The $\mathbf{r}$-cubical lattice and a generalization of the $\mathbf{cd}$-index

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Résumé

Dans cet article nous étudions des questions extrémales pour le treillis $\mathbf{r}$-cubique. Pour cela, nous généralisons l'index $\mathbf{cd}$ du treillis cubique à un index $\mathbf{r}$-$\mathbf{cd}$, que nous appellons $\Psi(\mathbf{r})$. Les coefficients de $\Psi(\mathbf{r})$ dénombrent les permutations d'André $\mathbf{r}$-signées augmentées, généralisant d'une manière naturelle les résultats de Purtill qui mettent en rapport l'index $\mathbf{cd}$ du trellis cubique et les permutations d'André. Le nombre de permutations d'André $\mathbf{r}$-signées augmentées est donné par une fonction génératrice trigonométrique. Nous déterminons les configurations extrémales maximisant la fonction de Möbius sur les ideaux rang-sélectionnés. Nous prouvons également que la configuration extrémale maximisant la fonction de Möbius pour les sélections de rangs arbitraires est le système des rangs alternants impairs, $\{1, 3, 5, \ldots\}$.

Abstract

In this paper we study extremal questions for the $\mathbf{r}$-cubical lattice. To do this we generalize the $\mathbf{cd}$-index of the cubical lattice to an $\mathbf{r}$-$\mathbf{cd}$-index, which we call $\Psi(\mathbf{r})$. The coefficients of $\Psi(\mathbf{r})$ enumerate augmented André $\mathbf{r}$-signed permutations, a natural generalization of Purtill's results relating the $\mathbf{cd}$-index of the cubical lattice and André permutations. The number of augmented André r-signed permutations is given by a trigonometric generating function. We determine the extremal configurations for maximizing the Möbius function of rank-selected upper and lower order ideals. Also we find the extremal configuration which maximizes the Möbius function of arbitrary rank selections is the odd alternating ranks, $\{1, 3, 5, \ldots\}$.

1 Introduction

The main purpose of this paper is to study extremal questions on the $\mathbf{r}$-cubical lattice $C^{\mathbf{r}}$. This lattice is a natural generalization of the the cubical lattice, that is, the face lattice of a cube. The cubical lattice C_n of order n may be described by taking the nth power of the first poset in Figure 1 and then adjoining a minimal element. The $\mathbf{r}$-cubical lattice is constructed in a similar manner, where we instead take a product of posets M_r, described in Figure 1. Such a lattice was first studied by Metropolis, Rota, Strehl, and White in [9]. They were interested in Dilworth compositions of the $\mathbf{r}$-cubical lattice.

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Figure 1: The Hasse diagrams of the poset M_r with $r = 2$ and $r = 5$.

A number of authors have recently been interested in extremal questions with the Möbius function of a subposet of a fixed poset. The boolean algebra B_n and the lattice of subspaces of an n-dimensional vector space over GF_q have been studied in [16], while the face lattice of an n-dimensional octahedron O_n and face lattices of convex polytopes have been studied in [14, 15]. As an example, for arbitrary rank selections from B_n, the Möbius function in absolute value (equivalently, the beta-invariant) is maximized by taking every other rank from the poset [16]. The techniques of Sagan, Yeh, and Ziegler used the fact that for rank selections from the boolean algebra one can instead study permutations in the symmetric group with certain descent sets. Their work for arbitrary rank selections is equivalent to results of Niven [11] and de Bruijn [5], who determined that the largest class of permutations in the symmetric group having a fixed descent set are the alternating permutations.

The **cd**-index of an Eulerian poset is a non-commutative polynomial which encodes the flag f-vector. Purtill proved that the **cd**-index of C_n has non-negative coefficients. By an observation of Stanley, this fact implies that among arbitrary rank selections from C_n, the alternating rank selection maximizes the Möbius function. Uniqueness of the alternating rank configuration for the arbitrary rank selection case follows from elementary properties of the **cd**-index (see [15]). This extremal technique motivated our definition of a more general **cd**-index for the **r**-cubical lattice, which we call $\Psi(C^{\mathbf{r}})$. We use this **r**-**cd**-index to solve the arbitrary rank selection question for the **r**-cubical lattice.

After giving an R-labeling of the **r**-cubical lattice, we define the notion of augmented **r**-signed permutations. This R-labeling enables us to study the question of maximizing the S-rank-selected beta-invariant, $\beta(S)$, over rank-selected ideals from the r-cubical lattice in terms of these permutations. We find that for rank-selected lower order ideals from the r-cubical lattice of rank $n+1$, the beta invariant attains a maximum when we take roughly the ranks 1 through $\frac{(r-1)(n+1)}{2r-1}$ of the poset. Similarly for rank-selected upper order ideals, the beta invariant attains a maximum when we take roughly the ranks $\frac{(n+r+2)}{r+1}$ through n of the poset.

Purtill [13, Sections 5 and 6] studied the non-negativity of the coefficients of the **cd**-index for the Boolean algebra and the cubical lattice by showing its coefficients count a class of permutations, called augmented André permutations and André signed permutations, respectively. Using André permutations he obtained recurrences for their respective **cd**-indexes. We instead prove these recurrences by a direct combinatorial argument (see equation (1) and Proposition 5.2). In Section 7, we extend Purtill's notion of André signed permutations to what we call augmented André **r**-signed permutations. This natural generalization enables us to show that the coefficients of $\Psi(\mathbf{r})$ enumerate augmented André **r**-signed permutations.

Finally, in Section 8 we maximize the beta invariant of the **r**-cubical lattice over arbitrary rank selections. We do this by showing that the **ab**-index of $C^{\mathbf{r}}$, $\Psi(C^{\mathbf{r}})$, has the strictly increasing alternating property. That is, we prove that the coefficient of the **ab**-word vw is larger than the coefficient of the **ab**-word vw^* when v ends in a different letter than w begins with and where w^* is obtained from w by uniformly exchanging the **a**'s and **b**'s. Niven established these kind of inequalities when he proved that the descent set having the most permutations is the alternating descent set [11]. Our inequalities imply that the coefficient of the alternating **ab**-word $\mathbf{baba}\cdots$ in $\Psi(C^{\mathbf{r}})$ is the largest. In other words, the set of ranks $\{1, 3, 5, \ldots\}$ is the unique extremal configuration for the **r**-cubical lattice.

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2 The ab-index

In this section we give a brief introduction to the **ab**-index and the **cd**-index. For all terminology and notation related to the **cd**-index, we will follow [18].

Let P be a finite, graded poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$. Denote the rank function of P by ρ. For $S \subseteq [n] = \{1, 2, \ldots, n\}$, we define the *$S$-rank-selected subposet* to be $P(S) = \{x \in P : \ \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. Let $\alpha(S) = \alpha_P(S)$ denote the number of maximal chains in $P(S)$ and let the *beta invariant* $\beta(S) = \beta_P(S)$ be defined by $\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T)$.

To encode the beta invariant of the poset P, we begin by defining a monomial in the non-commutative variables **a** and **b** by $u_S = u_1 \cdots u_n$, where u_i is **a** if $i \notin S$ and u_i is **b** if $i \in S$. (Later when we work with permutations, it will be helpful to think of **a** as "ascent" and **b** as "descent".) As an example, if $n = 5$ and $S = \{1, 4, 5\}$, then $u_S = \mathbf{baabb}$. Form a non-commutative polynomial, called the **ab**-*index*, by

$$\Psi(P) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

The degree of both **a** and **b** is defined to be one so that $\Psi(P)$ is homogeneous of degree n.

For an **ab**-word w we denote its length by $|w|$. Let 1 denote the unique word of length 0. Also the *complement* of the word w is the word formed by uniformly exchanging the letters **a** and **b**. We denote the complement of w by w^*.

Fine (refer to [2]) observed that if P is an Eulerian poset, then $\Psi(P)$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the **cd**-*index*. See Stanley [18] for an elementary proof of the existence of the **cd**-index for Eulerian posets. Since both **c** and **d** are symmetric in **a** and **b**, this implies the well-known property that for an Eulerian poset P of rank $n+1$, $\beta_P(S) = \beta_P(\overline{S})$, where $\overline{S}$ denotes the complement of S in the set $[n]$. In terms of a word w and its complement, this means that the coefficient of w is equal to the coefficient of w^* in $\Psi(P)$.

Stanley observed that if the **cd**-index has non-negative coefficients, then the two alternating words $\mathbf{aba}\cdots$ and $\mathbf{bab}\cdots$ maximize the coefficients of the **ab**-index [15]. This motivates the following definition.

Definition 2.1 *Let $\mathcal{L}$ be a linear combination of* **ab**-*words of length n. That is, $\mathcal{L} = \sum_{z \,:\, |z|=n} c(z) \cdot z$. We say that $\mathcal{L}$ has the* weakly increasing alternating property *if the following two conditions hold:*

1. *If v and w is a pair of words so that the last letter of the word v is different from the first letter of the word w and $|v| + |w| = n$, then $c(vw) \geq c(vw^*)$.*

2. *If w is a word of length n that begins with $\mathbf{b}$, then $c(w) \geq c(w^*)$.*

If all of the inequalities above are strict then we say $\mathcal{L}$ has the strictly increasing alternating property.

It is easy to see that if $\mathcal{L}$ has the strictly and $\mathcal{K}$ has the weakly increasing alternating property then their sum $\mathcal{L} + \mathcal{K}$ has the strictly increasing alternating property. Observe that $\mathcal{L}$ having the strictly increasing alternating property implies that the largest coefficient in $\mathcal{L}$ is the coefficient in front of the alternating word $\mathbf{baba} \cdots$.

We say that a linear combination of $\mathbf{ab}$-words, $\mathcal{L} = \sum_{z\,:\,|z|=n} c(z) \cdot z$, is *self-complementary* if for all words w, $c(w) = c(w^*)$.

Lemma 2.2 *If a linear combination $\mathcal{L}$ of $\mathbf{ab}$-words of length n can be expressed as a $\mathbf{cd}$-index with non-negative coefficients then $\mathcal{L}$ has the weakly increasing alternating property and is self-complementary. Moreover, if $\mathcal{L}$ can be expressed as a $\mathbf{cd}$-index with positive coefficients then the inequality $c(vw) > c(vw^*)$, where the last letter of the word v differs from the first letter of the word w, is a strict inequality. Hence, the $\mathbf{ab}$-words with largest coefficient are the two alternating words $\mathbf{aba} \cdots$ and $\mathbf{bab} \cdots$.*

Lemma 2.3 *Let $\mathcal{L}$ be a linear combination $\mathbf{ab}$-words of length n and $\mathcal{K}$ be a linear combination $\mathbf{ab}$-words of length m. If both $\mathcal{L}$ and $\mathcal{K}$ has the weakly increasing alternating property and $\mathcal{K}$ is self-complementary, then $\mathcal{L} \cdot \mathcal{K}$ also has the weakly increasing alternating property.*

3 R-labelings

An *edge-labeling* λ of a locally finite poset P is a map which assigns to each edge in the Hasse diagram of P an element from some poset Λ. For us Λ will always be a linearly ordered poset. In this case we say that λ is a *linear edge labeling* (see [6] for a further study of linear edge labelings). If x and y is an edge in the poset, that is, y covers x in P, then we denote the label on this edge by $\lambda(x, y)$. A maximal chain $x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ in an interval $[x, y]$ in P is called *rising* if the labels are weakly increasing with respect to the order of the poset Λ, that is, $\lambda(x_0, x_1) \leq_\Lambda \lambda(x_1, x_2) \leq_\Lambda \cdots \leq_\Lambda \lambda(x_{k-1}, x_k)$. An edge-labeling is called an *R-labeling* if for every interval $[x, y]$ in P there is a unique rising maximal chain in $[x, y]$.

Let P be a poset of rank $n+1$ with R-labeling λ. For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec \ldots \prec x_{n+1} = \hat{1}\}$ in P, the *descent set* of the chain c is $D(c) = \{i \ : \ \lambda(x_{i-1}, x_i) >_\Lambda \lambda(x_i, x_{i+1})\}$. Observe that $D(c)$ is a subset of the set $[n]$.

A result of Björner and Stanley [4, Theorem 2.7] says that if P is a graded poset of rank $n+1$, $S \subseteq [n]$, and P admits an R-labeling, then $\beta(S)$ equals the number of maximal chains in P having descent set S with respect to the given R-labeling λ. A consequence of this result and the definition of the $\mathbf{ab}$-index is that we may compute the $\mathbf{ab}$-index by considering an R-labeling of the poset.

Lemma 3.1 *Let P be a graded poset of rank $n+1$. If λ is an R-labeling of P, then the* **ab***-index of P is equal to*

$$\Psi(P) = \sum_{c} u_{D(c)},$$

where the sum is over all maximal chains c.

As an example, we give the standard R-labeling for the boolean algebra. Viewing B_n as the poset of all the subsets of $[n]$ ordered by inclusion, label the edge $A \subset B$ with the unique element in $B - A$. Observe the maximal chains in B_n correspond to permutations of the set $[n]$. It is now easy to give a recursion for the **ab**-index of the boolean algebra. Consider permutations on the set $[n+2]$, and let $i+1$ be the position where the element 1 or $n+2$ occurs first, reading from right to left. Note there are i elements from the set $\{2, \ldots, n+1\}$ to the right of this position. If $i = 0$ then these permutations are enumerated by $\Psi(B_{n+1}) \cdot \mathbf{c}$. If $1 \leq i \leq n$, they are enumerated by $\binom{n}{i} \cdot \Psi(B_{n+1-i}) \cdot \mathbf{d} \cdot \Psi(B_i)$. Thus

$$\Psi(B_{n+2}) = \Psi(B_{n+1}) \cdot \mathbf{c} + \sum_{i=1}^{n} \binom{n}{i} \cdot \Psi(B_{n+1-i}) \cdot \mathbf{d} \cdot \Psi(B_i), \tag{1}$$

where $\Psi(B_1) = 1$. This formula was established by Purtill in [13, Corollary 5.8] using André permutations.

Hence, by equation 1 we may compute

$$\begin{array}{rclrcl} \Psi(B_2) & = & \mathbf{c}, & \Psi(B_3) & = & \mathbf{c}^2 + \mathbf{d}, \\ \Psi(B_4) & = & \mathbf{c}^3 + 2 \cdot \mathbf{cd} + 2 \cdot \mathbf{dc}, & \Psi(B_5) & = & \mathbf{c}^4 + 3 \cdot \mathbf{c}^2\mathbf{d} + 5 \cdot \mathbf{cdc} + 3 \cdot \mathbf{dc}^2 + 4 \cdot \mathbf{d}^2. \end{array}$$

By the recursion (1) it is easy to see that the coefficients of each **cd**-monomial in $\Psi(B_n)$ are positive. Thus by Lemma 2.2 we conclude

Theorem 3.2 (Sagan, Yeh, and Ziegler [16]) *For arbitrary rank selections S from the boolean algebra B_n, the two unique extremal configurations for maximizing the beta invariant $\beta(S)$ are the following rank selections:*

$$\{1, 3, 5, \ldots\} \cap [n-1] \quad \textit{and} \quad \{2, 4, 6, \ldots\} \cap [n-1].$$

This theorem is implicit in the work of Niven [11] and de Bruijn [5], who studied permutations with a given descent set.

4 The r-cubical lattice

For r a positive integer, let M_r denote the poset formed from an r-element antichain and a maximal element $\hat{1}$, where each element of the antichain is covered by the maximal element $\hat{1}$. See Figure 1 for two examples. For a sequence of positive integers $\mathbf{r} = (r_1, \ldots, r_n)$, define the (multi-indexed) **r**-*cubical lattice* $C^{\mathbf{r}}$ to be $M_{r_1} \times M_{r_2} \times \cdots \times M_{r_n} \cup \{\hat{0}\}$. This is a graded poset of rank $n+1$. Indeed, this is a lattice since it is a finite join-semilattice. When $\mathbf{r} = (2, \ldots, 2)$ the **r**-cubical lattice will be the cubical lattice C_n, that is, the face lattice of the n-dimensional cube.

Another way to view the $\mathbf{r}$-cubical lattice is consider infinite sequences $A = (A_1, A_2, \ldots)$ of subsets from the set $[n] = \{1, 2, \ldots, n\}$, such that $A_j \cap A_k = \emptyset$ when $j \neq k$, and $i \notin A_j$ when $j > r_i$. Define the order relation by $A \leq B$ if $A_i \supseteq B_i$ for all $i = 1, 2, \ldots$, and adjoin a minimal element $\hat{0}$.

The Whitney numbers of the second kind for $C^{\mathbf{r}}$ are given by elementary symmetric functions. That is, the number of elements of rank $n + 1 - k$ in the $\mathbf{r}$-cubical lattice is the kth elementary symmetric function in the variables $r_1, r_2, \ldots, r_n$, for $k = 0, \ldots, n$: $e_k(r_1, \ldots, r_n)$.

The $\mathbf{r}$-cubical lattice has a very nice R-labeling described as follows: for the cover relation $A \prec B$, where $A \neq \hat{0}$, label the corresponding edge in the Hasse diagram by (i, a), where i is the unique index such that $A_i \neq B_i$, and let a be the singleton element in $A_i - B_i$. Also, for the relation $\hat{0} \prec B$ let the label be the special element G. Hence, the set of labels T_n are $\{G\} \cup \{(i, j) \,:\, 1 \leq j \leq n,\ 1 \leq i \leq r_j\}$.

So far we have not given a linear order on the set of labels T_n. We now do this. Choose any linear order of Λ which satisfies the following condition

$$(i, j) <_\Lambda G \Longrightarrow i < r_j, \qquad \text{and} \qquad (i, j) >_\Lambda G \Longrightarrow i = r_j. \tag{2}$$

That is, the labels above the element G in the ordering of Λ are those of the form (r_j, j).

Lemma 4.1 *Let Λ be a linear order on the set T_n satisfying condition (2). Then the above described edge-labeling for the r-cubical lattice is an R-labeling.*

5 Augmented r-signed permutations

Definition 5.1 *Let N be a finite set of cardinality n and let $\mathbf{r}$ be a vector which is indexed by the set N, that is, $\mathbf{r} = (r_i)_{i \in N}$. An augmented $\mathbf{r}$-signed permutation σ on the set N is a list of the form*

$$(G, (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)),$$

where $1 \leq i_m \leq r_{j_m}$ and $(j_1, j_2, \ldots, j_n)$ is a permutation of the elements in the set N. We will write $\sigma_0 = G$ and $\sigma_k = (i_k, j_k)$.

We view the elements $i_1, \ldots, i_n$ as signs; hence the name $\mathbf{r}$-signed permutation. Since we list the special element G first, we say that the permutation is augmented. Thus if we exclude the special element G, we may say that the permutations is *non-augmented.* Usually, we will consider the set $N = [n] = \{1, 2, \ldots, n\}$. For $0 \leq i \leq j \leq n$ we let $[i, j] = \{i, i + 1, \ldots, j\}$. We use the notation $\sigma|_{[i,j]}$ to denote the restricted permutation $\sigma|_{[i,j]} = (\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$.

Let Λ be a linear order on the set T_n. The descent set of an augmented $\mathbf{r}$-signed permutation, $\sigma = (G = \sigma_0, \sigma_1, \ldots, \sigma_n)$, with respect to Λ is the set $D_\Lambda(\sigma) = \{i \,:\, \sigma_{i-1} >_\Lambda \sigma_i\}$. The same definition also applies to non-augmented $\mathbf{r}$-signed permutations.

The maximal chains in the $\mathbf{r}$-cubical lattice correspond to augmented $\mathbf{r}$-signed permutations on the set $[n]$. Thus the number of augmented $\mathbf{r}$-signed permutations having a certain descent set is equal to the number of maximal chains with this same descent set.

We denote the **ab**-index of the **r**-cubical lattice by $\Psi(C^{\mathbf{r}}) = \Psi(\mathbf{r}) = \Psi(r_1, \ldots, r_n)$. For a vector $\mathbf{r} = (r_1, \ldots, r_n)$ and a positive integer s, we write $(\mathbf{r}, s)$ for the vector $(r_1, \ldots, r_n, s)$. Let $\overline{\mathbf{c}}_s = \mathbf{a} + (s-1)\mathbf{b}$, and $\overline{\mathbf{d}}_s = \mathbf{ab} + (s-1)\mathbf{ba}$. For N a finite subset of $\mathbb{P} = \{1, 2, \ldots\}$, define the vector $\mathbf{r}_N$ by $(r_{m_1}, \ldots, r_{m_n})$ where $N = \{m_1, \ldots, m_n\}$. Another useful notation is $\Pi(N) = \prod_{m \in N} r_m$.

Proposition 5.2 *The* **ab***-index of the* $(\mathbf{r}, s)$*-cubical lattice satisfies the following recurrence:*

$$\Psi(\mathbf{r}, s) = \Psi(\mathbf{r}) \cdot \overline{\mathbf{c}}_s + \sum_{\substack{I + J = [n] \\ I \neq \emptyset}} \Pi(I) \cdot \Psi(\mathbf{r}_J) \cdot \overline{\mathbf{d}}_s \cdot \Psi\left(B_{|I|}\right),$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ *and* $\Psi(C^{\emptyset}) = 1$.

The idea of the proof is to sum over all augmented **r**-signed permutations, and see where the element $n+1$ occurs. Before this element there is an augmented $\mathbf{r}_I$-signed permutation, and after it there is a non-augmented $\mathbf{r}_J$-signed permutation.

With the above recurrence we may compute:

$$\begin{aligned}
\Psi(C^{\emptyset}) &= 1 \\
\Psi(C^{p}) &= \overline{\mathbf{c}}_p \\
\Psi(C^{p,q}) &= \overline{\mathbf{c}}_p \overline{\mathbf{c}}_q + p \cdot \overline{\mathbf{d}}_q \\
\Psi(C^{p,q,r}) &= \overline{\mathbf{c}}_p \overline{\mathbf{c}}_q \overline{\mathbf{c}}_r + p \cdot \overline{\mathbf{d}}_q \overline{\mathbf{c}}_r + p \cdot \overline{\mathbf{c}}_q \overline{\mathbf{d}}_r + q \cdot \overline{\mathbf{c}}_p \overline{\mathbf{d}}_r + pq \cdot \overline{\mathbf{d}}_r \mathbf{c}.
\end{aligned}$$

When we set $\mathbf{r} = (2, \ldots, 2)$ in Proposition 5.2, we obtain

$$\Psi(C_{n+1}) = \Psi(C_n) \cdot \mathbf{c} + \sum_{i=1}^{n} \binom{n}{i} \cdot 2^i \cdot \Psi(C_{n-i}) \cdot \mathbf{d} \cdot \Psi(B_i), \tag{3}$$

where $\Psi(C_0) = 1$. This identity was first established by Purtill [13, Corollary 5.12]. Specializing to the cubical lattice we obtain

$$\begin{aligned}
\Psi(C_0) &= 1 & \Psi(C_1) &= \mathbf{c} \\
\Psi(C_2) &= \mathbf{c}^2 + 2 \cdot \mathbf{d} & \Psi(C_3) &= \mathbf{c}^3 + 4 \cdot \mathbf{cd} + 6 \cdot \mathbf{dc}.
\end{aligned}$$

By the recursion (3) it is easy to see that the coefficients of each **cd**-monomial in $\Psi(C_n)$ are positive. Thus by Lemma 2.2 we conclude

Theorem 5.3 (Readdy [15]) *For arbitrary rank selections S from the cubical lattice C_n, the two unique extremal configurations which maximize the beta-invariant $\beta(S)$ are the following rank selections:*

$$\{1, 3, 5, \ldots\} \cap [n] \quad \textit{and} \quad \{2, 4, 6, \ldots\} \cap [n].$$

6 Rank-selected ideals

Let P be a bounded poset of rank $n+1$. Recall that a *lower order ideal* is a subset $\mathcal{I}$ of P such that if $x \in \mathcal{I}$ and $y \leq x$ then $y \in \mathcal{I}$. An *upper order ideal* is similarly defined with $\leq$ replaced by $\geq$. A *rank-selected lower order ideal* is a lower order ideal of the form $P(S)$, where $S = [1,k] \subseteq [n]$. Likewise, we can define a *rank-selected upper order ideal.* In this section we are interested in maximizing $\beta(S)$ over rank selections S of the form $[1,k]$ and $[k,n]$ from the r-cubical lattice. This problem is equivalent to maximizing the Möbius function (in absolute value) over rank-selected lower order ideals and rank-selected upper order ideals, respectively.

Theorem 6.1 *For rank selections $[1,k] \subseteq [n]$ from the r-cubical lattice $C_n^{\mathbf{r}}$ of rank $n+1$, where $r \geq 2$, $\beta([1,k])$ attains a maximum when we take k to be $\left\lfloor \frac{(r-1)(n+1)}{2r-1} \right\rfloor$ or $\left\lceil \frac{(r-1)(n+1)}{2r-1} \right\rceil$.*

To prove the necessary inequalities in this theorem, consider the bipartite graph on the vertex set $\mathcal{B}_n^{\mathbf{r}}[1,k-1] \cup \mathcal{B}_n^{\mathbf{r}}[1,k]$. Say that two permutations are adjacent if by moving an element we can obtain one permutation from the other. Now by enumerating the edges in the graph, the inequalities follow.

We can also consider the question of maximizing $\beta(S)$ of the r-cubical lattice over rank-selected upper order ideals. Via an argument similar to that for Theorem 6.1, we find:

Theorem 6.2 *For rank selections $[k,n] \subseteq [n]$ from the r-cubical lattice of rank $n+1$, where $r \geq 2$, $\beta([k,n])$ attains a maximum when we take k to be $\left\lfloor \frac{n+r+2}{r+1} \right\rfloor$ or $\left\lceil \frac{n+r+1}{r+1} \right\rceil$.*

As a remark, numerical calculations strongly suggest that in Theorem 6.1, $\beta([1,k])$ attains a unique maximum when we take k to be $\left\lceil \frac{(r-1)n}{2r-1} \right\rceil$. When $r = 2$, it is known that $\beta([1,k])$ attains a unique maximum when we take k to be $\lceil \frac{n}{3} \rceil$, see [15, Lemma 2.2.1]. In fact, for arbitrary lower order ideals (i.e., not necessarily rank-selected) from the cubical lattice the same configuration maximizes the Möbius function in absolute value [15, Theorem 2.0.1]. It would be interesting to see if the same extremal result holds for arbitrary lower order ideals from the r-cubical lattice.

7 André permutations

Purtill showed a relation between the **cd**-index of the cubical lattice and André signed permutations [13]. In this section we define augmented André **r**-signed permutations and obtain a relation between these permutations and the **r**-**cd**-index of the **r**-cubical lattice. We study two sets of **r**-signed permutations, $\mathcal{A}^{\mathbf{r}}$ and $\mathcal{N}_0^{\mathbf{r}}$. The set $\mathcal{A}^{\mathbf{r}}$ corresponds to the **r**-cubical lattice and the set $\mathcal{N}_0^{\mathbf{r}}$ to the boolean algebra. The proofs of the two fundamental identities in this section (Proposition 7.2 and Theorem 7.3) follow by showing both sides of each identity satisfy the same recurrence. We also enumerate the number of augmented André r-signed permutations. When we set $\mathbf{r} = (2, \ldots, 2)$ the results of this section specialize to Purtill's work.

Define the set T to be $T = \{(i,j) \ : \ j \in \mathbb{P}, \ 1 \le i \le r_j\} \cup \{G\}$. Observe that the entries of $\mathbf{r}$-signed permutations are elements of T. Throughout what follows in this section we fix Λ, a linear order on the set T. Define $G <_\Lambda (r_1, 1) <_\Lambda (r_2, 2) <_\Lambda \cdots$. Order the labels of the form (i,j), where $i < r_j$, by $(i_1, j_1) <_\Lambda (i_2, j_2)$ if $j_1 > j_2$, or if $j_1 = j_2$ and $i_1 < i_2$. Finally say that $(i,j) <_\Lambda G$ if and only if $i < r_j$. The linear order Λ satisfies condition (2).

Definition 7.1 *Let $\mathbf{r}$ be a vector which is indexed by a finite set N with $|N| = n > 0$, that is, $\mathbf{r} = (r_i)_{i \in N}$. We may assume $N \subseteq \mathbb{P}$. We say an augmented $\mathbf{r}$-signed permutation $\sigma = (G = \sigma_0, \sigma_1, \ldots, \sigma_n)$ on the set N is an* augmented André $\mathbf{r}$-signed permutation *if the following three conditions are satisfied:*

1. *The permutation σ has no double descents, that is, there is no index i such that σ has a descent at the ith and $(i+1)$st positions.*
2. *For all $1 \le i < j \le n$, if $\sigma_{i-1} = \max_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$ and $\sigma_j = \min_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$, then there exists a k, with $i < k < j$, such that $\sigma_{i-1} <_\Lambda \sigma_k$.*
3. *For $x = \max N$, $(r_x, x) = \sigma_m$ for some $1 \le m \le n$ and $\sigma|_{[0,m-1]}$ is an augmented André $\mathbf{r}_J$-signed permutation on the set J, where $J = \{y \in N \ : \ (z,y) = \sigma_k$ for some $1 \le k \le m-1\}$.*

The permutation (G) is allowed to be an augmented André $\mathbf{r}$-signed permutation on the set $N = \emptyset$.

A non-augmented $\mathbf{r}$-signed permutation satisfying conditions 1 and 2 in Definition 7.1 is called an *non-augmented André $\mathbf{r}$-signed permutation.* (Note that for the non-augmented case we need to reformulate the beginning of condition 2 as, "For all $2 \le i < j \le n$ ...".) We denote the set of all augmented and all non-augmented André $\mathbf{r}$-signed permutations respectively by $\mathcal{A}^\mathbf{r}$ and $\mathcal{N}^\mathbf{r}$. Furthermore, we define the set of all non-augmented André $\mathbf{r}$-signed permutations which begin with its smallest element (with respect to the linear order Λ) by $\mathcal{N}_0^\mathbf{r}$. That is,

$$\mathcal{N}_0^\mathbf{r} = \{(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{N}^\mathbf{r} \ : \ \sigma_1 = \min_\Lambda\{\sigma_1, \sigma_2, \ldots, \sigma_n\}\}.$$

We will mainly work with the sets $\mathcal{A}^\mathbf{r}$ and $\mathcal{N}_0^\mathbf{r}$.

For σ a non-augmented André $\mathbf{r}$-signed permutation of an n-set, the *variation* of σ is given by $U(\sigma) = u_S$, where S is the descent set of σ taken with respect to Λ and u_S is the $\mathbf{ab}$-word defined in Section 2. The *reduced variation* of $\sigma \in \mathcal{N}_0^\mathbf{r}$, which we denote by $V(\sigma)$, is formed by replacing each $\mathbf{ab}$ in $U(\sigma)$ with $\mathbf{d}$ and then replacing each remaining $\mathbf{a}$ by a $\mathbf{c}$. Observe that this is always possible since an element in $\mathcal{N}_0^\mathbf{r}$ does not begin with an descent and cannot have any double descents. We recursively define the reduced variation $V(\sigma)$ for an augmented André $\mathbf{r}$-signed permutation σ on the set N. Assume that N has cardinality n. If $\sigma_m = (r_x, x) = (s, x)$, where $x = \max N$, then

$$V(\sigma) = \begin{cases} V(\sigma|_{[0,m-1]}) \cdot \overline{\mathbf{d}}_s \cdot V(\sigma|_{[m+1,n]}) & \text{if } m < n \\ V(\sigma|_{[0,n-1]}) \cdot \overline{\mathbf{c}}_s & \text{if } m = n, \end{cases}$$

with $V(G) = 1$. This definition makes sense since $\sigma|_{[m+1,n]}$ belongs to the set $\mathcal{N}_0^{\mathbf{r}_I}$.

Proposition 7.2 *For $\mathbf{r} = (r_1, \ldots, r_n)$, the following equality holds:*

$$\sum_{\sigma \in \mathcal{N}_0^{\mathbf{r}}} V(\sigma) = \Pi(N) \cdot \Psi(B_n).$$

We denote this sum by $V(\mathcal{N}_0^{\mathbf{r}})$.

Theorem 7.3 *There exists a bijection between the two sets $\mathcal{A}^{\mathbf{r},s}$ and $\mathcal{A}^{\mathbf{r}} \cup \bigcup \mathcal{A}^{\mathbf{r}_J} \times \mathcal{N}_0^{\mathbf{r}_I}$, where the union ranges over all $I + J = [n]$ with $I \neq \emptyset$, all the unions are disjoint, and $\times$ is the Cartesian product. Moreover, the following equality holds:*

$$\Psi(\mathbf{r}) = \sum_{\sigma \in \mathcal{A}^{\mathbf{r}}} V(\sigma).$$

We denote this sum by $V(\mathcal{A}^{\mathbf{r}})$ and call it the non-commutative augmented André $\mathbf{r}$-signed polynomial.

Theorem 7.4 *Let $\mathbf{r} = (r, \ldots, r)$. Then the exponential generating function of the number of augmented André $\mathbf{r}$-signed permutations is*

$$\sum_{n \geq 0} g_n \cdot \frac{x^n}{n!} = \sqrt[r]{\frac{1}{1 - \sin(rx)}}, \qquad \text{with} \qquad g_n \sim \frac{1}{\Gamma(\frac{2}{r})} \cdot \left(\frac{8}{\pi^2}\right)^{\frac{1}{r}} \cdot n^{\frac{2}{r}-1} \cdot \left(\frac{2r}{\pi}\right)^n \cdot n!.$$

8 Arbitrary rank selections

In this section we consider the problem of maximizing the beta invariant of the $\mathbf{r}$-cubical lattice over arbitrary rank selections. We will do so by showing that $\Psi(C^{\mathbf{r}_N})$ has the strictly increasing alternating property. We will assume that $r_1, r_2, \ldots$ are all positive integers greater than or equal to 2. Let $N = \{m_1, \ldots, m_n\}$ be a finite subset of $\mathbb{P}$ of cardinality n. For an $\mathbf{ab}$-word w of length n we define $\beta(w, N)$ to be the coefficient of w in the $\mathbf{ab}$-index $\Psi(C^{\mathbf{r}_N})$. Thus $\beta(w, N)$ is a symmetric function in the variables $r_{m_1}, \ldots, r_{m_n}$. Also, we let $\beta_B(w)$ be the coefficient of w in the $\mathbf{ab}$-index of the boolean algebra, $\Psi(B_{|w|+1})$. Thus we have the two identities:

$$\Psi(C^{\mathbf{r}_N}) = \sum_w \beta(w, N) \cdot w, \qquad \text{and} \qquad \Psi(B_{n+1}) = \sum_w \beta_B(w) \cdot w,$$

where w ranges over all $\mathbf{ab}$-words of length n. Observe that since the coefficients $\beta(w, N)$ enumerate augmented $\mathbf{r}$-signed permutations, we know that they are non-negative.

One may also conclude that $\beta(w, N)$ may be written as a linear combination of the elementary symmetric functions in the variables $r_{m_1}, \ldots, r_{m_n}$. Thus $\beta(w, N)$, viewed as function of r_{m_i}, will be a polynomial of degree one.

Since the $\mathbf{cd}$-index of the boolean algebra has positive coefficients (this may be verified by equation (1)), we deduce the following strict inequality:

$$\beta_B(vw^*) > \beta_B(vw) \quad \text{when} \quad v^{((1))} = w^{(1)}.$$

This fact will be useful to us later. Similarly, for the cubical lattice, that is when $\mathbf{r} = (2, \ldots, 2)$, we also know that the **cd**-index has positive coefficients. Thus the same strict inequality holds. Since the cubical lattice is Eulerian, we know that $\beta(w, [n])$ attains a maximum exactly when w is alternating. That is, when $w = \mathbf{baba}\cdots$ or $w = \mathbf{abab}\cdots$ with $|w| = n$.

Theorem 8.1 *Let $\mathbf{r} = (r_1, \ldots, r_n)$, where $r_1, \ldots, r_n \geq 2$, and at least one entry is greater than or equal to 3. Then the* **ab***-index of the* **r***-cubical lattice $C^{\mathbf{r}}$ has the strictly increasing alternating property.*

Corollary 8.2 *Let $\mathbf{r} = (r_1, \ldots, r_n)$, where $r_1, \ldots, r_n \geq 2$, and at least one entry is greater than or equal to 3. For arbitrary rank selections $S \subseteq [n]$ of the* **r***-cubical lattice $C^{\mathbf{r}}$, $\beta(S)$ attains a unique maximum when we take S to be $\{1, 3, 5, \ldots\} \cap [n]$.*

When $\mathbf{r} = (2, \ldots, 2)$ the lattice $C^{\mathbf{r}}$ is the cubical lattice C_n. As was observed in Theorem 5.3, this lattice has two extremal configurations, namely $\{1, 3, 5, \ldots\} \cap [n]$ and $\{2, 4, 6, \ldots\} \cap [n]$.

Proof of Theorem 8.1: The proof is by induction on n. Assume that the theorem holds for all values less than or equal to n, and we would like to prove it for $n+1$. Say that $r_{n+1} = s$. Consider the $(r_1, \ldots, r_n, s)$-cubical lattice, where $s \geq 2$. Let $\mathcal{K}$ denote the coefficient of the linear term in s in the expression $\Psi(\mathbf{r}, s)$. Then we may write $\Psi(\mathbf{r}, s) = \Psi(\mathbf{r}, 2) + (s-2) \cdot \mathcal{K}$. The theorem will follow once we are able to show that $\Psi(\mathbf{r}, 2)$ has the strictly increasing alternating property and $\mathcal{K}$ has the weakly increasing alternating property.

We begin by showing that $\Psi(\mathbf{r}, 2)$ has the strictly increasing alternating property. Recall the recursion formula for $\Psi(\mathbf{r}, s)$ in Proposition 5.2. We have that

$$\Psi(\mathbf{r}, 2) = \Psi(\mathbf{r}) \cdot \mathbf{c} + \sum_{\substack{I+J=[n] \\ I \neq \emptyset}} \Pi(I) \cdot \Psi(\mathbf{r}_J) \cdot \mathbf{d} \cdot \Psi\left(B_{|I|}\right). \tag{4}$$

By Lemma 2.3 we know that each term in equation (4) has the weakly increasing alternating property. Hence this sum has the weakly increasing alternating property. By a refined argument, which we omit, one obtains that $\Psi(\mathbf{r}, 2)$ has the strictly increasing alternating property.

Recall that $\mathcal{K} = [s]\Psi(\mathbf{r}, s)$, where $[s]$ denotes the coefficient of the linear term in the variable s. By the recursion formula in Proposition 5.2 we have

$$\mathcal{K} = \Psi(\mathbf{r}) \cdot \mathbf{b} + \sum_{\substack{I+J=[n] \\ I \neq \emptyset}} \Pi(I) \cdot \Psi(\mathbf{r}_J) \cdot \mathbf{ba} \cdot \Psi\left(B_{|I|}\right).$$

The proof that $\mathcal{K}$ has the weakly increasing alternating property follows from the induction hypothesis by a non-trivial argument, which we omit. □

The exponential generating function for the number of augmented r-signed permutations having **ab**-word $\mathbf{bab}\cdots$ (letting $h_n = \beta(\mathbf{bab}\cdots, [n])$) and the asymptotics are given by (see [6])

$$\sum_{n \geq 0} h_n \cdot \frac{x^n}{n!} = \frac{\sin((r-1)x) + \cos(x)}{\cos(rx)} \qquad \text{and} \qquad h_n \sim \frac{4}{\pi} \cdot \cos\left(\frac{\pi}{2r}\right) \cdot \left(\frac{2r}{\pi}\right)^n \cdot n!.$$

9 Concluding remarks

There are many related problems to study. For instance, are there other posets which have an **r**-**cd**-index? More generally, are there other extensions of the **cd**-index? The authors have found an example of a poset other than the **r**-cubical lattice which has an **r**-**cd**-index. In fact, it has the same **ab**-index as C_n^r. What other classes of posets will have their **ab**-index satisfying the strictly increasing alternating property? A poset that seems to fulfill a similar condition is the partition lattice Π_n. Our data suggests the **ab**-word with the largest coefficient in $\Psi(\Pi_n)$ is the word $\mathbf{bab}\cdots\mathbf{b}$ of length $n-2$.

Stanley proved the **cd**-index of Gorenstein* lattices has non-negative coefficients, see [18, Theorem 2.2]. This includes face lattices of convex polytopes. From this he observed the beta invariant will reach its maximum value, for arbitrary rank selections, by taking alternating rank selections. However, uniqueness of this result does not follow from his observation. By Lemma 2.2 it would be enough to show the **cd**-index of the face lattice of a convex polytope has positive coefficients. Stanley has conjectured that among all Gorenstein* lattices of rank n, the boolean algebra B_n minimizes all the coefficients of the **cd**-index [19, Conjecture 2.7].

The exponential generating function $\sqrt[r]{\frac{1}{1-rx}}$ enumerates r-multipermutations, see [12]. Notice that both this generating function and the one for the number of augmented André r-signed permutations are of the form $\sqrt[r]{f(rx)}$, where $f(x)$ is an exponential generating function. Is there a theory which explains generating functions of this form?

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