pyr}(V)) = \Psi(V) \cdot \mathbf{c} + G(\Psi(V)).$$

This theorem gives us a new recursion formula for the $\mathbf{cd}$-index of the Boolean algebra B_n different from the one Purtill obtained in [14]. It is

$$\Psi(B_{n+1}) = \Psi(B_n) \cdot \mathbf{c} + G(\Psi(B_n)).$$

Example 5.3 *Let* V *be the polytope in Example 4.5, with* $\mathbf{cd}$*-index* $\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}$. *We have*

$$\begin{array}{rcl} \Gamma(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) & = & 6 \cdot \mathbf{c}^2 \otimes 1 + 9 \cdot \mathbf{c} \otimes \mathbf{c} + 1 \otimes \mathbf{c}^2 + 8 \cdot \mathbf{d} \otimes 1 + 5 \cdot 1 \otimes \mathbf{d}. \\ G(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) & = & 6 \cdot \mathbf{c}^2\mathbf{d} + 9 \cdot \mathbf{cdc} + \mathbf{dc}^2 + 13 \cdot \mathbf{d}^2. \end{array}$$

Hence the $\mathbf{cd}$*-index of the pyramid of* V *is given by*

$$\Psi(\mathrm{Pyr}(V)) \;=\; \mathbf{c}^4 + 6 \cdot \mathbf{c}^2\mathbf{d} + 14 \cdot \mathbf{cdc} + 9 \cdot \mathbf{dc}^2 + 13 \cdot \mathbf{d}^2.$$

6 Other operations on polytopes

Let W be an n-dimensional convex polytope with vertex v. Let $\mathbf{u}$ be a vector such that $W \cap \{\mathrm{x} \in \mathbb{R}^n \ : \ \mathbf{u} \cdot \mathrm{x} \geq c\} = \{v\}$. The vertex figure V of W at the vertex v is defined as the polytope $V = W \cap \{\mathrm{x} \in \mathbb{R}^n \ : \ \mathbf{u} \cdot \mathrm{x} = c - \epsilon\}$, for small enough $\epsilon > 0$. We define the truncated polytope $\widehat{W}$ as the polytope $W \cap \{\mathrm{x} \in \mathbb{R}^n \ : \ \mathbf{u} \cdot \mathrm{x} \leq c - \epsilon\}$. The combinatorial structure of V and $\widehat{W}$ only depends on W and v, and not on $\mathbf{u}$, c, or ϵ.

Proposition 6.1 *Let W be a convex polytope and let v be a vertex of W. Assume that the vertex figure at v is the polytope V. Let $\widehat{W}$ be the polytope W with the vertex v cut off. Then the difference in the* $\mathbf{cd}$*-index of $\widehat{W}$ and W is given by*

$$\Psi(\widehat{W}) - \Psi(W) = D(w) - G(w) = G'(w),$$

where $w = \Psi(V)$.

Example 6.2 *Let W be a four-dimensional convex polytope such that at the vertex v it has the vertex figure V, where V is the three-dimensional polytope mentioned in Examples 4.5 and 5.3. Hence*

$$\begin{aligned} \Psi(\widehat{W}) - \Psi(W) &= D(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) - G(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) \\ &= \mathbf{c}^2\mathbf{d} + 6 \cdot \mathbf{cdc} + 9 \cdot \mathbf{dc}^2 + 13 \cdot \mathbf{d}^2. \end{aligned}$$

Another operation on polytopes is pasting two polytopes along a common facet. Let V and W be two polytopes such that they intersect in a facet F, that is, $V \cap W = F$. A corollary of Stanley's result [17, Lemma 2.1] is that the $\mathbf{cd}$-index of the union $V \cup W$ is given by

$$\Psi(V \cup W) = \Psi(V) + \Psi(W) - \Psi(F) \cdot \mathbf{c}.$$

The *Minkowski sum* of two subsets X and Y of $\mathbb{R}^n$ is defined as

$$X + Y = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^n \ : \ \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

Notably, the Minkowski sum of two convex polytopes is another convex polytope. For a vector x we denote the set $\{\lambda \cdot \mathrm{x} \ : \ 0 \leq \lambda \leq 1\}$ by $[\mathbf{0}, \mathrm{x}]$. We say that the non-zero vector x lies in general position with respect to the convex polytope V if the line $\{\lambda \cdot \mathrm{x} + \mathbf{u} \in \mathbb{R}^n \ : \ \lambda \in \mathbb{R}\}$ intersects the boundary of the polytope V in at most two points for all $\mathbf{u} \in \mathbb{R}^n$.

Proposition 6.3 *Let V be an n-dimensional convex polytope and x a non-zero vector that lies in general position with respect to the polytope V. Let H be a hyperplane orthogonal to the vector x, and let $\mathrm{Proj}(V)$ be the orthogonal projection of V onto the hyperplane H. Observe that $\mathrm{Proj}(V)$ is an $(n-1)$-dimensional convex polytope. Then the* $\mathbf{cd}$*-index of the Minkowski sum of V and $[\mathbf{0}, \mathrm{x}]$ is given by*

$$\Psi(V + [\mathbf{0}, \mathrm{x}]) = \Psi(V) + D(\Psi(\mathrm{Proj}(V))).$$

7 On Simsun permutations

Let S be a set such that $S \cup \{0\}$ is a linearly ordered set.

Definition 7.1 *An* augmented permutation π *of length n on S is a list* $\pi = (0 = s_0, s_1, \ldots, s_n)$, *where $s_1, \ldots, s_n$ are n distinct elements from the set S.*

The descent set of the augmented permutation π is the set $D(\pi) = \{i \ : \ s_{i-1} > s_i\}$. Observe the descent set of π is a subset of $[n] = \{1, \ldots, n\}$. We say that π has no double descents if there is no index i such that $s_i > s_{i+1} > s_{i+2}$. The *variation* of a permutation π is given by $U(\pi) = u_{D(\pi)}$, where u_S is the **ab**-monomial $u_1 \cdots u_n$ such that $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$.

Let $R_n(S)$ be the set of augmented permutations on the set S of length n so that any such permutation begins with an ascent and has no double descents. We let $R_0(S)$ be the singleton set containing the permutation (0). For an augmented permutation π in $R_n(S)$ we define the *reduced variation* of π, which we denote by $V(\pi)$, by replacing each **ab** in $U(\pi)$ with **d** and then replacing each remaining **a** by a **c**. For a subset T of $R_n(S)$ we define $V(T) = \sum_{\pi \in T} V(\pi)$.

We now ask the following question. Given an Eulerian poset P of rank $n+1$, is it possible to find in a canonical manner a linearly ordered set S and a subset T of $R_n(S)$ such that $\Psi(P) = V(T)$? Examples of such posets and permutation sets are the Boolean algebra and André permutations, and the cubical lattice and signed André permutations. See [7, 14]. For more refined identities using such a poset–permutation set correspondence, see [5, 10, 17].

We will now define three operations on permutations. These will give us a partial answer to our question. For a permutation $\pi = (0, s_1, \ldots, s_n)$ and an element x, we define the concatenation $\pi \cdot x = (0, s_1, \ldots, s_n, x)$. We extend this notion for a class T of permutations by $T \cdot x = \{\pi \cdot x \ : \ \pi \in T\}$. Let M be an element larger than all the elements in the linear order $S \cup \{0\}$. For T a subset of $R_n(S)$ we have that $T \cdot M \subseteq R_{n+1}(S \cup \{M\})$. Moreover, we have that $V(T \cdot M) = V(T) \cdot \mathbf{c}$.

We will now define the insert operation. Let M be as just defined and let m be an element smaller than all the elements in $S \cup \{0\}$. For $T \subseteq R_n(S)$ and $x \in \{m, M\}$, we define $\mathrm{Insert}(T, x)$ to be set of all augmented permutations $(0, s_1, \ldots, s_i, x, s_{i+1}, \ldots, s_n)$ such that

1. $(0, s_1, \ldots, s_n) \in T$,
2. $(0, s_1, \ldots, s_i, x, s_{i+1}, \ldots, s_n) \in R_{n+1}(S \cup \{x\})$,
3. if x is the maximal element M, then $i \neq n$, and
4. if x is the minimal element m, then $i \neq 0$.

That is, we insert x into the permutation $(0, s_1, \ldots, s_n) \in T$ such that no double descents occur and we do not allow the maximal element at the end nor the minimal element at the beginning of the permutation. Observe that we have $\mathrm{Insert}(T, M)$ and $\mathrm{Insert}(T, m) \subseteq R_{n+1}(S \cup \{M, m\})$.

Lemma 7.2 *For $T \subseteq R_n(S)$ we have $V(\mathrm{Insert}(T, M)) = G(V(T))$ and $V(\mathrm{Insert}(T, m)) = G'(V(T))$.*

Theorem 7.3 *Let P be an Eulerian poset of rank $n+1$. Let $S \cup \{0\}$ be a linearly ordered set, and let T be a subset of $R_n(S)$ such that $\Psi(P) = V(T)$. Introduce a new maximal element M and a minimal element m to the set $S \cup \{0\}$. Then the following identities hold:*

$$\begin{aligned} \Psi(P \times B_1) &= V\left(\operatorname{Insert}(T, M) \cup T \cdot M\right), \\ \Psi(P \diamond B_2) &= V\left(\operatorname{Insert}(T, M) \cup \operatorname{Insert}(T, m) \cup T \cdot M\right). \end{aligned}$$

Simion and Sundaram defined a class of permutations called simsun permutations; see [19, page 267] and [20]. We will now see how simsun permutations are closely related with the operations $\operatorname{Insert}(T, n)$ and $T \cdot n$ on permutations.

A *simsun permutation* π of length n is a augmented permutation $\pi = (0, s_1, \ldots, s_n)$ on the set $\{1, \ldots, n\}$ of length n such that for all $0 \leq k \leq n$ if we remove the k entries $n, n-1, \ldots, n-k+1$ from the permutation π, the resulting permutation does not have any double descents. Let $\mathcal{S}_n$ denote the set of all simsun permutations of length n. We have that $\mathcal{S}_n \subseteq R_n(\{1, \ldots, n\})$.

Similarly, we may define a *signed simsun permutation* π of length n as an augmented permutation of length n on the set $\{-n, \ldots, -1, 1, \ldots, n\}$ such that exactly one of the elements $+i$ and $-i$ occurs in the permutation and for all $0 \leq k \leq n$ if we remove the k entries $\pm n, \pm(n-1), \ldots, \pm(n-k+1)$ from the permutation π, the resulting permutation belongs to $R_{n-k}(\{-(n-k), \ldots, -1, 1, \ldots, n-k\})$. Let $\mathcal{S}_n^{\pm}$ denote the set of all signed simsun permutations of length n.

Corollary 7.4 *The sets of all simsun permutations and all signed simsun permutations satisfy the following recursions:*

$$\begin{aligned} \mathcal{S}_n &= \operatorname{Insert}(\mathcal{S}_{n-1}, n) \cup \mathcal{S}_{n-1} \cdot n, \\ \mathcal{S}_n^{\pm} &= \operatorname{Insert}(\mathcal{S}_{n-1}^{\pm}, n) \cup \operatorname{Insert}(\mathcal{S}_{n-1}^{\pm}, -n) \cup \mathcal{S}_{n-1}^{\pm} \cdot n. \end{aligned}$$

Thus we have $\Psi(B_{n+1}) = V(\mathcal{S}_n)$ and $\Psi(C_n) = V(\mathcal{S}_n^{\pm})$.

8 The shelling components of the simplex

Stanley [17] studies the shelling components of the simplex in order to obtain a formula for the **cd**-index of a simplicial Eulerian poset. Namely, if P is a simplicial Eulerian poset of rank $n+1$ with h-vector $(h_0, \ldots, h_n)$ then the **cd**-index of P is given by $\Psi(P) = \sum_{i=0}^{n-1} h_i \cdot \check{\Phi}_i^n$. By using the techniques we have developed, we now study the **cd**-polynomials $\check{\Phi}_i^n$.

Recall that B_n is the Boolean algebra, that is, all the subsets of $\{1, \ldots, n\}$ ordered by inclusion. Let c_i be the coatom $\{1, \ldots, n\} - \{n+1-i\}$. Similarly, for $i \neq j$ let $c_{i,j}$ be the element $\{1, \ldots, n\} - \{n+1-i, n+1-j\}$, that is, $c_{i,j}$ is the intersection of the two sets c_i and c_j. Define the poset $B'_{n,i}$ for $1 \leq i \leq n-1$ by

$$B'_{n,i} = \bigcup_{j=1}^{i} [\emptyset, c_j] \cup \{\{1, \ldots, n\}\}.$$

That is, $B'_{n,i}$ consists of the maximal element $\{1, \ldots, n\}$ and all the elements below the coatoms $c_1, \ldots, c_i$. Since the elements $c_{j,k}$, where $1 \le j \le i$ and $i+1 \le k \le n$, are only covered by one element in $B'_{n,i}$, we know that $B'_{n,i}$ is not an Eulerian poset. However we can obtain an Eulerian poset by adding an element γ in the following manner. Let $B_{n,i}$ be the poset $B'_{n,i} \cup \{\gamma\}$, where the coatom γ covers all elements $c_{j,k}$ with $1 \le j \le i$ and $i+1 \le k \le n$. The poset $B_{n,i}$ is Eulerian. Observe that $B_{n,1} = B_{n-1} * B_2$ and $B_{n,n-1} = B_n$. Stanley defines $\check{\Phi}^n_i$ by the relation

$$\Psi(B_{n,i}) = \check{\Phi}^{n-1}_0 + \cdots + \check{\Phi}^{n-1}_{i-1}.$$

That is, $\check{\Phi}^n_0 = \Psi(B_{n+1,1}) = \Psi(B_n) \cdot \mathbf{c}$, and for $1 \le i \le n-1$, $\check{\Phi}^n_i = \Psi(B_{n+1,i+1}) - \Psi(B_{n+1,i})$.

We now state the main result of this section.

Theorem 8.1 *The following recursion holds for $\check{\Phi}^n_i$: $G(\check{\Phi}^n_i) = \check{\Phi}^{n+1}_{i+1}$.*

Stanley conjectured [17, Conjecture 3.1] that the reduced variation of certain classes of permutations is equal to $\check{\Phi}^n_i$. This conjecture was proved by Hetyei in [10]. We now present a slightly modified result of this kind. It follows easily by Theorem 8.1 and the techniques of Section 7. Let $\mathcal{S}_{n,k}$ be the set of simsun permutations of length n ending with the element k.

Corollary 8.2 *The reduced variation of the set $\mathcal{S}_{n,k}$ is given by $V(\mathcal{S}_{n,k}) = \check{\Phi}^n_{n-k}$.*

9 Concluding Remarks

There are a number of questions that appear at this point in the research. We put forward a few of them.

In Section 8 we found new properties that hold for the **cd**-index of the shelling components of the simplex. In [5] the **cd**-index of shelling components of the cube have been studied. Are there any identities between the **cd**-indexes of the shelling components of the cube involving coproducts?

Stanley conjectured that among all Gorenstein* lattices of rank n, the Boolean algebra B_n minimizes all the coefficients of the **cd**-index, [18, Conjecture 2.7]. We present the following generalization:

Conjecture 9.1 *Let F be a polytope of dimension $d-1$. Then among all d-dimensional polytopes having F as a facet, the pyramid of F minimizes all the coefficients of the* **cd***-index.*

Let L be a linear functional on the Newtonian coalgebra V. Then the linear map D^L defined on V by

$$D^L(x) = \sum_x x_{(1)} \cdot L(x_{(2)}) \cdot x_{(3)}$$

is a coderivation on V. That is, D^L satisfies the relation $\Delta \circ D^L = (D^L \otimes 1 + 1 \otimes D^L) \circ \Delta$. In the Newtonian coalgebras $\mathcal{P}$, $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{F}$ are there any coderivations which have a combinatorial interpretation?

Let V and W be two convex polytopes in $\mathbb{R}^n$. The Minkowski sum $V+W$ is also a convex polytope. Assume that we know the **cd**-index of the two polytopes V and W. This does not give us enough information to compute the **cd**-index of the Minkowski sum $V+W$. What more information do we need of V and W in order to compute $\Psi(V+W)$? Recently, the authors together with Louis Billera have found an answer in the case when one of the polytopes is a line segment.

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