New enumerative results on two-dimensional directed animals

Mireille Bousquet-Mélou*

Abstract. We list several open problems concerning the enumeration of directed animals on two-dimensional lattices. We show that all these problems are special cases of two central problems: calculating the position-generating function and the perimeter and area generating function of square lattice animals.

We propose a possible direction for solving these two problems: we extend Dhar's correspondence between hard particle gas models and enumeration of animals according to the area, and show that each of the main two generating functions is, essentially, the density of a gas model given by the stationary distribution of a probabilistic transition.

We are able to compute the density of certain distributions. We thus obtain new bivariate generating functions for directed animals on the square lattice and on the triangular lattice respectively. We derive from these two results the generating functions of animals on decorated square and triangular lattices, as well as the average number of loops in directed animals as conjectured by A. R. Conway.

Résumé. Nous commençons par recenser les problèmes ouverts concernant l'énumération des animaux dirigés sur réseaux de dimension 2. Nous montrons qu'il existe deux questions centrales, auxquelles se ramènent les autres problèmes : l'une d'elles est relative à la position des cellules d'un animal sur réseau carré, l'autre à son périmètre de site.

Pour tenter de répondre à ces deux questions, nous étendons le lien décrit par Dhar entre les modèles de gaz à particules dures et l'énumération suivant l'aire des animaux dirigés : nous démontrons que chacune des séries génératrices principales s'exprime simplement en fonction de la densité d'un modèle de gaz donné par la distribution stationnaire d'une transition probabiliste.

Dans certains cas, cette densité se calcule facilement. Nous obtenons ainsi une nouvelle série génératrice bi-variée pour les animaux dirigés, sur réseaux carré et triangulaire. Nous déduisons de ces deux résultats les séries génératrices des animaux sur réseaux décorés, ainsi que le nombre moyen de boucles dans un animal de taille fixée, qui avait été conjecturé par A. R. Conway.

1 Introduction

An *animal* on a graph G is a finite connected set of vertices. In other words, any two vertices of an animal A are connected through a path of G having all its vertices in A (Figure 1). On a periodic infinite graph, animals are usually defined up to a translation. These simple combinatorial objects are of interest in statistical physics. For instance, they are the main ingredient of cell growth models. Moreover, enumerating animals according to the *perimeter* and the *area* permits to solve the (site) percolation model on the underlying graph G [6].

*LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, FRANCE.
`bousquet@labri.u-bordeaux.fr`
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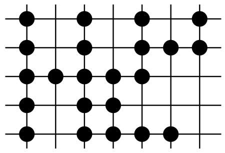

Figure 1: An animal on the square lattice.

Enumerating these animals seems to be a very difficult problem. To our knowledge, the most precise result is the following [14]: there exists a constant K such that, if a_n denotes the number of square lattice animals having n vertices, then $a_n^{1/n}$ tends to K when n tends to infinity. Finding lower and upper bounds for K is difficult, and not even the first digit of K is known:

$$3,87 < K < 4,65.$$

The enumeration of certain *directed animals* is more tractable. Let G be an oriented graph having a distinguished vertex O, called the *origin*. If an edge goes from v to w, then v is said to be a *father* of w. A *directed animal* on G is a finite set of vertices A, containing O, such that any vertex of A can be reached from O through an oriented path of G having all its vertices in A. The origin O is the *source* of A. The vertices of A are called *cells*, and the number of cells is the *area* of the animal. The *neighbours* of A are the vertices that do not belong to A but have a father in A. The *(site) perimeter* of A is its number of neighbours. This definition generalizes the notion of (undirected) animals given above since unoriented graphs can be seen as a special kind of oriented graphs, by replacing each unoriented edge by a pair of oriented edges. From now on, we only deal with directed animals, and the word "directed" will be often omitted.

We study in this paper directed animals on the square, triangular and honeycomb lattices. Examples are given in Figure 2. The edges are oriented upwards in all lattices. The leftmost animal has area 12 and perimeter 10.

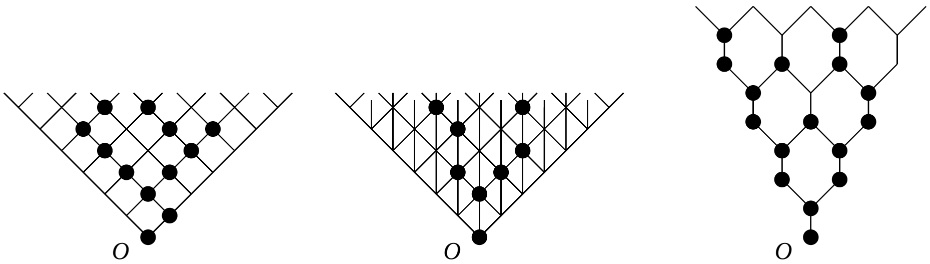

Figure 2: Directed animals on two-dimensional lattices.

Directed animals are related to directed (site) percolation models. Moreover, as shown by Dhar [11], enumerating directed animals according to area on certain graphs is equivalent to solving a *hard particle gas model* on other graphs. A hard particle gas model is a statistical lattice model in which two adjacent vertices cannot be simultaneously occupied by molecules of gas. A combinatorial proof of this equivalence has also been given using the notion of *heaps of pieces* [4, 15].

Actually, the correspondence between directed animals and gas models is not only a motivation for studying animals. It is also a very efficient way of enumerating them according to the area: the few exact known results can be obtained by solving the corresponding gas model. The main two results are the area generating functions of directed animals on the square lattice and on the three-dimensional *next-nearest neighbour* lattice drawn in Figure 3(a) [11]. For square lattice animals, there exists, besides the gas model argument, a very simple and nice combinatorial proof based on the notion of *heaps of pieces* [4, 15]. However, this combinatorial method has not (yet) been extended to animals in three dimensions, for which the very difficult solution of the corresponding gas model, called the *hard hexagon model*, remains the unique enumeration technique [2].

Proposition 1.1 [10, 11, 13] — *The area generating function of directed animals on the square lattice is*

$$S_0(t) = \frac{1}{2}\left(\left(1-\frac{4t}{1+t}\right)^{-1/2} - 1\right).$$

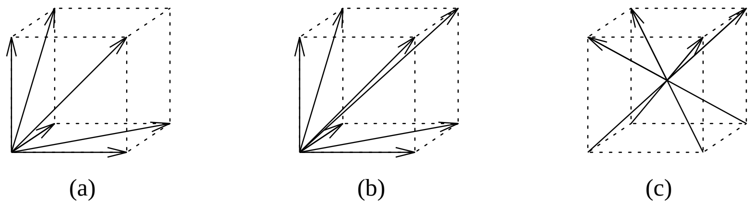

Figure 3: Three-dimensional oriented lattices.

Looking upon animals as heaps of pieces suggests immediately that the area generating function of directed animals on the triangular lattice is obtained by replacing t by $t/(1-t)$ in the area generating function of square lattice animals. Similarly, the same substitution, performed on the generating function of animals on the lattice of Figure 3(a) gives the area generating function of animals on the lattice of Figure 3(b). Actually, we can obtain a slight refinement of the area generating function. Let v be a cell of a directed animal A on the triangular lattice; we say that v is *only supported at the center* if the vertex placed just below in the same column is the only father of v lying in A. The number of such cells is denoted $c(A)$.

Proposition 1.2 — *The generating function of directed animals on the triangular lattice, counted according to their area and number of cells only supported at the center is*

$$\begin{aligned}\tilde{T}_0(t,w) &= \sum_A t^{|A|} w^{c(A)} \\ &= \frac{1}{2}\left(\left(1-\frac{4t}{1+t-tw}\right)^{-1/2} - 1\right).\end{aligned}$$

In particular, the area generating function of directed animals on the triangular lattice is

$$T_0(t) = \frac{1}{2}\left((1-4t)^{-1/2} - 1\right). \qquad (1)$$

Finally, Conway, Brak and Guttmann [8] have conjectured simple algebraic identities for the generating functions for directed animals on *decorated square lattices*. These conjectures have been proved by Ali [1]. More details on these lattices are given in the following sections, together with a new proof of Ali's results. We also introduce some decorated triangular lattices and give simple algebraic expressions for the generating functions of directed animals on these lattices.

The animals above are sometimes called *site-animals* to distinguish them from *bond-animals*, which are connected sets of edges. More precisely, a (directed) bond-animal A on an oriented graph G is a finite set of edges such that each edge of A belongs to an oriented path of G starting from O and having all its edges in A. The *area* of a bond-animal is its number of edges. The *neighbours* of A are edges that do not belong to A, but whose starting point belongs to an edge of A. The *(bond) perimeter* of A is its number of neighbours. The enumeration of bond-animals according to the perimeter and the area is related to bond-percolation models. When there is no risk of confusion, we will continue to use simply the word "animal" to denote site-animals.

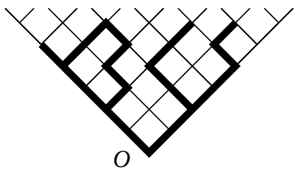

Figure 4: A directed bond-animal on the square lattice (area 21, perimeter 20).

As far as we know, we have mentionned *all* exact known results concerning the enumeration of animals. By elimination, one can obtain an infinite list of open problems. Here is, however, a tentative classification.

• The area is the only statistic by which exact enumerations are available. One could try to take into account other parameters, such as the perimeter, given its crucial role in percolation models. Conway has shown that the perimeter generating function of square lattice directed animals is not algebraic [7].

• Nothing is known about bond-animals.

• The honeycomb lattice raises serious difficulties, for site-animals on this lattice do not behave like animals on the square or triangular lattice. In particular, their area generating function does not seem to be algebraic.

• Other lattices could be studied, in two and three (and more...) dimensions. In particular, the enumeration of directed animals on the lattice of Figure 3(c) corresponds to the famous unsolved *hard square model*.

We focus in this paper on directed animals on two-dimensional lattices. We list in the following section several open problems concerning their enumeration. We show that all these problems (including the enumeration of bond-animals on the square lattice, the enumeration of site-animals on the honeycomb lattice, etc.) are special cases of two central problems: calculating the *position-generating function* and the *perimeter and area generating function* of square lattice animals.

How can we compute these generating functions? A natural idea is to extend one of the two methods for enumerating square lattice animals according to the area, i.e., the link with hard particle models on the one hand, and the notion of heaps on the other hand. It turns out that the notion of heaps is not, at least at first sight, easily generalized. Its main drawback is that is turns the arrangement of rows of an animal upside down whereas both the position of cells and the perimeter are closely linked to this arrangement. Therefore, we have concentrated on Dhar's idea, and have extended it to take into account additional parameters.

Our central result is that the position-generating function and the area and perimeter generating function of square lattice directed animals — and, consequently, all generating functions mentionned in Section 2 — are, essentially, the density of a certain one-dimensional gas model. The distribution of this gas is the *stationary distribution* of a simple *probabilistic transition.*

These transitions are described in Section 3. They are characterized by four parameters p_1, p_2, p_3 and p_4. When

$$p_1 p_4 (1 - p_2)(1 - p_3) = p_2 p_3 (1 - p_1)(1 - p_4), \tag{2}$$

the stationary distribution and its density have simple expressions. Alas, the position-generating function is related to the transition $(p_1, p_2, p_3, p_2 p_3 / p_1)$ while the area and perimeter generating function is related to the transition (p_1, p_2, p_2, p_2), and neither of these transitions satisfy (2)... However, when $p_3 = 0$, the first transition satisfies (2) and is combinatorially significant. We thus obtain a new bivariate generating function for directed animals on the square lattice. We derive from this result the generating functions of animals on decorated square lattices, as well as the average number of *loops* in animals of given area as conjectured by Conway [7].

Similarly, we obtain for animals on the triangular lattice a new bivariate generating function, from which we derive the generating functions of animals on decorated triangular lattices as well as the average number of loops in animals of given area as conjectured by Conway [7]. These results do not follow from the corresponding square lattice results.

To finish, here is now the history of this paper. I had conjectured the bivariate generating functions of Propositions 4.1 and 4.4 for the square and triangular lattices respectively. Since I was not able to prove them "combinatorially", I tried to extend Dhar's method, and discovered that all open problems on two-dimensional lattices were equivalent to the solution of a gas model. Although I could only solve this model in some special cases — thus proving the two conjectures — I believe that the *general* correspondence between directed animals and gas models is worth being presented, and could lead in the future to the solution of other open questions.

2 A survey of open problems

2.1 The position of cells in a square lattice animal

Let us consider a directed animal A on the square lattice. Let $v \neq O$ be a cell of A. Three cases occur, illustrated by Figure 5: in the first case, we say that the cell v is *only supported on the right*, in the second case, that it is *only supported on the left*, and in the third case, that v is a *loop* (in Figure 5, the vertices belonging to the animal are denoted by black circles, and the others by crosses). We denote $r(A)$ (resp. $\ell(A)$) the number of cells of A only supported on the right (resp.

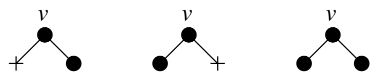

Figure 5: The three cases for the square lattice.

left). The number of loops of A is $b(A) = |A| - r(A) - \ell(A) - 1$.

Definition 2.1 — *The* position-generating function *of directed animals on the square lattice is*

$$S_1(t,u,v) = \sum_A t^{|A|} u^{r(A)} v^{\ell(A)}. \tag{3}$$

The loop-generating function *of directed animals on the square lattice is*

$$\begin{aligned} S_\ell(t,w) &= \sum_A t^{|A|} w^{b(A)} \qquad (4) \\ &= w^{-1} S_1(tw, w^{-1}, w^{-1}). \end{aligned}$$

The sums are over all directed animals of the square lattice.

The following propositions show the interest of two specializations of the series $S_1(t,u,v)$, namely $S_1(t,u,u)$ (or $S_\ell(t,w)$) and $S_1(t,u,1)$. We compute $S_1(t,u,1)$ in Section 4.

Proposition 2.2 — *Let $S_b(t,x)$ be the area and perimeter generating function of bond-animals on the square lattice:*

$$S_b(t,x) = \sum_A t^{|A|} x^{p(A)}.$$

We have

$$S_b(t,x) = \frac{x}{t} S_\ell\left(tx, 2 + \frac{t}{x}\right),$$

where $S_\ell(t,w)$ is the loop-generating function of site-animals, defined by (4).

In their attempt to understand why the generating functions for animals on the square and honeycomb lattices do not have similar behaviours, Conway, Brak and Guttmann have introduced a new class of lattices, called *strange lattices* [8] or *decorated square lattices* [1]. The $n-$*decorated square lattice* is obtained from the usual square lattice by adding n new vertices on each SE-NW edge. The edges of the new lattice are oriented upwards (see Figure 6). We define the $n-$*decorated triangular lattice* in a similar way.

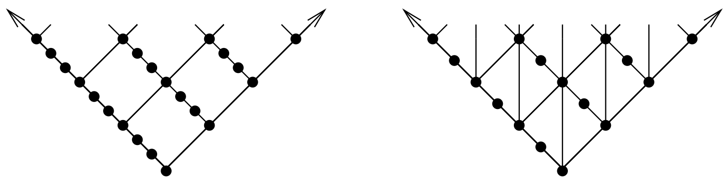

Figure 6: Square and triangular lattices with 2 and 1 decorations.

Proposition 2.3 — *Let $S_{d,n}(x,y)$ be the area generating function of animals on the $n-$decorated square lattice:*

$$S_{d,n}(x,y) = \sum_A x^{|S(A)|} y^{|A|-|S(A)|},$$

where $|S(A)|$ is the number of cells of A on the underlying square lattice. We have

$$S_{d,n}(x,y) = S_1\left(x\frac{1-y^{N+1}}{1-y}, y^N \frac{1-y}{1-y^{N+1}}, 1\right).$$

Proposition 2.4 — *Let b_n denote the total number of loops in all animals of area n. Let $S_m(t)$ be the generating function of the sequence $(b_n)_n$:*

$$S_m(t) = \sum_n b_n t^n = \sum_A b(A) t^{|A|}.$$

We have:

$$S_m(t) = t \frac{\partial S_0}{\partial t}(t) - S_0(t) - 2 \frac{\partial S_1}{\partial u}(t, 1, 1)$$

where the series $S_0(t)$ and $S_1(t, u, v)$ are respectively defined in Proposition 1.1 *and Equation* (3).

2.2 The site perimeter of directed animals

Let $S_2(t, x)$ (resp. $H_2(t, x)$) be the area and perimeter generating function of animals on the square (resp. honeycomb) lattice. No formula for these series is known. Moreover, no formula is known for $H_2(t, 1)$, whereas $S_2(t, 1) = S_0(t)$ is given in Proposition 1.1.

Proposition 2.5 [12] — *The area and perimeter generating functions of animals on the square and honeycomb lattices are related as follows:*

$$H_2(t, x) = tx + S_2(t^2, x(1 + t)). \tag{5}$$

The number of cells only supported on the right is undirectly related to the site perimeter. Let A be an animal on the square lattice. Let us call *right neighbour* of A any neighbour of A lying to the north-east of a cell of A.

Lemma 2.6 — *Let A be a square lattice animal. The number of right neighbours of A is $1 + r(A)$.*

2.3 The position of cells in a triangular lattice animal

Let A be an animal on the triangular lattice. Let $v \neq O$ be a cell of A. Seven cases occur now, defined in Figure 7. We denote $r(A)$ (resp. $\ell(A)$, $c(A)$) the number of cells of A only supported

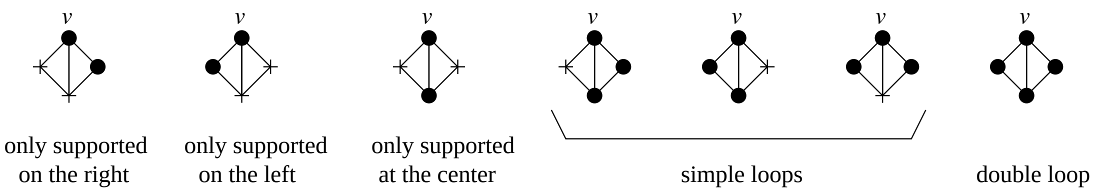

Figure 7: The seven cases for the triangular lattice.

on the right (resp. on the left, at the center). The number of simple loops is denoted $sb(A)$. The number of double loops is $db(A) = |A| - r(A) - \ell(A) - c(A) - sb(A) - 1$.

Definition 2.7 — *The* position-generating function *of directed animals on the triangular lattice is*

$$T_1(t, u, v, w, x) = \sum_A t^{|A|} u^{r(A)} v^{\ell(A)} w^{c(A)} x^{sb(A)}. \tag{6}$$

The loop-generating function *of directed animals on the triangular lattice is*

$$T_\ell(t,x,y) = \sum_A t^{|A|} x^{sb(A)} y^{db(A)} \tag{7}$$

$$= y^{-1} T_1(ty, y^{-1}, y^{-1}, y^{-1}, xy^{-1}).$$

The sums are over all directed animals of the triangular lattice.

We compute $T_1(t,u,1,1,1)$ in Section 4 and find that it is equal to $T_1(t,1,1,u,1)$, which means that

$$\sum_A t^{|A|} u^{r(A)} = \sum_A t^{|A|} u^{\ell(A)} = \sum_A t^{|A|} u^{c(A)}. \tag{8}$$

This is not trivial at all. Again, the series $T_1(t,u,v,w,x)$ has two interesting specializations.

Proposition 2.8 — *Let $T_b(t,x)$ be the area and perimeter generating function of bond-animals on the triangular lattice:*

$$T_b(t,x) = \sum_A t^{|A|} x^{p(A)}.$$

We have

$$T_b(t,x) = \frac{x}{t} T_\ell \left(tx^2, 2 + \frac{t}{x}, 3 + 3\frac{t}{x} + \frac{t^2}{x^2} \right),$$

where $T_\ell(t,x,y)$ is the loop-generating function of site animals, defined by (7)*.*

Proposition 2.9 — *Let $T_{d,n}(x,y)$ be the area generating function of animals on the $n-$decorated triangular lattice:*

$$T_{d,n}(x,y) = \sum_A x^{|T(A)|} y^{|A|-|T(A)|},$$

where $|T(A)|$ is the number of cells of A on the underlying triangular lattice. We have

$$T_{d,n}(x,y) = T_1 \left(x\frac{1-y^{N+1}}{1-y}, y^N \frac{1-y}{1-y^{N+1}}, 1, 1, 1 \right).$$

Proposition 2.10 — *Let $T_m(t)$ be the generating function of the total number of loops of animals of given area:*

$$T_m(t) = \sum_A b(A) t^{|A|}.$$

We have

$$T_m(t) = t\frac{\partial T_0}{\partial t}(t) - T_0(t) - 2\frac{\partial T_1}{\partial u}(t,1,1,1,1) - \frac{\partial T_1}{\partial w}(t,1,1,1,1)$$

where the generating functions $T_0(t)$ and $T_1(t,u,v,w,x)$ are defined by (1) *and* (6) *respectively.*

Remark. According to (8), this identity can also be written as

$$T_m(t) = t\frac{\partial T_0}{\partial t}(t) - T_0(t) - 3\frac{\partial T_1}{\partial u}(t,1,1,1,1).$$

3 Directed animals and gas models

3.1 Animals of bounded width

Consider a cyclic oriented square lattice having N cells in each row (Figure 8). Edges are oriented away from the center, and the vertices of the first row are labelled with $1, 2, \ldots, N$. It is convenient to consider that the labels belong to the ring $\mathbb{Z}/N\mathbb{Z}$. In this section, we deal with animals that may have a source formed of several vertices.

Definition 3.1 — *Let $C \subset [N] = \{1, 2, \ldots, N\}$. A directed animal A* of source C *is a finite set of vertices containing C such that any vertex of A is connected to (at least) one cell of C through a path starting from C and having all its vertices in A.*

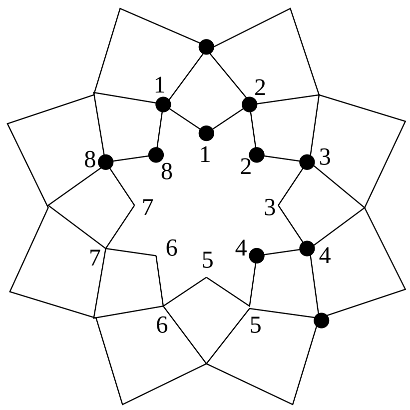

Figure 8: A cyclic oriented square lattice (width 8) and an animal of source $\{1, 2, 4, 8\}$.

All definitions given above can be extended to bounded animals (area, perimeter, loops, cells only supported on the right, etc). Since the lattice has now a finite width, we can write a *finite* system of equations satisfied by the generating functions of animals having a given source. All of them are rational functions.

Lemma 3.2 — *For $C \subset [N]$, let $S_1^{(N)}(C)$ and $S_2^{(N)}(C)$ be respectively the position-generating function and the area and perimeter generating function of animals of source C. By convention, $S_1^{(N)}(\emptyset) = S_2^{(N)}(\emptyset) = 1$. We have*

$$S_1^{(N)}(C) = t^{|C|} \sum_{D \subset \mathcal{N}(C)} S_1^{(N)}(D) u^{|D \cap \mathcal{N}_\ell(C)|} v^{|D \cap \mathcal{N}_r(C)|} \tag{9}$$

where $\mathcal{N}(C) = C \cup \{i+1 : i \in C\}$, $\mathcal{N}_\ell(C) = \{i \in C : i-1 \not\in C\}$ and $\mathcal{N}_r(C) = \{i+1 \not\in C : i \in C\}$. Similarly,

$$S_2^{(N)}(C) = t^{|C|} \sum_{D \subset \mathcal{N}(C)} S_2^{(N)}(D) x^{|\mathcal{N}(C) \setminus D|}. \tag{10}$$

The generating functions $S_1^{(N)}(C)$ and $S_2^{(N)}(C)$ are clearly related to the generating functions $S_1(t, u, v)$ and $S_2(t, x)$ defined in Section 2 by

$$\lim_{N \to \infty} S_1^{(N)}(\{1\}) = S_1(t, u, v) \quad \text{and} \quad \lim_{N \to \infty} S_2^{(N)}(\{1\}) = S_2(t, x).$$

3.2 Local transitions

Consider the one-dimensional cyclic lattice drawn in Figure 9. Let $\mathcal{L}_0$ (resp. $\mathcal{L}_1$) be the set of vertices of the internal (resp. external) row. Label the vertices in each row with $1, 2, \ldots, N$.

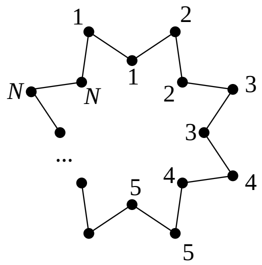

Figure 9: The one-dimensional cyclic lattice.

A *cell distribution* on $\mathcal{L}_1$ assigns each subset of $\mathcal{L}_1$ with a probability of being the set of *occupied vertices*. The *density* of a cell distribution is $1/N$ times the average number of occupied vertices. Let p_1, p_2, p_3 and p_4 be real numbers lying in $[0,1]$. A distribution of cells on $\mathcal{L}_1$ induces a distribution of cells on $\mathcal{L}_0$ via the *local transition* (p_1, p_2, p_3, p_4): given two consecutive vertices of $\mathcal{L}_1$, say i and $i+1$, the vertex of $\mathcal{L}_0$ lying between them will bear a cell

- with probability p_1 if i and $i+1$ are both unoccupied
- with probability p_2 if i is occupied and $i+1$ is unoccupied
- with probability p_3 if i is unoccupied and $i+1$ is occupied
- with probability p_4 if i and $i+1$ are both occupied.

This transition is schematized in Figure 10, in which a black circle denotes an occupied vertex.

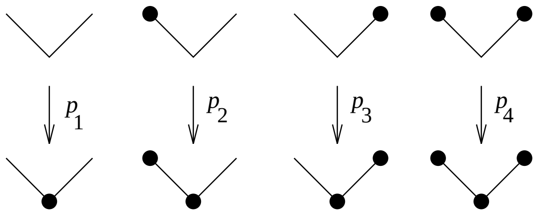

Figure 10: A local transition.

If the distribution of cells thus obtained on $\mathcal{L}_0$ is the same as the original one on $\mathcal{L}_1$, it is said to be *stationary for the transition* (p_1, p_2, p_3, p_4).

Proposition 3.3 — *Suppose that* $p_1(1-p_1)(1-p_4)(1-p_2)(1-p_3) \neq 0$ *and*

$$p_1 p_4 (1-p_2)(1-p_3) = p_2 p_3 (1-p_1)(1-p_4). \tag{11}$$

Then the cell distribution given by

$$\forall D \subset \mathcal{L}_1, \qquad P(D \text{ occupied}, \ \mathcal{L}_1 \setminus D \text{ empty}) = \frac{1}{Z}\left(\frac{p_1}{1-p_4}\right)^{|D|}\left(\frac{(1-p_1)(1-p_4)}{(1-p_2)(1-p_3)}\right)^{|\bar{\mathcal{N}}_r(D)|}$$

where $\bar{\mathcal{N}}_r(D) = \{i \in D : i+1 \notin D\}$ *and*

$$Z = \sum_{D \subset \mathcal{L}_1}\left(\frac{p_1}{1-p_4}\right)^{|D|}\left(\frac{(1-p_1)(1-p_4)}{(1-p_2)(1-p_3)}\right)^{|\bar{\mathcal{N}}_r(D)|},$$

is stationary for the transition (p_1, p_2, p_3, p_4)*. Its density is a rational function in the* p_i*'s. When* n *tends to infinity, it tends to*

$$\frac{1}{2}\left[1 - \frac{1-p_1-p_4}{1-p_4}\left(\left(\frac{1-p_1-p_4}{1-p_4}\right)^2 + 4\frac{p_1(1-p_1)}{(1-p_2)(1-p_3)}\right)^{-1/2}\right].$$

3.3 Connections between the two models

Using (9) and (10), we have generalized Dhar's argument [11] and shown that the position-generating function and the area and perimeter generating function of directed animals on the square lattice can be expressed in terms of the density of a gas model given by the stationary distribution of a local transition. Unfortunately, we can not compute this density in the most general case, but only in a particular case to which Section 4 is devoted.

Proposition 3.4 — *Let* $\rho(p_1, p_2, p_3, p_4)$ *denote the density of the gas model given by the stationary distribution of the transition* (p_1, p_2, p_3, p_4)*.*

- *The position-generating function of one-source directed animals on the cyclic square lattice is*

$$\sum_A t^{|A|}u^{r(A)}v^{\ell(A)} = \frac{1-u-v}{uv}\rho(p_1, p_2, p_3, p_2p_3/p_1)$$

  where

$$p_1 = \frac{tuv}{1-u-v}, \quad p_2 = \frac{tu(1-u)}{1-u-v} \quad \text{and} \quad p_3 = \frac{tv(1-v)}{1-u-v}.$$

- *The area and perimeter generating function of one-source directed animals on the cyclic square lattice is*

$$\sum_A t^{|A|}x^{p(A)} = 1 - x - \rho(p_1, p_2, p_2, p_2)$$

  where

$$p_1 = 1 - x - t \quad \text{and} \quad p_2 = 1 - x.$$

Using the ideas of Section 2, we can express several generating functions for one-source directed animals (that is, animals of source $\{1\}$) in terms of the density of some gas models. Each of these models is given by the (unique) stationary distribution of a local transition. The following table shows to which transition each enumerative problem is related.

Lattice	Site/Bond	Parameters	Transition
square	site	$\|A\|$, $r(A)$ and $\ell(A)$	$p_1, p_2, p_3, p_2p_3/p_1$
square	site	$\|A\|$ and $r(A)$	$p_1, p_2, 0, 0$
square	site	$\|A\|$ and $b(A)$	$p_1, p_2, p_2, p_2^2/p_1$
square	bond	$\|A\|$ and $p(A)$	idem
square	bond	$\|A\|$	$1, p, p, p^2$
square	site	$\|A\|$ and $p(A)$	p_1, p_2, p_2, p_2
honeycomb	site	$\|A\|$ and $p(A)$	idem
honeycomb	site	$\|A\|$	$p(1-p), p, p, p$

Note that the transition $(p_1, p_2, 0, 0)$ is the only transition of this table that satisfies identity (11). This will be used in the next section to find new generating functions. A similar study can be carried out for animals on the triangular lattice.

4 Explicit results

We give here the generating functions for directed animals on the square and triangular lattices, according to their area and number of cells only supported on the right. These results are new and can be used to obtain the generating functions of animals on decorated lattices, as well as the mean number of loops in animals of given area. The generating functions involved here are quadratic. Surprisingly, we do not have any "combinatorial" proofs of these formulas.

4.1 The square lattice

Proposition 4.1 — *The generating function for directed animals on the square lattice, according to their area and number of cells only supported on the right is*

$$S_1(t,u,1) = \frac{1}{2}\left(\left(1 - \frac{4t}{(1+t)(1+t-tu)}\right)^{-1/2} - 1\right).$$

Remark. According to Lemma 2.6, $uS_1(t,u,1)$ is the generating function of directed animals, counted according to the area and the number of right neighbours.

Corollary 4.2 — *The area generating function of animals on the $n-$decorated square lattice is*

$$S_{d,n}(x,y) = S_1\left(x\frac{1-y^{N+1}}{1-y}, y^N\frac{1-y}{1-y^{N+1}}, 1\right),$$

where $S_1(t,u,1)$ is given in the proposition above.

Remark. This result had already been proved by Ali [1]. His proof is also inspired by Dhar's method, but requires is a two-dimensional Ising model, whereas we only need a one-dimensional model here.

Corollary 4.3 — *The generating function of the total number of loops of animals of given area is*

$$S_m(t) = \sum_A b(A) t^{|A|} = \frac{1}{2}\left(1 - \frac{1-4t+t^2+4t^3}{(1+t)^{1/2}(1-3t)^{3/2}}\right).$$

This implies that, in animals of area n,

- *the average number of cells supported only on the right (left) is asymptotically $4n/9$,*
- *the average number of loops is asymptotically $n/9$.*

4.2 The triangular lattice

Proposition 4.4 — *The generating function for directed animals on the triangular lattice, according to their area and number of cells only supported on the right is*

$$T_1(t,u,1,1,1) = \frac{1}{2}\left(\left(1 - \frac{4t}{1+t-tu}\right)^{-1/2} - 1\right).$$

Remark. Comparing with Proposition 1.2 shows that the parameters "number of cells only supported on the right" and "number of cells only supported at the center" have the same distribution on animals of fixed area. This is not obvious at all, and it would be nice to find a more direct proof.

Corollary 4.5 — *The area generating function of animals on the $n-$decorated triangular lattice is*

$$T_{d,n}(x,y) = T_1\left(x\frac{1-y^{N+1}}{1-y}, y^N\frac{1-y}{1-y^{N+1}}, 1, 1, 1\right),$$

where $T_1(t,u,1,1,1)$ is given in the proposition above.

Corollary 4.6 — *The generating function of the total number of loops of animals of given area is*

$$T_m(t) = \sum_A b(A) t^{|A|} = \frac{1}{2}\left(1 - \frac{1-6t+6t^2}{(1-4t)^{3/2}}\right).$$

This implies that, in animals of area n,

- *the average number of cells supported only on the right (left) is $\frac{n(n-1)}{2(2n-1)}$,*
- *the average number of cells supported only at the center is $\frac{n(n-1)}{2(2n-1)}$,*
- *the average number of loops is $\frac{(n-1)(n-2)}{2(2n-1)}$.*

Remarks. 1. Uniform generation of random directed animals suggests that they are in general very "thin" [9]. Corollaries 4.3 and 4.6 give a measure of thinness. If random animals were compact, then they would have lots of loops whereas only one fourth of the cells are loops.

2. The generating functions of animals on the decorated square lattice and the results of Corollaries 4.3 and 4.6 were conjectured by Andrew Conway in his Ph.D. thesis [7]. He rightly says that results should be easier to prove once the answer is known...

References

[1] A. A. Ali, *Enumeration of directed site animals on the decorated square lattices*, preprint, Tata Institute of Fundamental Research, Bombay, India.

[2] R. J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[3] J. Bétréma and J.-G. Penaud, *Animaux et arbres guingois*, Theoret. Comput. Sci. **117** (1993), 67–89.

[4] J. Bétréma and J.-G. Penaud, *Modèles avec particules dures, animaux dirigés et séries en variables partiellement commutatives*, Rapport No. 93–18, LaBRI, Université Bordeaux 1, France (1993).

[5] J. Blease, Series expansions for directed bond-percolation problem, *J. Phys. C: Solid. State Phys.* **10** (1977) 917–924.

[6] M. Bousquet-Mélou, *Percolation models and animals*, Technical Report No. 993-95, LaBRI, Université Bordeaux 1, France (1995).

[7] A. R. Conway, *On the behaviour of self avoiding walks and related problems*, Ph.D. thesis, University of Melbourne, Australia (1993).

[8] A. R. Conway, R. Brak and A. J. Guttmann, *Directed animals on two-dimensional lattices*, J. Phys. A: Math. Gen. **26** (1993) 3085–3091.

[9] A. Denise and J.-G. Penaud, *Génération aléatoire d'animaux dirigés*, Actes des Journées "Polyominos et pavages" (P. Aigrain ed.) Lyon (1992) 61–69.

[10] D. Dhar, *Equivalence of the two-dimensional directed site animal problem to Baxter's hard square lattice gas model*, Phys. Rev. Lett. **49** (1983) 959–962.

[11] D. Dhar, *Exact solution of a directed-site animals-enumeration problem in three dimensions*, Phys. Rev. Lett. **51** (1983) 853–855.

[12] D. Dhar, M. Phani and M. Barma, *Enumeration of directed site animals on two-dimensional lattices*, J. Phys. A: Math. Gen. **15** (1982) L279–L284.

[13] D. Gouyou-Beauchamps and G. Viennot, *Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem*, Adv. in Appl. Math. **9** (1988) 334–357.

[14] D. A. Klarner and R. L. Rivest, *A procedure for improving the upper bound for the number of n-ominoes*, Canad. J. Math. **13** (1973) 585–602.

[15] X. G. Viennot, *Heaps of pieces I: Basic definitions and combinatorial lemmas*, Combinatoire énumérative (G. Labelle and P. Leroux eds.), Lecture Notes in Math. **1234** Springer-Verlag, Berlin (1986) 321–350.