LATTICE PATH PROOFS FOR DETERMINANTAL FORMULAS FOR SYMPLECTIC AND ORTHOGONAL CHARACTERS

(Shortened version. The full-length article will appear elsewhere)

MARKUS FULMEK AND CHRISTIAN KRATTENTHALER

Abstract. We give bijective proofs for Jacobi–Trudi-type and Giambelli-type identities for symplectic and orthogonal characters. These proofs are based on interpreting King and El-Sharkaway's symplectic tableaux, Proctor's odd and intermediate symplectic tableaux, Proctor's orthogonal tableaux, and Sundaram's odd orthogonal tableaux in terms of certain families of nonintersecting lattice paths. This work is intended to be the counterpart of the Gessel–Viennot proof of the Jacobi–Trudi identities for Schur functions for the case of symplectic and orthogonal characters.

Résumé. On donne des démonstrations bijectives des identités de type Jacobi–Trudi et Giambelli pour les caractères symplectiques et orthogonaux. Ces démonstrations sont basées sur une interprétation des tableaux symplectiques de King et El-Sharkaway, des tableaux symplectiques impairs et symplectiques intermédiaires de Proctor, des tableaux orthogonaux de Proctor, et des tableaux orthogonaux impairs de Sundaram en termes de certaines familles de chemins qui ne s'entrecoupent pas. Le but de ce travail est de proposer, dans le cas des caractères symplectiques et orthogonaux, des démonstrations analogues à la preuve de Gessel et Viennot des identités de Jacobi–Trudi pour les fonctions de Schur.

1. Introduction

Schur functions, the irreducible general linear characters, can be combinatorially defined by means of semistandard Young tableaux (see [5, I, (5.12)]). There are several determinant formulas for Schur functions. Those which are relevant for this paper are the Jacobi–Trudi identity and its dual form, and the Giambelli identity (see [5, I, (3.4), (3.5), p. 30, Ex. 9]). In their well-known (yet unpublished) paper [3] Gessel and Viennot give a beautiful bijective proof for the Jacobi–Trudi identities for Schur functions (see also [12, sec. 7+1]). It bases on interpreting semistandard tableaux as families of nonintersecting lattice paths. As was shown by Stembridge [12, sec. 9], also the Giambelli identity allows a bijective proof by using nonintersecting lattice paths.

There are Jacobi–Trudi-type and Giambelli-type determinant formulas for irreducible symplectic and orthogonal characters (see [2, Prop. 24.22, Cor. 24.24, Prop. 24.44, Prop. 24.33, (24.47)]), too. Since there are also tableaux descriptions for symplectic and orthogonal characters, it is natural to ask for bijective proofs of the symplectic and orthogonal Jacobi–Trudi and Giambelli identities. First attempts in this direction for the Jacobi–Trudi identities were made by Bressoud and Wei [1], and more successfully by Okada [6]. However, the determinant formulas that Okada proved bijectively are different from the Jacobi–Trudi identities (but are interesting in their own right). Additional algebraic steps were necessary to prove the Jacobi–Trudi identities themselves.

We solve the problem completely for the symplectic case and partially for the orthogonal case. For the bijective proofs of the symplectic identities we utilize lattice path interpretations of the tableaux given by King and El-Sharkaway [4]. Particularly nice is the bijective proof for the *dual* symplectic Jacobi–Trudi identity, which combines the Gessel–Viennot method with a modified reflection principle. The bijective proof for the "ordinary" symplectic Jacobi–Trudi identity is more elaborate. On the other hand, the bijective proof of the symplectic Giambelli identity is almost trivial. In addition, we are able to provide bijective proofs for all of Proctor's [7, 9, 10] Jacobi–Trudi identities for his odd symplectic and intermediate symplectic characters.

There are several candidates for orthogonal tableaux. We show that Proctor's orthogonal tableaux [10, 8] are the "right" tableaux for proving the dual orthogonal Jacobi–Trudi identity. As might be surprising at first sight, these tableaux have a very natural lattice path interpretation. The proof itself is very illustrative. It also employs the Gessel–Viennot method and some kind of reflection argument. Unfortunately, we were not able to use the same tableaux in order to bijectively prove the "ordinary" orthogonal Jacobi–Trudi identity and the orthogonal Giambelli identity. As a substitute, by using Sundaram's tableaux [14] at least we are able to give bijective proofs for the odd orthogonal Giambelli identity and for determinant formulas that are only slight variations of the odd orthogonal Jacobi–Trudi identities.

2. Some Definitions

By *paths* we always mean lattice paths in the plane integer lattice $\mathbb{Z}^2$ consisting of unit horizontal and vertical steps in the positive direction, unless we explicitly allow other steps. Given points u and v, we denote the set of all lattice paths from u to v by $\mathcal{P}(u,v)$. If $\mathbf{u}=(u_1,\dots,u_m)$ and $\mathbf{v}=(v_1,\dots,v_m)$ are vectors of points, we denote the set of all m-tuples $(P_1,\dots,P_m)$ of paths, where P_i runs from u_i to v_i, $i=1,\dots,m$, by $\mathcal{P}(\mathbf{u},\mathbf{v})$. A set of paths is said to be *nonintersecting* if no two paths of this set have a point in common. Otherwise it is called *intersecting*. If to each horizontal edge a in $\mathbb{Z}^2$ a weight $w(a)$ is assigned, the weight $w(P)$ of a path P is defined to be the product of the weights of all its horizontal steps. The weight $w(\mathbf{P})$ of an m-tuple $\mathbf{P}=(P_1,\dots,P_m)$ is defined to be the product $\prod_{i=1}^m w(P_i)$ of the weights of all the paths in the m-tuple. Given any weight function w defined on a set $\mathcal{A}$, by the generating function $\mathbf{GF}(\mathcal{A})$ we mean $\sum_{x\in\mathcal{A}} w(x)$.

3. Bijective proofs for symplectic identities

Bijective proof of the symplectic dual Jacobi–Trudi identity. Let $\lambda=(\lambda_1,\dots,\lambda_r)$ be a partition of length $r\le n$. A (semistandard) tableau T of shape λ with entries from $\{1,2,\dots\}$ is called a *symplectic* (*semistandard*) *tableau* (see [7, 13]) if it obeys the additional constraint

$$T_{i,j}\ge 2i-1. \tag{3.1}$$

Let $\mathbf{x}=\langle x_1,x_1^{-1},x_2,x_2^{-1},\dots,x_n,x_n^{-1}\rangle$. The weight of a symplectic tableau T is given by

$$\mathbf{x}^T=\prod_i x_i^{\left|\left\{T_{j,k}=2i-1\right\}\right|-\left|\left\{T_{j,k}=2i\right\}\right|}. \tag{3.2}$$

The *symplectic character* associated to λ is combinatorially defined by (see [7, 13]) $sp_{2n}(\lambda,\mathbf{x})=\sum_T \mathbf{x}^T$, where the sum is over all symplectic tableaux T of shape λ with entries $\le 2n$.

We are going to sketch a bijective proof of the symplectic "e-formula" (see [2, Cor, 24.24])

$$sp_{2n}(\lambda,\mathbf{x})=|e_{\lambda'_j-j+i}(\mathbf{x})-e_{\lambda'_j-j-i}(\mathbf{x})|_{\lambda_1\times\lambda_1}. \tag{3.3}$$

Here, $e_m(\mathbf{x})$ denotes the elementary symmetric function of order m in the variables $\mathbf{x}$. As usual, λ' denotes the partition conjugate to λ. Formula (3.3) is the symplectic analogue of the dual form of the Jacobi–Trudi identity for Schur functions.

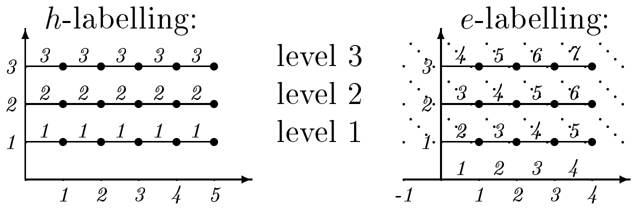
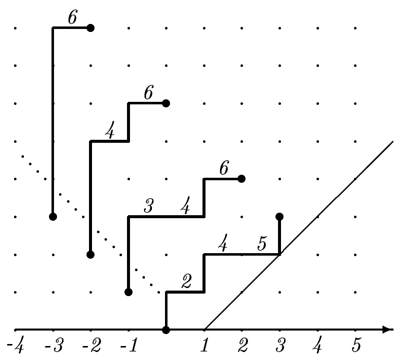


Figure 1

First we interpret symplectic tableaux in terms of lattice paths. Let T be a symplectic tableau of shape λ with entries $\le 2n$, where λ is a partition of length $r \le n$. With T we associate a λ_1-tuple $(P_1, \dots, P_{\lambda_1})$ of nonintersecting lattice paths, where P_j runs from $u_j = (-j+1, j-1)$ to $v_j = (\lambda'_j - j + 1, 2n - \lambda'_j + j - 1)$, by reading P_j off the j-th column. Here we use the e-labelling of horizontal edges which is e.g. explained in [11, ch. 4] (see Figures 1 and 2).

Obviously, T obeys the symplectic constraint if and only if the first path of the associated λ_1-tuple of paths does not cross the line $y = x - 1$. The following figure gives an example for $n = 3$ and $\lambda = (4, 3, 2)$.



$T =$

2	3	4	6
4	4	6	
5	6		

Figure 2

If we define the weight of a horizontal edge with e-label (cf. Figure 1) $2i-1$ to be x_i, and the weight of a horizontal edge with e-label $2i$ to be x_i^{-1} then the correspondence depicted in Figure 2 is weight-preserving with respect to the weight (3.2). Therefore the left side of (3.3) can be interpreted as the generating function for all λ_1-tuples $(P_1, \dots, P_{\lambda_1})$ of nonintersecting lattice paths, where P_j runs from $(-j+1, j-1)$ to $(\lambda'_j - j + 1, n - \lambda'_j + j - 1)$ and does not cross $y = x - 1$.

Next we give the lattice path interpretation of the right side of (3.3). Let $\mathbf{R}$ denote the reflection in the line $y = x - 2$. For a permutation $\sigma \in \boldsymbol{S}_{\lambda_1}$ denote $(v_{\sigma(1)}, \dots, v_{\sigma(\lambda_1)})$ by $\mathbf{v}_\sigma$. For $\varepsilon \in \{1, -1\}^{\lambda_1}$ denote by $\mathbf{u}^{(\varepsilon)}$ the λ_1-tuple of points whose j-th component is u_j if $\varepsilon_j = 1$ and $\mathbf{R}\,(u_j)$ if $\varepsilon_j = -1$. We consider the set $\bigcup_{\sigma \in \boldsymbol{S}_{\lambda_1}, \varepsilon \in \{1,-1\}^{\lambda_1}} \mathcal{P}(\mathbf{u}^{(\varepsilon)}, \mathbf{v}_\sigma)$. It is easy to see that the determinant in (3.3) can be written as a certain generating function for this set,

$$\left| e_{\lambda'_j - j + i}\,(\mathbf{x}) - e_{\lambda'_j - j - i}\,(\mathbf{x}) \right|_{\lambda_1 \times \lambda_1} = \sum_{\sigma \in \boldsymbol{S}_{\lambda_1}, \varepsilon \in \{1,-1\}^{\lambda_1}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\varepsilon)\, \mathbf{GF}\left(\mathcal{P}\left(\mathbf{u}^{(\varepsilon)}, \mathbf{v}_\sigma\right)\right),$$

where $\operatorname{sgn}(\varepsilon) = \prod_{i=1}^{\lambda_1} \varepsilon_i$. In order to establish (3.3), we have to give an involution that in the right-hand side sum cancels all contributions of λ_1-tuples of paths in $\bigcup_{\sigma \in \boldsymbol{S}_{\lambda_1}, \varepsilon \in \{1,-1\}^{\lambda_1}} \mathcal{P}(\mathbf{u}^{(\varepsilon)}, \mathbf{v}_\sigma)$ that are either intersecting or contain a path that crosses the line $y = x - 1$.

This is done by a combination of the Gessel–Viennot method [3, 12, sec. 1] and a modified reflection principle. A path P crossing the line $y = x - 1$ must meet the line $y = x - 2$. Let s be the last meeting point. Now, to P's initial portion up to s apply the following modified reflection in $y = x-2$, which is exemplified in Figure 3. All points $(x, y) \in P$ with $x+y \equiv 0 \pmod 2$ (call them *even points*) are reflected in the usual way, i.e., $(x, y) \mapsto (y + 2, x - 2)$, and so are all *odd* points ($x + y \equiv 1 \pmod 2$) whose adjacent steps are both vertical or both horizontal. The remaining case is a "kink" in an odd point p, i.e., a horizontal step of the path meets a vertical one in p. Here the whole kink is shifted until its even points have reached their new (reflected) positions.

Now we are able to describe the desired weight-preserving and sign-reversing involution. For a λ_1-tuple $(P_1, \dots, P_{\lambda_1})$ containing a path meeting the forbidden line $y = x-2$, choose i minimal such that P_i meets $y = x - 2$, and replace P_i's portion up to the last meeting point with the line $y = x - 2$ by its *modified* reflection. Clearly, this mapping is weight-preserving and sign-reversing. It reverses sign since it changes the sign $\operatorname{sgn}(\varepsilon)$ of ε while leaving σ invariant. On λ_1-tuples of paths that do not contain any path crossing $y = x - 1$ but are intersecting we apply the Gessel-Viennot involution. Clearly, we thus obtain again a λ_1-tuple of paths that does not contain any path crossing $y = x - 1$ but is intersecting. It is easy to see that this mapping is a weight-preserving and sign-reversing involution. Thus only those λ_1-tuples remain that are nonintersecting and none of their paths cross the line $y = x - 1$. But as was exhibited above, these λ_1-tuples correspond to symplectic tableaux. □

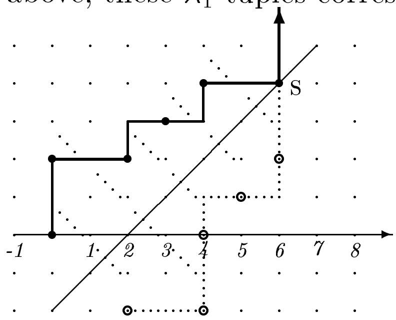


Figure 3

Bijective proof of the symplectic Jacobi–Trudi identity. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of length $r \le n$. A bijective proof for the symplectic "h-formula", the symplectic analogue of the "ordinary" Jacobi–Trudi identity for Schur functions, (see [2, Prop. 24.22])

$$sp_{2n}(\lambda, \mathbf{x}) = \left| h_{\lambda_i - i + 1}(\mathbf{x}) \quad \vdots \quad h_{\lambda_i - i + j}(\mathbf{x}) + h_{\lambda_i - i - j + 2}(\mathbf{x}) \right|_{r \times r}, \tag{3.4}$$

is more difficult. Here, $h_m(\mathbf{x})$ denotes the complete homogeneous symmetric function of order m in the variables $\mathbf{x}$. We shall give only a rough sketch of the algorithm without rigorous argumentation. This time we encode symplectic tableaux of shape λ in terms of r-tuples $(P_1, \dots, P_r)$ of nonintersecting lattice paths where P_i runs from $u_i = (r - i, 1)$ to $v_i = (\lambda_i + r - i, 2n)$, by reading P_i from the i-th row of the tableau. Here we use the usual h-labelling of horizontal edges [11, ch. 4] (see Figure 1). The symplectic constraint simply says that P_i must not contain a horizontal step strictly below the line $y = 2i - 1$. We define the weight of a horizontal edge with h-label $2i - 1$ to be x_i, and the weight of a horizontal edge with h-label $2i$ to be x_i^{-1} in order that this correspondence is weight-preserving with respect to the weight (3.2).

Let now $\mathbf{R}$ denote the reflection in the line $x = r - 1$. We shall consider the set $\bigcup_{\sigma\in \boldsymbol{S}_r,\, \varepsilon\in\{1\}\times\{1,-1\}^{r-1}} \mathcal{P}(\mathbf{u}^{(\varepsilon)}, \mathbf{v}_\sigma)$. The notation $\mathbf{u}^{(\varepsilon)}$ has to be understood in the same sense as above, only the reflection $\mathbf{R}$ has a different meaning now. The determinant in (3.4) can be written as

$$\left| h_{\lambda_i-i+1}(\mathbf{x}) \quad \vdots \quad h_{\lambda_i-i+j}(\mathbf{x}) + h_{\lambda_i-i-j+2}(\mathbf{x}) \right|_{r\times r} = \sum_{\sigma\in \boldsymbol{S}_r,\, \varepsilon\in\{1\}\times\{1,-1\}^{r-1}} \operatorname{sgn}(\sigma)\, \mathbf{GF}\left(\mathcal{P}\left(\mathbf{u}^{(\varepsilon)}, \mathbf{v}_\sigma\right)\right).$$

The following algorithm performs a cancellation in the set $\bigcup_{\sigma\in \boldsymbol{S}_r,\, \varepsilon\in\{1\}\times\{1,-1\}^{r-1}} \mathcal{P}(\mathbf{u}^{(\varepsilon)}, \mathbf{v}_\sigma)$ such that only those r-tuples of paths survive for which $\varepsilon = (1,\dots,1)$, and where P_i does not contain any horizontal step strictly below the line $y = 2i-1$. In the final step we shall apply Gessel–Viennot involution to cancel all intersecting r-tuples. Thus only those r-tuples will remain which correspond to symplectic tableaux.

Let $(P_1,\dots,P_r)$ be an r-tuple of paths where there is either a path P_i containing a horizontal step below $y = 2i-1$ or where the associated ε differs from $(1,1,\dots,1)$.

At first, we adjoin to each path P_i a set M_i of variables. M_i is defined by

$$M_i = \begin{cases} \emptyset & \text{if } \varepsilon_i = 1 \\ \{x_1, x_1^{-1}, \dots, x_{i-1}, x_{i-1}^{-1}\} & \text{if } \varepsilon_i = -1. \end{cases} \tag{3.5}$$

We are thus concerned with objects of the form

$$[(P_1, M_1), \dots, (P_r, M_r), \sigma, \varepsilon], \tag{3.6}$$

where $(P_1,\dots,P_r)$ is an r-tuple of paths with associated σ and ε, and the M_i's are subsets from the sequence $\mathbf{x}$. The values ε_i actually are encoded in the sets M_i. Note that for the above object the x-coordinate u_i^x of the starting point of P_i can be expressed in M_i's cardinality, $u_i^x = |M_i| + r - i$.

The algorithmic involution consists of several steps, each yielding "intermediate" objects. This set of intermediate objects, which we denote by $\mathcal{O}$, is the set of all objects of the form

$$[(P_1, M_1), \dots, (P_r, M_r), \sigma], \tag{3.7}$$

subject to

$$P_i \text{ runs from } (u_i^x, 1) \text{ to } (\lambda_{\sigma(i)} + r - \sigma(i), 2n) \tag{3.8}$$

$$M_i \subseteq \{x_1, x_1^{-1}, \dots, x_{i-1}, x_{i-1}^{-1}\} \tag{3.9}$$

$$|M_i| = u_i^x - r + i. \tag{3.10}$$

Together, conditions (3.9) and (3.10) imply $r - i \le u_i^x \le r + i - 2$.

We define the *weight* of an intermediate object in $\mathcal{O}$ to be the weight of the r-tuple of the paths times the product of all variables contained in the multiset union of the sets.

On the set $\mathcal{O}$ we define two operations, $\mathbf{A}$ and $\mathbf{B}$. First we define operation $\mathbf{A}$. Let k be minimal such that either P_k contains a horizontal step at height $h < 2k-1$ or its adjoined set M_k contains the h-th component of $\mathbf{x}$ ($x_{(h+1)/2}$ if h is odd and $x_{h/2}^{-1}$ if h is even); let h be minimal, too. Now replace (P_k, M_k) by

$$\begin{cases} (P_k \cup h, M_k \setminus \{h\}) & \text{if } h \in M_k, \\ (P_k \setminus h, M_k \cup \{h\}) & \text{if } h \notin M_k. \end{cases}$$

This has to be read in the following sense. If the h-th component of $\mathbf{x}$ is in M_k, remove it and insert a horizontal step in P_k at height h, fixing the end point (this shifts the starting point one unit to the left). If not, remove one horizontal step at height h in P_k, fixing the end point (this shifts the starting point one unit to the right) and adjoin the h-th component of $\mathbf{x}$ to M_k.

Two paths in an intermediate object may have a common starting point, or the *reflection* of a starting point in the line $x = r-1$ may coincide with the starting point of another path. For these two cases, we define operation $\mathbf{B}$. Let (k,l) be a (lexicographic) minimal pair of distinct indices so that either $u_k^x = u_l^x$ (starting points of P_k, P_l coincide) or $u_k^x + u_l^x = 2r-2$ (starting points of P_k, P_l lie symmetrically with respect to $x = r-1$).

In the first case, simply interchange P_k and P_l, and multiply σ with the transposition (l,k). This results in

$$[\cdots, (P_k', M_k), \dots, (P_l', M_l), \dots, \sigma \circ (l,k)], \tag{3.11}$$

where $P_k' = P_l$ and $P_l' = P_k$. Clearly, conditions (3.8), (3.9) and (3.10) are satisfied.

In the second case, again interchange paths P_k and P_l, and multiply σ with the transposition (l,k); in addition, replace M_k, M_l by their "complements" M_k', M_l', resulting in

$$[\cdots, (P_k', M_k'), \dots, (P_l', M_l'), \dots, \sigma \circ (l,k)], \tag{3.12}$$

where $P_k' = P_l$, $P_l' = P_k$ and $M_k' := \{x_1, x_1^{-1}, \dots, x_{k-1}, x_{k-1}^{-1}\} \setminus M_k^{-1}$, $M_l' := \{x_1, x_1^{-1}, \dots, x_{l-1}, x_{l-1}^{-1}\} \setminus M_l^{-1}$. Here, M_k^{-1} means the set $\{y^{-1} : y \in M_k\}$, and the same for M_l^{-1}.

Now we start with an object defined by (3.6) and (3.5), and forget the ε, thus obtaining an object $[(P_1, M_1), \dots, (P_r, M_r), \sigma]$, satisfying (3.8)–(3.10). Hence, it is an element of $\mathcal{O}$. We define a weight-preserving mapping by applying operation $\mathbf{A}$ to this object, then $\mathbf{B}$, then $\mathbf{A}$ again, etc., (formally: $\mathbf{B}^{[0 \text{ or } 1]} \circ (\mathbf{A} \circ \mathbf{B})^n \circ \mathbf{A}$), as long as the particular operation is applicable.

It is not difficult to see that the algorithm terminates with an application of $\mathbf{A}$. Let $[(\bar{P}_1, \bar{M}_1), \dots, (\bar{P}_r, \bar{M}_r), \bar{\sigma}]$, be the terminal object. It is easy to show that $\bar{u}_i^x \in \{r-i, r+i-2\}$. ($\bar{u}_i^x$ denotes the x-coordinate of the starting point of $\bar{P}_i$.) Besides, if $\bar{u}_i^x = r-i$ we have $\bar{M}_i = \emptyset$, and if $\bar{u}_i^x = r+i-2$ we have $\bar{M}_i = \{x_1, x_1^{-1}, \dots, x_{i-1}, x_{i-1}^{-1}\}$. In the first case we set $\bar{\varepsilon}_i = 1$, in the latter $\bar{\varepsilon}_i = -1$. We thus obtain an object

$$[(\bar{P}_1, \bar{M}_1), \dots, (\bar{P}_r, \bar{M}_r), \bar{\sigma}, \bar{\varepsilon}]. \tag{3.13}$$

Thus $(P_1, \dots, P_r)$ is mapped to $(\bar{P}_1, \dots, \bar{P}_r)$. It is true that the permutation $\bar{\sigma}$ exactly corresponds to the permutation of the end points of $\bar{P}_1, \dots, \bar{P}_r$, and $\bar{\varepsilon}$ corresponds to the "right" reflections of the starting points of $\bar{P}_1, \dots, \bar{P}_r$. It can be shown that this algorithm is weight-preserving and sign-reversing (i.e. that σ and $\bar{\sigma}$ differ in sign).

Among the remaining r-tuples we apply the Gessel–Viennot involution thus cancelling all intersecting r-tuples. It can be shown that, altogether, this defines a weight-preserving and sign-reversing involution. $\square$

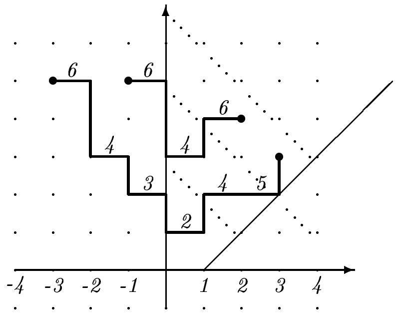
Bijective proof of the symplectic Giambelli identity. The symplectic analogue of the Giambelli identity for Schur functions reads (see [2, (24.47)])

$$sp_{2n}((\alpha|\beta), \mathbf{x}) = |sp_{2n}((\alpha_i|\beta_j), \mathbf{x})|_{m \times m}. \tag{3.14}$$

Here, $(\alpha|\beta)$ is the Frobenius notation for partitions (cf. [5]) and m is the number of cells in the main diagonal of (α, β). Again, we consider paths in the integer lattice consisting of horizontal steps in the positive direction. The direction of a vertical step, however, is only *positive* if it is *strictly to the right* of the y-axis, and is *negative* if it *does not* lie strictly to the right of the y-axis.

To the left of the horizontal axis we consider a "shifted" h-labelling by assigning label $i+1$ to a step at height i, while we consider the "usual" e-labelling to the right. Given a tableau of shape $(\alpha|\beta)$ with entries from $\{1, 2, \ldots, 2n\}$, we associate to its i-th principal hook a path P_i from $(-\alpha_i, 2n-1)$ to $(\beta_i + 1, 2n - \beta_i - 1)$ by interpreting the entries of the hook (read from "right to bottom") as labels of the corresponding steps of P_i.

Figure 4 shows an example (with $n = 3$) of this correspondence between nonintersecting lattice paths and symplectic tableaux.



2	3	4	6
4	4	6	
5	6		

Figure 4

As is clear from the picture, the symplectic constraint translates into the condition that the first path must not cross the line $y = x - 1$. Obviously, the set of all m-tuples of lattice paths subject to his condition (where m denotes the rank of the partition λ) is invariant under Gessel–Viennot involution. Therefore the same arguments as in Stembridge's [12, sec. 9] proof of the "ordinary" Giambelli identity establish (3.14). $\square$

Bijective proofs of Jacobi–Trudi type identities for Proctor's odd and intermediate symplectic characters. In [7, 9, 10] Proctor introduces odd symplectic characters, and, more generally, intermediate symplectic characters that interpolate between Schur functions and (even) symplectic characters. These characters can also be described combinatorially by means of tableaux which (roughly speaking) obey the symplectic constraint (3.1) for the first few rows and an additional rather intricate constraint involving the jeu de taquin. In certain special cases, he gives Jacobi–Trudi identities for these characters [7, p. 317], [9, Prop. 8.1], [10, App. A.2]. It comes as no surprise that, because in these cases the second constraint is either superfluous or very simple, the above proof methods for the symplectic e-formula (3.3) and the symplectic h-formula (3.4) also suffice to yield bijective proofs of Proctor's Jacobi–Trudi identities.

4. Bijective proofs for orthogonal identities

In [10, 8] Proctor gave tableaux interpretations for orthogonal characters. In fact, Proctor defined two slightly different types of orthogonal tableaux, *coarse orthogonal tableaux* (to be defined below) and *fine orthogonal tableaux*. The number of coarse tableaux of shape λ equals the dimension of the irreducible representation of the orthogonal group indexed by λ. However, the coarse tableaux are not very well suited for describing the irreducible characters. This task is better performed by the *fine* orthogonal tableaux. Orthogonal characters are indexed by N-orthogonal partitions [10, 8]. These are partitions with $\lambda'_1 + \lambda'_2 \leq N$. They obey the orthogonal "e-formula", the orthogonal analogue of the dual form of the Jacobi–Trudi identity,

$$o_N(\lambda,\mathbf{x}) = \left| e_{\lambda'_j-j+1}(\mathbf{x}) \quad \vdots \quad e_{\lambda'_j-j+i}(\mathbf{x}) + e_{\lambda'_j-j-i+2}(\mathbf{x}) \right|_{\lambda_1\times\lambda_1}. \tag{4.1}$$

Here, $\mathbf{x} = \langle x_1, x_1^{-1}, \dots x_n, x_n^{-1} \rangle$ if N is even, and $\mathbf{x} = \langle x_1, x_1^{-1}, \dots x_n, x_n^{-1}, 1 \rangle$ if N is odd. Setting all x_i equal to 1, one obtains

$$o_N(\lambda,(1^N)) = \left| e_{\lambda'_j-j+1}((1^N)) \quad \vdots \quad e_{\lambda'_j-j+i}((1^N)) + e_{\lambda'_j-j-i+2}((1^N)) \right|_{\lambda_1\times\lambda_1}. \tag{4.2}$$

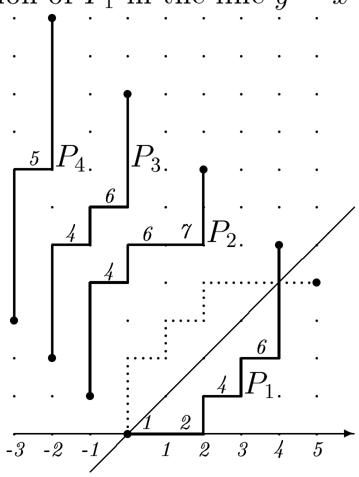
((1^N) is the abbreviation for the N-tuple $(1,1,\dots,1)$.) As is well-known, when we replace in $o_N(\lambda,\mathbf{x})$ each x_i by 1 we obtain the dimension of the corresponding irreducible representation. As mentioned above, this dimension is equinumerous with the coarse orthogonal tableaux of the shape λ with entries $\leq N$. Therefore, we may write

$$o_N(\lambda,(1^N)) = |\{T : \ T \text{ coarse orthogonal}, T_{i,j} \leq N\}|. \tag{4.3}$$

We are able to give a bijective proof for (4.1) by using Proctor's fine orthogonal tableaux. However, there is not enough space to describe it here. Instead, we describe a bijective proof for the weaker (4.2) by using coarse orthogonal tableaux and the interpretation (4.3). The proof idea for (4.1) is the same, however the argumentation has to be more careful because we have to take care of weights which is not the case for our proof of (4.2).

Bijective proof of the orthogonal dimension formula (4.2). Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be an N-orthogonal partition of length r. A tableau is called *coarse orthogonal* if it satisfies the value p case of the coarse orthogonal condition for all $p \in \mathbb{N}$. A tableau T of shape λ is said to satisfy the *value p case of the coarse orthogonal condition* if the number of entries $\leq p$ in T's first two columns does not exceed p.

By reading P_j from the j-th column of the tableau and using the e-labelling, a coarse orthogonal tableau of shape λ corresponds to a λ_1-tuple $(P_1, \dots, P_{\lambda_1})$ of nonintersecting lattice paths, where P_j runs from $u_j = (-j+1, j-1)$ to $v_j = (\lambda'_j - j + 1, N - \lambda'_j + j - 1)$, and where the reflection of P_1 in the line $y = x$ does not intersect P_2. This is exemplified in Figure 5.



1	4	4	5
2	6	6	
4	7		
6			

Figure 5

Let us denote the reflection in the line $y = x$ by $\mathbf{R}$. For the bijective proof of (4.2) we consider the set $\bigcup_{\sigma\in \mathcal{S}_{\lambda_1}, \varepsilon\in\{1\}\times\{1,-1\}^{\lambda_1-1}} \mathcal{P}(\mathbf{u}^{(\varepsilon)}, \mathbf{v}_\sigma)$. It is easy to see that the determinant in (4.2) equals the weighted sum $\sum_{\mathbf{P}} \operatorname{sgn}(\sigma)$ where the sum is over all λ_1-tuples $\mathbf{P}$ of paths in this set.

We shall give a sign-reversing involution that cancels all λ_1-tuples $(P_1,\dots,P_{\lambda_1})$ that are either intersecting or contain two paths P_i, P_j, $i \neq j$, such that P_i intersects $\mathbf{R}\,(P_j)$. Suppose that this had been done. The remaining λ_1-tuples are nonintersecting and for all pairs P_i, P_j the reflected path P_i and the path $\mathbf{R}\,(P_j)$ do not intersect. In particular, the paths P_i, $i \geq 2$, neither meet P_1 nor $\mathbf{R}\,(P_1)$. A moment's thought shows that, because of the order of the end points (v_{j+1} lies in the North-West of v_j, $j = 1, 2, \dots, \lambda_1 - 1$, and in addition v_2 lies in the North-West of $\mathbf{R}\,(v_1)$; the latter is true because λ is an N-orthogonal partition), this implies that the end point of P_1 must be v_1. Since for $l \geq 2$ the end points v_l lie in the North-West of v_1 and $\mathbf{R}\,(v_1)$, the complete paths P_i, $i \geq 2$, must lie in the North-West of P_1 and $\mathbf{R}\,(P_1)$, not intersecting any of them. Since also $P_2, \dots, P_{\lambda_1}$ are nonintersecting, because of the order of the starting and end points, for $i \geq 2$ P_i must run from u_i to v_i. In other words, the remaining λ_1-tuples are those for which $\sigma = \mathrm{id}$ and $\varepsilon = (1, \dots, 1)$, which are nonintersecting, and where $\mathbf{R}\,(P_1)$ for $i \geq 2$ does not meet P_i. This would prove the assertion.

Now we define the involution. First cancel all intersecting λ_1-tuples by the Gessel–Viennot involution. For a remaining λ_1-tuple, look for the lowest level $x + y = c$ containing a point p of intersection *above* $y = x$ of P_i and $\mathbf{R}\,(P_j)$. Choose (i, j) to be minimal in lexicographic order. Now, as in the Gessel–Viennot involution, interchange terminal portions of $\mathbf{R}\,(P_j)$ and P_i beginning from p, obtaining P_i' and $P_j{}'$. Then reflect back P_i', thus obtaining $\mathbf{R}\,(P_i')$ and P_j'. The following figure illustrates this operation.

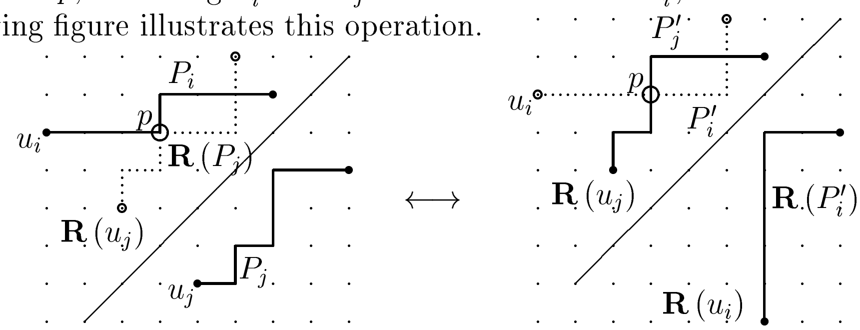


Figure 6

Thus, from the original λ_1-tuple of paths we obtain a new λ_1-tuple by replacing P_i by $\mathbf{R}\,(P_i')$ and P_j by P_j'. It does not introduce a new point of (ordinary) intersection in the resulting λ_1-tuple. In fact, there cannot be such a point above level $x + y = c$; and there is no such point below, because c was chosen to be minimal. This mapping is sign-reversing since the associated permutations differ by the transposition (i, j). And it is straight-forward to verify that it is an involution. $\square$

Bijective proofs for odd orthogonal identities by means of Sundaram's odd orthogonal tableaux. Sundaram's odd orthogonal tableaux [14, 13] are tableaux with entries from the alphabet $1 < 2 < \cdots < 2n < \infty$ in the usual sense, except that column strictness does not extend to symbol ∞ (i.e., the symbol ∞ may occur *more than once in a column*), which obey the symplectic constraint (3.1), and in addition have at most one entry ∞ in each *row*. The weight of a Sundaram tableaux is defined by (3.2), entries ∞ do not contribute to the weight. Since the difference with symplectic tableaux is not too big, slight modifications of the above proofs for the symplectic Jacobi–Trudi identities (3.3) and (3.3) can be used to prove the odd orthogonal Jacobi–Trudi type identities

(4.4)

$$o_{2n+1}(\lambda, \mathbf{x}) = \left|\sum_{k\geq 0} e_{\lambda'_i-i+j-k}\left(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\right) - \sum_{k\geq 0} e_{\lambda'_i-i-j-k}\left(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\right)\right|_{\lambda_1\times\lambda_1}$$

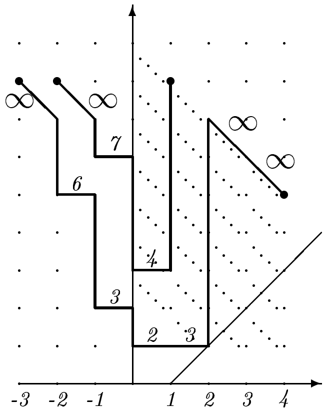
and

$$o_{2n+1}(\lambda, \mathbf{x}) = |h'_{\lambda_i-i+1} \;\vdots\; h'_{\lambda_i-i+j}(\mathbf{x}) + h'_{\lambda_i-i-j+2}(\mathbf{x})|_{r\times r}, \tag{4.5}$$

where $h'_k = h_k - h_{k-2}$ and $\mathbf{x} = \langle x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1\rangle$. The proof of the odd orthogonal Giambelli identity (see [2, (24.47)])

$$o_{2n+1}((\alpha|\beta), \mathbf{x}) = |o_{2n+1}((\alpha_i|\beta_j), \mathbf{x})|_{m\times m}, \tag{4.6}$$

by means of Sundaram's tableaux is similar to the one given for the symplectic Giambelli identity (3.14) (see Figure 4). The lattice path interpretation of Sundaram's tableaux is the same as in the symplectic case, besides the entries ∞ are interpreted as downward diagonal steps. This correspondence maps Sundaram's tableaux to sets of nonintersecting lattice paths that contain at most one diagonal step (with label ∞) in the left half-plane, and that do not cross the line $y = x - 1$. We give an example for $n = 4$ in Figure 7. Application of the Gessel–Viennot method immediately implies (4.6).



2	3	6	∞
4	4	7	∞
∞			
∞			

Figure 7

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Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Vienna, Austria