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The Combinatorics of Permutation Polytopes

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ABSTRACT. We study the facial structure of three important permutation polytopes: the *permutohedron* P_n, defined as the convex hull in $\mathbf{R}^n$ of all permutations of the vector $(1, 2, \ldots, n)$, the *Birkhoff* or *assignment* polytope B_n, defined as the convex hull in $\mathbf{R}^{n^2}$ of all $n \times n$ permutation matrices, and the *asymmetric traveling salesman* polytope T_n, defined as the convex hull of those $n \times n$ permutation matrices corresponding to n-cycles. The polytope P_n, a generic projection of B_n, admits a simple combinatorial description for its lattice of faces and for the faces themselves. We study specialized projections of B_n and some of their combinatorial properties. These *multipermutohedra* have vertices corresponding to permutations of a multiset of n (not necessarily distinct) elements. We consider what polytopes may arise as faces of the various permutation polytopes. Faces of P_n are products of smaller permutohedra. Characterizing the face lattice of B_n as the lattice of elementary bipartite graphs, it follows, for example, that every pair of vertices of B_n is contained in a cubical face. On the other hand, T_n has *every* 0-1 d-polytope as a face, for $d \sim \log n$.

1. *Introduction*

We start by defining the basic permutation polytopes and giving some of their elementary properties. Let $\mathcal{S}_n$ denote the symmetric group of degree n. Given $\sigma \in \mathcal{S}_n$, we define the corresponding $n \times n$ permutation matrix $X_\sigma \in \mathbf{R}^{n^2}$ by

$$(X_\sigma)_{ij} := \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

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With this definition we have $e_i X_\sigma = e_{\sigma(i)}$, where e_j denotes the jth unit vector in $\mathbf{R}^n$. For example, if $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, then $X_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

We denote by B_n the *Birkhoff polytope* (or the *assignment polytope*) of order n in $\mathbf{R}^{n^2}$ defined by

$$B_n := \text{conv}\{\ X_\sigma \mid \sigma \in \mathcal{S}_n\}.$$

A detailed study of this polytope is given in [**6**] (see also [**7**, chapter 5]). We begin with the well known *Birkhoff-von Neumann Theorem*, characterizing the Birkhoff polytope as be the polytope of all doubly stochastic $n \times n$ matrices (see [**3**]).

PROPOSITION 1.1. *B_n is an $(n-1)^2$ dimensional polytope with $n!$ vertices having the following inequality description:*

$$\begin{aligned} B_n \quad = \quad & \{\ X = (x_{ij}) \in \boldsymbol{R}^{n^2} \mid x_{ij} \geq 0\ ;\ 1 \leq i\ ,\ j \leq n, \\ & \sum_{j=1}^{n} x_{ij} = 1 \ for\ i = 1, \ldots, n \ and \ \sum_{i=1}^{n} x_{ij} = 1 \ for\ j = 1, \ldots, n\}. \end{aligned}$$

Proof: That each X_σ is a vertex of B_n is immediate. For example, consider maximizing the inner product $\langle X, X_\sigma \rangle$ over $X \in B_n$.

Let A_n denote the set on the right side. Since elements of A_n satisfy $2n-1$ independent linear equations, it follows that $dim\, A_n = n^2 - (2n-1) = (n-1)^2$. It is clear that $B_n \subseteq A_n$ since $X_\sigma \in A_n$ for each $\sigma \in \mathcal{S}_n$. To show that $B_n = A_n$, we show, by induction on n, that every vertex of A_n is a permutation matrix. If X is a vertex of A_n, then X must lie on at least $(n-1)^2$ facets, hence it must have at least $(n-1)^2$ entries zero. This implies that X must have a row with $(n-1)$ zeroes. So, $x_{ij} = 1$ for some i, j. Deleting the ith row and jth column from X leaves us with a $(n-1) \times (n-1)$ matrix $X(i,j)$ which is a vertex of A_{n-1} and therefore a permutation matrix by induction. This shows that X must also be a permutation matrix. □

Elements of B_n are called doubly stochastic matrices of order n. B_2 is a one dimensional polytope (an edge) while B_3 is a simplicial 4 polytope with 6 vertices, every pair of which forms an edge. If Δ_i denotes the i-simplex, then the dual of B_3 is $\Delta_2 \times \Delta_2$.

One face of B_n of some interest here is the face D_n generated by the *derangements* (permutations with no fixed points). That is,

$$D_n = \{\ X \in B_n \mid x_{ii} = 0,\ i = 1, \ldots, n\ \}.$$

Here, $dim\, D_n = (n-1)^2 - n = n^2 - 3n + 1$ for $n \geq 3$.

Next, define the *permutohedron* $P_n \subset \mathbf{R}^n$ to be the convex hull of all permutations of the vector $(1, 2, \ldots, n)$. Formally,

$$P_n = \text{conv}\{\ x_\sigma \mid \sigma \in \mathcal{S}_n\},$$

where $x_\sigma = (\sigma^{-1}(1), \ldots, \sigma^{-1}(n)) = (1, 2, \ldots, n)\, X_\sigma$. Alternatively, P_n is the image of B_n under the projection $\mathbf{R}^{n^2} \to \mathbf{R}^n$ defined by $X \mapsto (1, 2, \ldots, n)\, X$. Since $P_n \subset \{x \in \mathbf{R}^n \mid x_1 + \cdots + x_n = \binom{n+1}{2}\}$, it follows that its dimension is at most $n - 1$. To see that $dim\, P_n = n - 1$, one can check, for example, that the n points $(1, 2, \ldots, n)$ and $(1, 2, \ldots, n)\, X_{(i,i+1)}$, where $i = 1, 2, \ldots, n - 1$, are affinely independent. (Here $(i, i + 1)$ is an adjacent transposition.) The study of this polytope goes back at least to the early part of this century [**15**] (see also [**18**]).

An inequality description for P_n, given by Rado [**13**] in 1952, will be proved in a slightly more general context in Theorem 3.1. To motivate this description, define, for $S \subset [n] := \{1, 2, \ldots, n\}$ and $x \in \mathbf{R}^n$, $x(S) := \sum_{i \in S} x_i$. Now for any $x \in P_n$, it is clear that when $|S| = s$ the quantity $x(S)$ can be no smaller than $\binom{s+1}{2}$, the sum of the s smallest coordinates of the vector $(1, 2, \ldots, n)$. It turns out that these inequalities, together with the equation above, give a minimal complete inequality description of P_n.

PROPOSITION 1.2. *The permutohedron P_n has the inequality description*

$$P_n = \{x \in \boldsymbol{R}^n \mid x_1 + \cdots + x_n = \binom{n+1}{2},\ x(S) \geq \binom{s+1}{2},\ \textit{for all } S \subset [n]\ \},$$

where $s = |S|$. □

The inequalities in Proposition 1.2, lead quickly to a description of the lattice of faces of P_n. To define a face of codimension k, one needs k linearly independent equations to hold among the inequalities

$$x(S) \geq a_S := \binom{|S|+1}{2}, \tag{1}$$

say for proper subsets $S_1, \ldots, S_k \subset [n]$. It turns out to be necessary and sufficient that these sets are nested.

PROPOSITION 1.3. *$F \subset P_n$ is a face of codimension k if and only if equality holds in (1) precisely for k distinct proper subsets lying in a chain $S_1 \subset \cdots \subset S_k \subset [n]$.*

Proof: Notice that the quantities a_S define a supermodular function on subsets of $[n]$, that is, for sets S and T

$$a_{S \cup T} + a_{S \cap T} \geq a_S + a_T,$$

with equality if and only if either $S \subset T$ or $T \subset S$. Now, suppose equality holds in (1) for S and T for some $x \in P_n$. Then

$$a_S + a_T = x(S) + x(T) = x(S \cup T) + x(S \cap T) \geq a_{S \cup T} + a_{S \cap T},$$

and so either $S \subset T$ or $T \subset S$.

Thus equality holds in (1) for k proper subsets if and only if they form a chain $S_1 \subset \cdots \subset S_k \subset [n]$. The resulting system of equations is necessarily linearly independent. □

Proposition 1.3 leads directly to a description of the face lattice of P_n as the lattice of chains of subsets in $[n]$, ordered by refinement (equivalently, as the dual to the face lattice of the barycentric subdivision of the $(n-1)$-dimensional simplex Δ_{n-1}). This can be found in [**7**, §3.2] and is stated explicitly in [**4**, p.490-91]. We give an equivalent description in terms of ordered partitions.

By an *ordered partition* of $[n]$ we mean an indexed family $\pi = (A_1, \ldots, A_k)$ of mutually disjoint nonempty subsets $A_i \subset [n]$ such that $A_1 \cup \cdots \cup A_k = [n]$. We denote by Π_n the poset of all ordered partitions of $[n]$, ordered by refinement. For example, in Π_4, $(\{1,4\},\{3\},\{2\}) \leq (\{1,3,4\},\{2\})$. If we include in Π_n an element $\widehat{0} \leq \pi$ for every ordered partition π, then Π_n forms a lattice. Note that Π_n is *ranked* in the sense that the length of the longest chain in Π_n between $\widehat{0}$ and $\pi = (A_1, \ldots, A_k)$ is $n-k$; thus $r(\pi) = n-k$ defines a rank function. Finally, the interval $[\widehat{0}, \pi]$ in Π_n is isomorphic to $\Pi_{n_1} \times \cdots \times \Pi_{n_k}$, where $n_i = |A_i|$.

Proposition 1.4. *The face lattice of P_n is isomorphic to Π_n.*

Proof: For a face $F \subset P_n$ of codimension k, let $S_1 \subset \cdots \subset S_k \subset [n]$ be the chain given in Proposition 1.3. For notational purposes, let $S_0 = \emptyset$ and $S_{k+1} = [n]$. Then $F \mapsto \pi(F) = (A_1, \ldots, A_{k+1})$, where $A_i = S_i \setminus S_{i-1}$, defines a map π from the lattice of faces of P_n to Π_n. Note that $dim\, F = n-1-k = r(\pi(F))$. It follows from Proposition 1.3 that π is a bijection and both π and π^{-1} are order preserving. □

From the structure of intervals in Π_n, we obtain the following description of faces in P_n.

Corollary 1.1. *Faces of P_n are all of the form $P_{n_1} \times \cdots \times P_{n_k}$, where $n_1 + \cdots + n_k = n$.* □

Let $S(n,i)$ be the number of partitions of an n set into i blocks (a *Stirling number of the second kind*, see [**17**, p.34]). If f_{n-i} is the number of $(n-i)$ dimensional faces of $P(S)$, then

$$f_{n-i} = i\,!\, S(n,i) = \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^n.$$

If we denote by $F(\pi) = F(A_1, \ldots, A_k)$ the face of P_n corresponding to the ordered partition $\pi = (A_1, \ldots, A_k) \in \Pi_n$, then if $S_i = A_1 \cup \cdots \cup A_i$, it follows that

$$F(A_1, \ldots, A_k) = \mathrm{conv}\{x_\sigma \mid \sigma([|S_i|]) = S_i,\ i = 1, \ldots, k\}. \tag{2}$$

That is, $x_\sigma \in F(A_1, \ldots, A_k)$ if and only if σ is of the form

$$\sigma = \begin{pmatrix} 1\ 2\ \cdots\cdots n \\ A_1\ A_2 \cdots A_k \end{pmatrix}.$$

Thus the ordered partition $(\{i_1\}, \ldots \{i_n\})$ corresponds to the vertex of P_n defined by the permutation $\sigma = \begin{pmatrix} 1 & \ldots & n \\ i_1 & \ldots & i_n \end{pmatrix}$. Edges of P_n correspond to ordered partitions into $n-1$ parts, that is, those consisting of one 2-set and the rest singletons. If $\pi = (A_1, \ldots, A_{n-1})$ is such an ordered partition, with $|A_i| = 2$, then $F(\pi)$ has two vertices corresponding to permutations σ_1 and σ_2 that differ by the adjacent transposition $(i, i+1)$, that is, $\sigma_2 = \sigma_1(i, i+1)$. This description of edges of P_n should be compared with that obtained by considering P_n as the *zonotope* generated by differences $e_i - e_j$ of unit vectors in $\mathbf{R}^n$ (see [**5**, Examp. 2.2.5] or [**18**]).

2. *The Birkhoff Polytope*

We consider now the lattice of faces of the Birkhoff polytope B_n. For convenience, we shall often denote a vertex X_σ by σ. With each vertex $\sigma \in B_n$, we associate the bipartite graph $G(\sigma)$ that is defined to be the matching on the complete bipartite graph $K_{n,n}$ with the edge set $\{(i, \sigma(i)) : i = 1, \ldots, n\}$. If F is a face of B_n, then define $G(F)$ to be the union of $G(\sigma)$ over all the vertices $\sigma \in F$. $G(F)$ has the property that it has $2n$ nodes and every edge of $G(F)$ is in some complete matching in $G(F)$. Such graphs are called *elementary graphs* (see [**10**, chapter 4]; the definition of elementary graphs given there also requires them to be connected, but we will not require that here). Conversely, if $G \subseteq K_{n,n}$ is an elementary graph, define

$$F(G) := \mathrm{conv}\{\sigma \mid G(\sigma) \subseteq G\} = B_n \cap \{X \in \mathbf{R}^{n^2} \mid x_{ij} = 0 \ \textit{if}\ (i,j) \notin G\}.$$

We see that $F(G)$ is an intersection of facets of B_n and therefore is a face of B_n. Clearly, these correspondences are inverse to one another, and if F_1, F_2 are faces of B_n, then $F_1 \subseteq F_2$ if and only if $G(F_1) \subseteq G(F_2)$; see [**10**] for details.

This yields the following description of the face lattice of B_n. Since the empty graph on $2n$ nodes is an elementary graph, the set of elementary subgraphs of $K_{n,n}$, ordered by inclusion of edges, is a lattice. Here the join operation is defined by the union of the edge sets, while the meet of two elementary graphs is defined to be the join of all elementary graphs less than both (which may be empty).

THEOREM 2.1. *The face lattice of B_n is isomorphic to the lattice of all elementary subgraphs of $K_{n,n}$ ordered by inclusion.* □

The minimal nonempty elementary graphs are the matchings, corresponding to vertices of B_n. Covering the matchings are graphs consisting of a single (even) cycle and otherwise only edges of a matching. These correspond to edges of B_n, leading to the following characterization.

COROLLARY 2.1. *Permutations σ and π correspond to an edge of B_n if and only if $\sigma^{-1}\pi$ is a cycle.* □

Thus, any two vertices of B_3 must form an edge.

We say that a connected bipartite graph G has a *ear decomposition* if G has the following representation:

$$G = x \cup P_1 \cup \cdots \cup P_k \tag{3}$$

where x is an edge and for $i = 1, \ldots, k$, P_i is a path of odd length connecting 2 nodes of different color classes of $G_{i-1} := x \cup P_1 \cup \cdots \cup P_{i-1}$. We take $G_0 = x$. P_i is called the ith ear of G. If $P_i = \{(r_1, r_2), \ldots, (r_{2j-1}, r_{2j})\}$, then we define $\{(r_1, r_2), (r_3, r_4), \ldots, (r_{2j-1}, r_{2j})\}$ and $\{(r_2, r_3), \ldots, (r_{2j-2}, r_{2j-1})\}$ to be the set of *odd* and *even* edges of P_i respectively. It can be shown that $k = |E(G)| - |V(G)| + 1$. We note the following well known result (see [**10**, Theorem 4.1.6]):

THEOREM 2.2. *A bipartite graph G is elementary if and only if each component of G has a ear decomposition.* □

If G is an elementary subgraph of $K_{n,n}$, then we define the *degree* of G to be n. By Theorem 2.1, the sublattice of elementary subgraphs of G is ranked. We define the *rank* of G to be the length of a maximal chain of *nonempty* elementary subgraphs of G. Thus a matching will have rank 0, and so $rank(G) = dim\, F(G)$. If G_1 and G_2 are elementary bipartite, then we say that $G_1 \approx G_2$ if $F(G_1) \approx F(G_2)$, that is, if $F(G_1)$ and $F(G_2)$ have isomorphic face lattices. (In this case, $F(G_1)$ and $F(G_2)$ are said to be *combinatorially equivalent.*)

We wish to show that the rank of an elementary graph G is the total number of ears in ear decompositions of all the components of G. First take G to be a connected elementary graph with the ear decomposition as given in (3). This induces a chain of elementary graphs

$$G'_0 \subset G'_1 \subset \cdots \subset G'_k = G$$

where $G'_i = G_i \cup M_i$ and M_i is the union of even edges in the ears $P_{i+1}, \ldots, P_k$ (i.e. M_i is a matching on the node set $V(G) \setminus V(G_i)$, see Figure 1).

It is clear that G'_{i+1} covers G'_i for each i in this chain (i.e. there is no elementary graph G' such that $G'_i \subset G' \subset G'_{i+1}$). Hence the rank of G is $k = |E(G)| - |V(G)| + 1$.

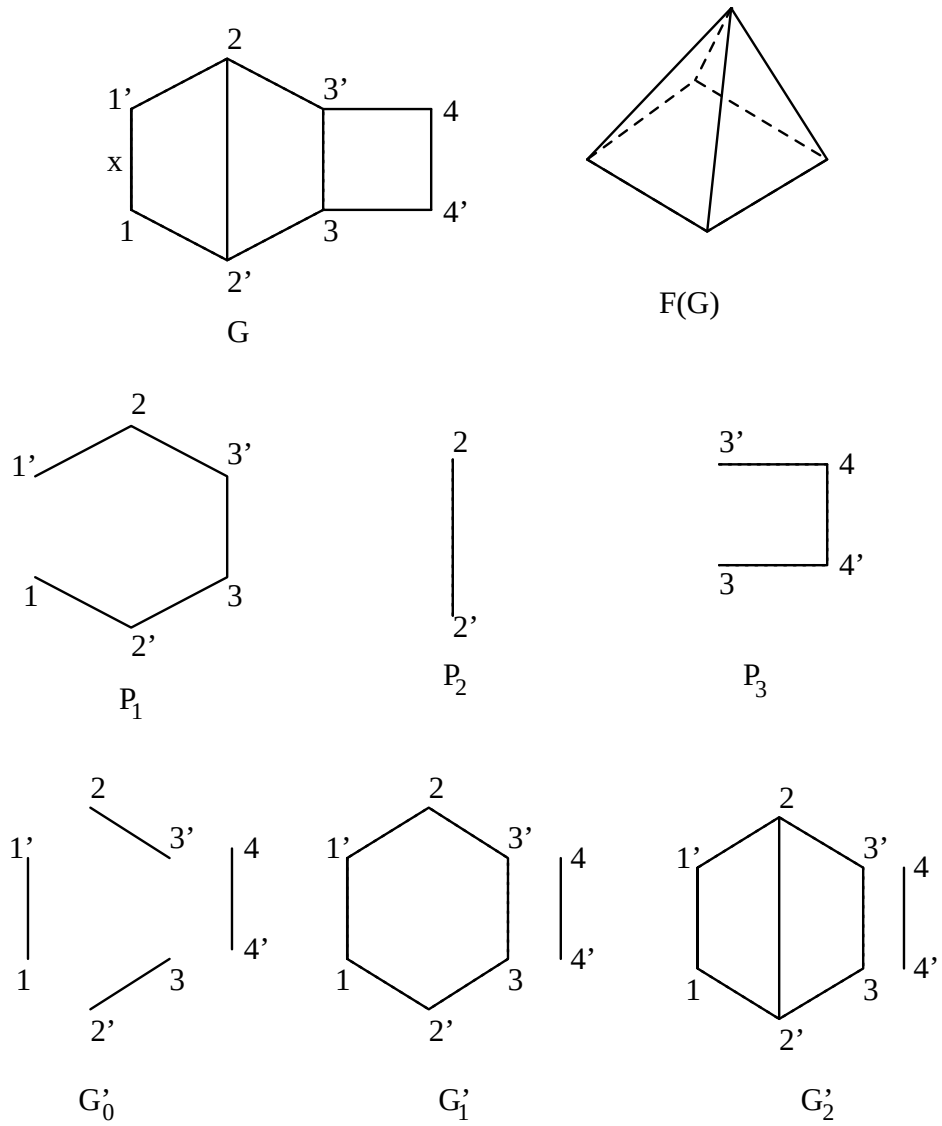

FIGURE 1. An ear decomposition of an elementary bipartite graph G of rank 3 and the chain of subgraphs of G.

Now let $G \subseteq K_{n,n}$ be an elementary graph with components $G_1, \ldots, G_k$ where G_i is elementary bipartite on v_i nodes, e_i edges and rank $r_i = e_i - v_i + 1$. Then we claim that the rank of G is

$$r := r_1 + \cdots + r_k = e_1 - v_1 + 1 + \cdots + e_k - v_k + 1 = |E(G)| - 2n + k.$$

This is again seen by exhibiting a (maximal) chain $G'_0 \subset G'_1 \subset \cdots \subset G'_r = G$ of length r where the first r_1 graphs construct G_1, the next r_2 construct G_2, and so on. This observation and Theorem 2.1 imply the following result:

COROLLARY 2.2. *If F is a face of B_n and $G(F)$ has e edges and k components, then $\dim F = e - 2n + k$.* □

Using the elementary graph structure, we derive some properties of B_n.

THEOREM 2.3. *Any d-polytope that appears as a face of some B_n does so for all $n \geq 2d$.*

Proof: Equivalently we show that if G is an elementary graph of rank d then $G \approx \widetilde{G}$ where $\widetilde{G}$ is an elementary subgraph of $K_{2d,2d}$ (that is, $\widetilde{G}$ has degree $2d$).

We prove this by induction on d. The result is trivially true for $d = 1$ since an elementary graph of rank 1 is a cycle. Now assume the result to be true for $j < d$ and let G be an elementary graph of rank d. If G has components $G_1, \ldots, G_k$ of positive ranks $d_1, \ldots, d_k$ (so that $d = d_1 + \cdots + d_k$) then by the induction hypothesis we can find an elementary graph $\widetilde{G} \subseteq K_{2d,2d}$ with components $\widetilde{G}_1, \ldots, \widetilde{G}_k$ such that $G_i \approx \widetilde{G}_i$ for each i so that $G \approx \widetilde{G}$.

So assume that G is connected with v nodes and e edges. If the degree of each node in G is at least 3, then the equations $e - v + 1 = d$ and $e \geq 3v/2$ yield $v \leq 2(d-1)$, that is, the degree of G is at most $d-1$. If G has a node of degree 2 incident on edges e_1, e_2, then contracting e_1 and e_2 in G yields an elementary graph G_1 such that $G_1 \approx G$. Repeating this process, we finally arrive at an elementary graph $\overline{G} \approx G$ which has every node of degree at least 3 and thus at most $2(d-1)$ nodes. Thus $rank\, \overline{G} = d = \bar{e} - \bar{v} + 1$, where $\bar{v}$, $\bar{e}$ are the numbers of vertices and edges of $\overline{G}$.

The process of edge contraction may lead to $\overline{G}$ having multiple edges (i.e., having more than one edge joining a pair of nodes). If $\overline{E} := \{\bar{e}_1, \ldots, \bar{e}_m\}$ is a set of multiple edges in $\overline{G}$, then $\overline{G} \setminus \overline{E}$ is a connected elementary graph having m fewer edges and rank $(\bar{e} - m) - \bar{v} + 1 = d - m \geq 0$. Hence $m \leq d$. Now we subdivide the edges $\bar{e}_1, \ldots, \bar{e}_m$ by placing 2 nodes on each edge to get an elementary graph $\widetilde{G} \approx \overline{G} \approx G$, where $\widetilde{G}$ has at most $2(2d-1)$ nodes and hence has degree less than $2d$. □

For $S = \{\sigma_1, \ldots, \sigma_k\} \subseteq \mathcal{S}_n$, let $F(S) = F(\sigma_1, \ldots, \sigma_k)$ be the smallest face in B_n containing $\sigma_1, \ldots, \sigma_k$.

THEOREM 2.4. *For $S \subseteq \mathcal{S}_n$, with $1 \leq |S| = k \leq n$,*

$$dim\, F(S) \leq (k-2)n + \left\lfloor \frac{n}{k} \right\rfloor .$$

This bound is always tight.

Proof: Let $G(S) = G(F(S))$, the union of the matchings $G(\sigma_1), \ldots, G(\sigma_k)$. Let $G(S)$ have components $G_1, \ldots, G_r$ and $G'_1, \ldots, G'_s$ of degrees $l_1, \ldots, l_r$ and $l'_1, \ldots, l'_s$, where $l_1, \ldots, l_r \geq k$ and $l'_1, \ldots, l'_s < k$ and

$$l_1 + \cdots + l_r + l'_1 + \cdots + l'_s = n.$$

Let $c = r + s$ be the number of components of $G(S)$. Since $G(S)$ is the union of k matchings, the components $G_1, \ldots, G_r$ can have at most $kl_1, \ldots, kl_r$ edges. $G'_1, \ldots, G'_s$ can have at most ${l'_1}^2, \ldots, {l'_s}^2$ edges, so

$$e \leq \sum_{i=1}^{r} kl_i + \sum_{i=1}^{s} {l'_i}^2,$$

where e is the number of edges of $G(S)$. Thus

$$\begin{aligned} e + c &\leq k \sum_{i=1}^{r} l_i + \sum_{i=1}^{s} {l'_i}^2 + r + s \\ &= k(n - \sum_{i=1}^{s} l'_i) + \sum_{i=1}^{s} {l'_i}^2 + r + s \\ &= kn - \sum_{i=1}^{s} [l'_i(k - l'_i) - 1] + r \\ &\leq kn + r \leq kn + \left\lfloor \frac{n}{k} \right\rfloor . \end{aligned}$$

By Corollary 2.2,

$$dim\, F(S) = e + c - 2n \leq (k-2)n + \left\lfloor \frac{n}{k} \right\rfloor ,$$

proving the result.

To show that this bound is tight, given n and k, let $n = qk + r$ with $0 \leq r < k$. Let G consist of q components, $q-1$ of which are complete bipartite graphs $K_{k,k}$ and one component of degree $k + r$ that is the union of k disjoint matchings. Then $G = G(S)$, where $S \subseteq \mathcal{S}_n$, $|S| = k$, and has $(q-1)k^2 + (k+r)k$ edges. Using Corollary 2.2, we compute $rank\, G = (k-2)n + \lfloor \frac{n}{k} \rfloor$. □

Note that if F is a face of B_n and $G(F)$ is not connected, then F is the product of faces. In particular, if $G(F)$ has components $G_1, \ldots, G_k$, with G_i bipartite with bipartition $\{I_i, J_i\}$ (so that $\{I_1, \ldots, I_k\}$ and $\{J_1, \ldots, J_k\}$ are partitions of $[n]$), denote by $B^{(i)}$, the Birkhoff polytope defined on the coordinates $\{x_{kl} \mid k \in I_i, l \in J_i\}$, itself a face of B_n. Since $G(F)$ is elementary, each G_i is elementary bipartite on $\{I_i, J_i\}$. Hence, $F_i := F(G_i)$ is a face of $B^{(i)}$. Then $F = F_1 \times \cdots \times F_k$,

embedded as block diagonal matrices in $\mathbf{R}^{n^2}$ (up to permutation of rows and columns).

If $\sigma, \pi \in \mathcal{S}_n$, then $G(\sigma, \pi) := G(\sigma) \cup G(\pi)$ is a union of two matchings which is a set of disjoint cycles and edges. Hence, by the above remark, $F_{\sigma,\pi} := F(G(\sigma, \pi))$ is a k-dimensional cube, where k is the number of cycles in the graph. This leads to a direct generalization of Corollary 2.1, first observed in [**8**].

PROPOSITION 2.1. *If $\sigma^{-1}\pi = \prod_{i=1}^{k} C_i \in \mathcal{S}_n$ where $C_1, \ldots, C_k$ are disjoint cycles, then the smallest face $F_{\sigma,\pi}$ of B_n containing both σ and π is a k-cube, where $k \leq \lfloor \frac{n}{2} \rfloor$. The vertices of $F_{\sigma,\pi}$ are given by $\sigma \, \Pi_{i \in S} C_i$, for $S \subseteq [k]$.* □

Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ and, for $S \subseteq [k]$, define $\mathcal{C}(S) := \Pi_{i \in S} C_i$. We shall associate $F_{\sigma,\pi}$ with the unit k-cube, with $\sigma\mathcal{C}(S)$ corresponding to the vertex with support S (so σ corresponds to the origin and π to $(1, 1, \ldots, 1)$). We note here that the k-cube $F_{\sigma,\pi}$ is actually a *zonotope* (Minkowski sum of line segments) generated by k mutually orthogonal segments $z_i \in \mathbf{R}^{n^2}$. Each z_i is supported on a cycle of the graph $G(\sigma, \pi)$ and has coordinates $+1$ or -1 alternately for edges of $G(\sigma)$ and $G(\pi)$. For convenience, we shall often denote $F_{\sigma,\pi}$ by $F(\sigma, \mathcal{C})$.

This discussion above leads to the following restrictions on faces of B_n.

THEOREM 2.5. *Let F be any face of B_n. Then*

(i) *every pair of vertices of F lie on a cubical face of dimension at most $\lfloor \frac{n}{2} \rfloor$, and*

(ii) *the complement of any facet of F is a face of F.*

Proof: Assertion (i) follows from Proposition 2.1. To prove (ii), we note that any facet of F is of the form $F \cap \{x_{ij} = 0\}$ for some i, j. The set of all vertices of F not lying on this facet form a face $F \cap \{x_{ij} = 1\}$. □

The only 2-polytopes satisfying the conditions of Theorem 2.5 are the triangle and the quadrilateral. The 3-polytopes allowed are the tetrahedron, square pyramid, triangular prism and the cube. These are all the possible 2- or 3-dimensional faces of B_n. However these properties are not sufficient to characterize faces of B_n completely. For example, the pyramid over B_3 satisfies both conditions of Theorem 2.5 but cannot be a face of any B_n (see [**6**, Theorem 4.2, p. 348]).

3. *Permutohedra and Multipermutohedra*

We saw in §1 that the permutohedron P_n can be defined as the image of B_n under the projection $\mathbf{R}^{n^2} \to \mathbf{R}^n$ defined by $X \mapsto (1, 2, \ldots, n)\, X$. We consider now the situation where the projection is defined by $X \mapsto a\, X$ for a general vector $a = (a_1, a_2, \ldots, a_n)$; equivalently, we consider the polytope defined as the convex hull of all permutations of a general point $a \in \mathbf{R}^n$. When $a_1 < a_2 < \cdots < a_n$, the combinatorial structure of the resulting polytope is the same as that of P_n. When some of the a_i's are equal, the structure of this polytope can vary markedly.

We adapt a slightly different point of view here than we did with P_n. We define the permutation polytope formed by taking the convex hull of all permutations of a multiset, that is, a "set" with repeated elements. When all the elements of the multiset are distinct we get the permutohedron.

Let $a_1 < a_2 < \cdots < a_n$ be n distinct numbers ($n > 1$) and consider the multiset

$$M := \{a_1^{k_1}, a_2^{k_2}, \ldots, a_n^{k_n}\} = \{\bar{a}_1, \ldots, \bar{a}_N\} \qquad k_i > 0,\ i = 1, \ldots, n \tag{4}$$

with $\bar{a}_1 \le \bar{a}_2 \le \cdots \le \bar{a}_N$ and $N = \sum_{i=1}^{n} k_i$. We think of M as containing k_1 copies of a_1, k_2 copies of a_2, and so on. Let $P(M) \subset \mathbf{R}^N$ be the permutation polytope formed by taking the convex hull of all permutations of M. More specifically, $P(M)$ is the image of B_N under the projection $X \mapsto (\bar{a}_1, \ldots, \bar{a}_N)\, X$ from $\mathbf{R}^{N^2}$ to $\mathbf{R}^N$. We call $P(M)$ the *multipermutohedron* on the multiset M. It can be obtained by contracting certain faces of the permutohedron of dimension $N-1$. We find all the facets for this polytope and determine its face lattice. We also find an expression for the number of distinct d-dimensional multipermutohedra. We first derive a description of this polytope in terms of inequalities, generalizing Proposition 1.2.

THEOREM 3.1. *The multipermutohedron $P(M)$ has the inequality description*

$$P(M) = \{x \in \boldsymbol{R}^N \mid x([N]) = \sum_{i=1}^{N} \bar{a}_i,\ x(S) \ge \sum_{i=1}^{|S|} \bar{a}_i,\ \textit{for all } S \subset [N]\ \}.$$

Before proving this theorem, we recall a result of Schur. We say that the vector $x = (x_1, \ldots, x_m)$ is *majorized* by the vector $y = (y_1, \ldots, y_m)$ (written $x \prec y$) if (after reordering if necessary) $x_1 \le x_2 \le \ldots \le x_m$ and $y_1 \le y_2 \le \ldots \le y_m$ and $\sum_{i=1}^{k} x_i \ge \sum_{i=1}^{k} y_i$ for $1 \le k \le m-1$ and $\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i$.

The following result can be found in [**7**, p.228].

LEMMA 3.1. *A vector x is majorized by a vector y if and only if there is a a doubly stochastic matrix X such that $x = yX$.* □

Proof of Theorem 3.1: The inequalities in Theorem 3.1 describe the set of all points majorized by the vector $\bar{a} := (\bar{a}_1, \ldots, \bar{a}_N)$. Lemma 3.1 implies that this is the set of all points of the form $\bar{a}\, X$ as X runs over all points in B_N, that is, $P(M)$. □

The theorem also shows that $dim\, P(M) = N - 1$. Not all of the inequalities given by (3.1) define facets of $P(M)$. The following theorem gives the inequalities necessary to describe $P(M)$.

THEOREM 3.2. *: The facets of $P(M)$ are given by intersections with the supporting hyperplanes*

$$H_S = \{\ x \in \boldsymbol{R}^N \mid x(S) = \sum_{i=1}^{|S|} \bar{a}_i\ \}$$

where $S \subset \{1, 2, \ldots, N\}$ is such that $k_1 + 1 \leq |S| \leq N - k_n - 1$ or $|S| = 1$ (if $k_n < N - 1$), or $|S| = N - 1$ (if $k_1 < N - 1$).

Proof: By Theorem 3.1, H_S is a supporting hyperplane to $P(M)$. Let $s = |S|$, $k_1 + 1 \leq s \leq N - k_n - 1$ (or $s = 1, N - 1$). Let $M_s := \{\bar{a}_1, \ldots, \bar{a}_s\}$, $\bar{M}_s := M \setminus M_s$. Then

$$F_S := H_S \cap P(M) = P(M_s) \times P(\bar{M}_s)$$

where $P(M_s)$ and $P(\bar{M}_s)$ are multipermutohedra defined on coordinates indexed by S and $[N] \setminus S$ respectively. Hence

$$dim\, F_S = dim\, P(M_s) + dim\, P(\bar{M}_s) = |M_s| - 1 + |\bar{M}_s| - 1 = N - 2$$

For other values of $|S|$, $P(M_s)$ or $P(\bar{M}_s)$ is 0-dimensional, implying that F_S has dimension less than $N - 2$. □

COROLLARY 3.1. *Faces of $P(M)$ are products of lower dimensional multipermutohedra, corresponding to ordered partitions of the set $[N]$. Each facet of $P(M)$ corresponds to an ordered partition into two blocks.* □

Example 1: Let $M = \{1, 2, 2, 3\}$. Then $P(M) \subset \mathbf{R}^4$ is given by

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 8 \\ 1 \leq x_i &\leq 3 \qquad i = 1, \ldots, 4 \\ x_i + x_j &\geq 3 \qquad 1 \leq i < j \leq 4 \end{aligned}$$

Figure 2 shows $P(\{1, 2, 2, 3\})$. It has 14 facets corresponding to these 14 inequalities, which correspond, in turn, to the 14 ordered partitions of the set $\{1, 2, 3, 4\}$ into two blocks. □

In general, the correspondence between faces of $P(M)$ and ordered partitions of $[N]$ need not be one-to-one. In Proposition 1.4, we saw that the face lattice of P_N was isomorphic to Π_N, the lattice of ordered partitions of $[N]$. Each ordered partition $\pi = (A_1, \ldots, A_m)$ in Π_N defines a face $F(\pi)$ of P_N as given by (2). A similar correspondence can be obtained between Π_N and the face lattice of $P(M)$ in the following way.

Given $\sigma \in \mathcal{S}_N$, define $\bar{a}_\sigma := (\bar{a}_{\sigma^{-1}(1)}, \ldots, \bar{a}_{\sigma^{-1}(N)}) \in P(M)$. For an ordered partition $\pi = (A_1, \ldots, A_m) \in \Pi_N$, define

$$F(\pi) = \text{conv}\{\bar{a}_\sigma \mid \sigma([s_i]) = S_i,\ i = 1, \ldots, m\}. \tag{5}$$

where $S_i = A_1 \cup \cdots \cup A_i$ and $s_i := |S_i|$. However, this map is not in general one to one. For example, if $|S_2| \leq k_1$, then the ordered partitions $(A_1, \ldots, A_m)$

and $(A_1 \cup A_2, A_3, \ldots, A_m)$ define the same face of $P(M)$. So, the faces of $P(M)$ correspond to a collection of ordered partitions rather than a single partition. We now define this correspondence more precisely.

Define the numbers $r_j := k_1 + \cdots + k_j$ for $j = 1, \ldots, n$ and let $r_0 = s_0 = 0$. For $\pi \in \Pi_N$ given above, define the ordered partition $\widehat{\pi} \geq \pi$ by taking the union of the sets $A_{i+1}, \ldots, A_k$ whenever

$$s_{i-1} < r_j \leq s_i < \cdots < s_k \leq r_{j+1} < s_{k+1}.$$

In other words, $\widehat{\pi}$ is the *coarsest* partition defining the same face of $P(M)$ as π.

Given $\pi_1, \pi_2 \in \Pi_N$ we say $\pi_1 \sim \pi_2$ if $\widehat{\pi}_1 = \widehat{\pi}_2$. Then $\sim$ is an equivalence relation on Π_N. Let $[\pi]$ be the equivalence class containing π. We order the classes by inclusion of their corresponding faces: we say $[\pi] \leq [\gamma]$ if there exists $\pi_1 \in [\pi]$, $\gamma_1 \in [\gamma]$ such that π_1 is a refinement of γ_1. We have $F(\pi) \subseteq F(\gamma)$ if and only if $[\pi] \leq [\gamma]$. Thus Π_N/M, the set of equivalence classes in Π_N with this ordering, is the face lattice of $P(M)$.

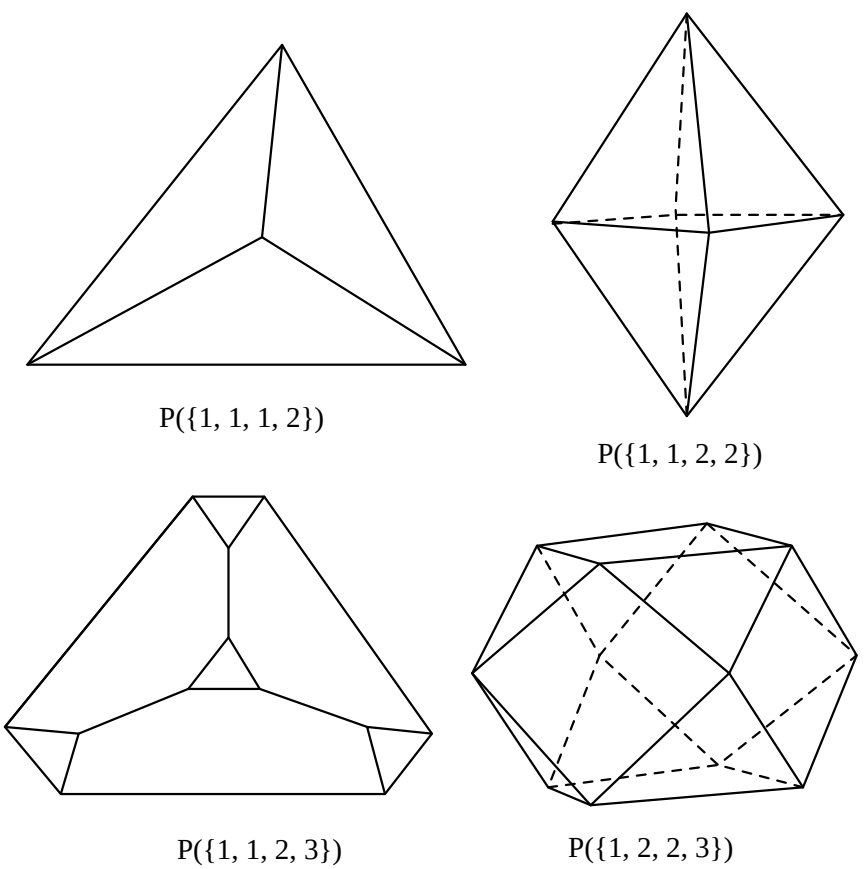

FIGURE 2. Some 3-dimensional multipermutohedra.

We observe that, when $M = \{a_1^{k_1}, a_2^{k_2}, \ldots, a_n^{k_n}\}$, the lattice Π_N/M depends only on the values of k_i and not on the a_is. Hence we have shown the following.

THEOREM 3.3. *If $M = \{a_1^{k_1}, \ldots, a_n^{k_n}\}$ and $M' = \{b_1^{k_1}, \ldots, b_n^{k_n}\}$ are multisets with $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$, then $P(M)$ is combinatorially equivalent to $P(M')$.* □

Thus we may assume from now on that $a_i = i$ and that $M = \{1^{k_1}, \ldots, n^{k_n}\}$. Since the face lattice of $P(M)$ does depend on k_i and $dim\, P(M) = N - 1 = \sum k_i - 1$, we could get different multipermutohedra of the same dimension by fixing N and varying k_i. We will count the number of combinatorially distinct multipermutohedra of each dimension.

Define a polytope P to be *decomposable* if there exist polytopes P_1, P_2 of positive dimensions such that $P \approx P_1 \times P_2$. P is said to be *indecomposable* otherwise.

LEMMA 3.2. *For any multiset M, $P(M)$ is indecomposable.*

Proof: We prove this by induction on $N = k_1 + \cdots + k_n$. The result is straightforward for $N \leq 2$, so assume $N > 2$. We may assume that $n \geq 2$; otherwise $P(M)$ is a point. By Theorem 3.2, either the inequalities $x_i \geq a_1$ or $x_i \leq a_n$ define N facets of $P(M)$ that are themselves multipermutohedra on $M_1 := M \setminus \{a_1\}$ or $M_n := M \setminus \{a_n\}$, and hence are indecomposable by the induction hypothesis. On the other hand, the facets of any product $P_1 \times P_2$ are decomposable unless P_1 or P_2 is a 1-polytope. However in this case, $P_1 \times P_2$ has at most two indecomposable facets. This proves the result. □

For $r < s$, we define the submultiset $M_{r,s} := \{\bar{a}_r, \ldots, \bar{a}_s\} \subset M$.

LEMMA 3.3. *The indecomposable faces of $P(M)$ are precisely those of the form $P(M_{r,s})$.*

Proof: For a face F of $P(M)$, choose $\pi = (A_1, \ldots, A_k)$ to be a finest ordered partition of $[N]$ such that $F = F(\pi)$ as in (5). If all except one of the A_i contain only a single element, then we obtain a face of the form $P(M_{r,s})$, which is indecomposable by Lemma 3.2. Otherwise, π defines a face that is a product of multipermutohedra of positive dimension and hence is decomposable. □

THEOREM 3.4. *Let $m, n > 1$ and consider the multisets $K = \{1^{k_1}, \ldots, n^{k_n}\}$ and $R = \{1^{r_1}, \ldots, m^{r_m}\}$. Then we have $P(K) \approx P(R)$ if and only if $m = n$ and either $(k_1, \ldots, k_n) = (r_1, \ldots, r_n)$ or $(k_1, \ldots, k_n) = (r_n, \ldots, r_1)$.*

Proof: Let $k = (k_1, \ldots, k_n)$ and $r = (r_1, \ldots, r_m)$. For sufficiency, the case $k = r$ is clear. In the case $k = (r_n, \ldots, r_1)$, an isomorphism between the face lattices of $P(K)$ and $P(R)$ is obtained by reversing the order of the blocks of any ordered partition.

We prove necessity by induction on $N = k_1 + \cdots + k_n$. Since $m, n > 1$, the smallest case is $N = 2$, for which $k = r = (1, 1)$. Assume $N > 2$ and $P(K) \approx P(R)$. We first show that $m = n$. Suppose $m < n$. If $k_1 > 1$ (respectively $k_n > 1$), then $P(K_{2,N})$ (respectively $P(K_{1,N-1})$) is, by Lemma 3.3, an indecomposable facet of $P(K)$ that is a multipermutohedron on n symbols. This should correspond to an indecomposable facet of $P(R)$, which by induction is also a multipermutohedron on n symbols, which is impossible. Similarly if $r_1 = 1$, then $P(R_{2,N})$ is an indecomposable facet of $P(R)$ on $m-1$ symbols, but there exist no such facets in $P(K)$ since $m < n$ and facets of $P(K)$ require at least $n - 1$ symbols. Finally, if $k_1 = k_n = 1$ and $r_1 > 1$, then $P(K)$ has more facets than $P(R)$, since every ordered partition of $[N]$ into two blocks gives a facet of $P(K)$ while this is not the case in $P(R)$. Hence $m \geq n$; similarly $n \geq m$, and so $m = n$. The case $m = n = 2$ is treated directly, so we assume $n > 2$.

By Lemma 3.3, $P(K_{2,N})$ is an indecomposable facet of $P(K)$. This implies either $P(K_{2,N}) \approx P(R_{1,N-1})$ or $P(K_{2,N}) \approx P(R_{2,N})$, and using induction, we are left with four possibilities.

Case 1: $(r_1, \ldots, r_{n-1}, r_n - 1) = (k_n, \ldots, k_2, k_1 - 1)$. This proves the result.

Case 2: $(r_1, \ldots, r_{n-1}, r_n - 1) = (k_1 - 1, k_2, \ldots, k_n)$, which implies that

$$r = (k_1 - 1, k_2, \ldots, k_{n-1}, k_n + 1). \tag{6}$$

Now we consider the face $P(K_{1,N-k_n})$ of $P(K)$. Since this is an indecomposable face of $P(K)$ on $n-1$ symbols of dimension $N - k_n - 1$, it must correspond to $P(R_{1,N-r_n})$ or $P(R_{r_1+1,N})$. The former has dimension $N - r_n - 1 = N - k_n$ so it must be $P(R_{r_1+1,N})$. This implies that either $(r_2, \ldots, r_n) = (k_{n-1}, \ldots, k_1)$ or $(r_2, \ldots, r_n) = (k_1, \ldots, k_{n-1})$. In the first case, (6) gives $k_n + 1 = r_n = k_1 = r_1 + 1$, so $r_1 = k_n$ and $r = (k_n, \ldots, k_1)$. In the second case, (6) implies

$$k_1 = r_2 = k_2 = r_3 = \cdots = k_{n-1} = r_n = k_n + 1$$

and so $r = (k_1 - 1, k_1, \ldots, k_1) = (k_n, \ldots, k_1)$, proving the result.

Case 3: $(r_1 - 1, r_2, \ldots, r_n) = (k_1 - 1, k_2, \ldots, k_n)$ showing that $r = k$.

Case 4: $(r_1 - 1, r_2, \ldots, r_n) = (k_n, \ldots, k_2, k_1 - 1)$ which implies that $r = (k_n + 1, k_{n-1}, \ldots, k_2, k_1 - 1)$. Again we show that $k_j = k_{n-j+1}$ for each $j > 1$ and $k_1 = k_n + 1$ by a method similar to Case 2, thus showing that $r = k$. □

THEOREM 3.5. *The number of combinatorially distinct d-dimensional multipermutohedra is $2^{d-1} + 2^{\lfloor \frac{d-1}{2} \rfloor} - 1$.*

Proof: By Theorem 3.4, we need to count the number of compositions of $d+1$ into two or more parts, modulo the equivalence given by reversing the order of the terms.

Let ℓ denote the number of symmetric compositions $c_1 + \cdots + c_k = d + 1$, that is those for which $(c_1, \ldots, c_k) = (c_k, \ldots, c_1)$. Since the total number of

compositions of $d+1$ is 2^d, the number we seek is $2^{d-1}+\ell/2-1$. We show that $\ell = 2^{\lfloor \frac{d+1}{2} \rfloor}$, the number of compositions of $\lfloor \frac{d+1}{2} \rfloor + 1$. To construct a bijection between compositions of $\lfloor \frac{d+1}{2} \rfloor + 1$ and symmetric compositions of $d+1$, we send $(d_1, \ldots, d_s)$ to $(d_1, \ldots, d_{s-1}, c_s, , d_{s-1}, \ldots, d_1)$, where

$$c_s = \begin{cases} 2d_s - 2 & \text{if } d \text{ is odd;} \\ 2d_s - 1 & \text{if } d \text{ is even.} \end{cases}$$

□

In dimension 2, there are two multipermutohedra, the triangle $P(\{1,1,2\})$ and the hexagon $P(\{1,2,3\})$. Four of the five 3-dimensional multipermutohedra are shown in Figure 2. The fifth, the permutohedron $P(\{1,2,3,4\}) = P_4$, can be formed by taking a regular octahedron, choosing two points on each edge, so that the edge is divided into three segments of equal length, and forming the the convex hull. $P(\{1,2,2,3\})$ can be formed by taking the convex hull of the midpoints of the edges of a cube.

It would be interesting to obtain an expression for the diameter of $P(M)$ in terms of $k_1, \ldots, k_n$. Since the edges of $P(M)$ correspond to ordered partitions of $[N]$ into $N-1$ sets, two vertices of $P(M)$ are adjacent if and only if they differ in two coordinates whose entries are a_i, a_{i+1} and a_{i+1}, a_i for some i.

Multipermutohedra were first studied by Schoute [**15**], who constructed them all from the simplex by a sequence of simple operations. An independent treatment of multipermutohedra appears in [**16**], although we have not seen this work. A different generalization of the permutohedron can be found in [**11**], where instead of varying the point to be permuted, one considers the effect of permuting by various subgroups of $\mathcal{S}_n$. In this case, for a given proper subgroup, the combinatorial type of the resulting polytope can vary with the point a, even if it is chosen generically. Another approach is that of Barvinok [**1**], in which more general representations of $\mathcal{S}_n$ are considered (here we have taken the standard representation on $\mathbf{R}^n$), and one forms the *orbit polytope* generated by the images under the representation of a given point in the space.

4. *The Asymmetric Traveling Salesman Polytope*

The traveling salesman polytope arises in the study of the optimization problem of the same name. Given the complete graph with n vertices K_n, along with a directed distance d_{ij} between each ordered pair of vertices (i,j) (so it may be that $d_{ij} \neq d_{ji}$), the *traveling salesman problem* is to determine the shortest directed (Hamiltonian) tour $i_1, i_2, \ldots, i_n, i_1$ through all the vertices. The total length of the tour $i_1, i_2, \ldots, i_n, i_1$ is given by the inner product $\langle X_\sigma, D \rangle$, where $D = (d_{ij})$ and $\sigma = (i_1, i_2, \ldots, i_n) \in \mathcal{S}_n$, a cycle of length n. One approach to getting an approximate, sometimes optimal, solution to traveling salesman problems is to linearize the problem by taking the convex hull of all such X_σ

and seeking to minimize the linear functional $\langle X, D\rangle$ over all X in the resulting polytope. Given an (approximate) linear description of this polytope, one can find an (approximate) solution to the underlying discrete optimization problem. The traveling salesman problem is an NP hard combinatorial problem, with little expectation that there will be an efficient solution method. Most of the study of the polytope of tours over nearly forty years has been towards obtaining a complete description of the linear inequalities that describe its facets. See [**9**] for an extensive discussion of the status of this problem.

To give a formal definition, we begin by letting

$$\mathcal{T}_n := \{\sigma \in \mathcal{S}_n \mid \sigma \textit{ is a cycle of length } n\}.$$

The *asymmetric traveling salesman polytope* of order n is defined by

$$T_n := \text{conv}\{X_\sigma \mid \sigma \in \mathcal{T}_n\}.$$

Note that

$$T_n \subset B_n \subset \mathbf{R}^{n^2}$$

and so if F is a face of B_n and V is the set of vertices of F that are also vertices of T_n then $F \cap \text{conv}(V)$ is a face of T_n; we call this the face *induced* by F.

We can use Proposition 2.1 to induce some faces of T_n. Let $\sigma \in \mathcal{S}_n$ and $\mathcal{C} = \{C_1, \ldots, C_k\}$, where $C_1, \ldots, C_k \in \mathcal{S}_n$ are disjoint cycles. Then $F(\sigma, \mathcal{C})$, the face of B_n generated by σ and $\pi = \sigma C_1 \cdots C_k$, is a k-cube, and if V is the collection of vertices of T_n in this cube, then $\text{conv}(V) = F(\sigma, \mathcal{C}) \cap T_n$ is a face of T_n. Call this face $F'(\sigma, \mathcal{C})$. We shall identify $F'(\sigma, \mathcal{C})$ with a 0-1 polytope embedded in the k-cube. We note that

$$F(\sigma, \mathcal{C}) = B_n \cap \{X \in \mathbf{R}^{n^2} \mid x_{ij} = 0 \textit{ if } (i,j) \notin G(\sigma, \pi)\}$$

and hence

$$F'(\sigma, \mathcal{C}) = T_n \cap \{X \in \mathbf{R}^{n^2} \mid x_{ij} = 0 \textit{ if } (i,j) \notin G(\sigma, \pi)\}.$$

One of the earliest papers on the polytope T_n is that of Heller [**8**], who observed most of Proposition 2.1 and used it to derive a tight upper bound on the dimension of the smallest face generated by a pair of vertices. We extend this bound in the following.

PROPOSITION 4.1. *Every set of k vertices of T_n is contained in a face of dimension at most $(k-2)n + \lfloor \frac{n}{k} \rfloor$ for $1 \le k \le n-1$.*

Proof: This follows immediately from Theorem 2.4 and the above procedure of generating faces of T_n from those of B_n. □

Proposition 4.1 reduces to Heller's tight bound for $k = 2$ and for certain values of n [**8**]. In particular, if $n \ge 8$ and $n = 4q + r$, $0 \le r < 4$, $r \ne 2$, then setting $\sigma = (1, \ldots, n)$ and $\mathcal{C} = \{C_1, \ldots, C_{2q+1}\}$, where $C_i = (i, 2q + i)$ for $i \le 2q$ and $C_{2q+1} = (n - r + 1, \ldots, n)$, makes $\dim F'(\sigma, \mathcal{C}) = \dim F(\sigma, \mathcal{C}) = \lfloor \frac{n}{2} \rfloor$. This

follows from showing that if $S \subseteq [2q]$ then $\sigma\mathcal{C}(S) \in \mathcal{T}_n$ if and only if $|S|$ is even. In the case $n = 4q + 2$, Heller showed that every pair of vertices lies in a face of dimension at most $2q = \lfloor n/2 \rfloor - 1$, and this bound is tight.

The following example, due to Rao [**14**], shows the potential for inducing interesting faces of T_n.

Example 2: Let $n = 9$, $\sigma = (1,2,3,4,5,6,7,8,9)$ and $\mathcal{C} = \{C_1, C_2, C_3\}$ where $C_1 = (1,7,4)$, $C_2 = (2,8,5)$ and $C_3 = (3,9,6)$. Then $F(\sigma, \mathcal{C}) \subset B_9$ is a 3-cube and the other vertices of this cube are given by

$\sigma_1 = \sigma C_1 = (1,8,9)(2,3,4)(5,6,7) \sim (1,0,0)$
$\sigma_2 = \sigma C_2 = (1,2,9)(3,4,5)(6,7,8) \sim (0,1,0)$
$\sigma_3 = \sigma C_3 = (1,2,3)(4,5,6)(7,8,9) \sim (0,0,1)$
$\sigma_{12} = \sigma C_1 C_2 = (1,8,6,7,5,3,4,2,9) \sim (1,1,0)$
$\sigma_{23} = \sigma C_2 C_3 = (1,2,9,7,8,6,4,5,3) \sim (0,1,1)$
$\sigma_{13} = \sigma C_1 C_3 = (1,8,9,7,5,6,4,2,3) \sim (1,0,1)$
$\sigma_{123} = \sigma C_1 C_2 C_3 = (1,8,6,4,2,9,7,5,3) \sim (1,1,1)$

We note that 5 of the vertices of this cube are vertices of T_9 and thus $F'(\sigma, \mathcal{C})$ is a bipyramid over a triangle as shown in Figure 3. □

It turns out that *any* 0-1 polytope can appear as a face of T_n in the above manner. The remainder of this section is devoted to outlining a proof of this assertion. Since any 0-1 d-polytope is affinely equivalent to one with vertices in the d-cube $\mathcal{I}_d$, such a polytope can be described by which vertices of $\mathcal{I}_d$ it *omits*. The idea of the proof is to produce a d-cube face of B_n such that the vertices that are not n-cycles are precisely the ones we wish to omit.

We will usually refer to vertices of $\mathcal{I}_d$ by their corresponding supports, which are subsets of $[d]$. If V is a subset of vertices of $\mathcal{I}_d$, then by $\mathcal{I}_d \setminus V$ we mean the convex hull of the vertices of $\mathcal{I}_d$ that are not in V. We first deal with the case $V = \emptyset$. Unless otherwise noted, in what follows d is a positive integer, $n = 3d$ and $\sigma = (1,2,\ldots,n) \in \mathcal{T}_n$.

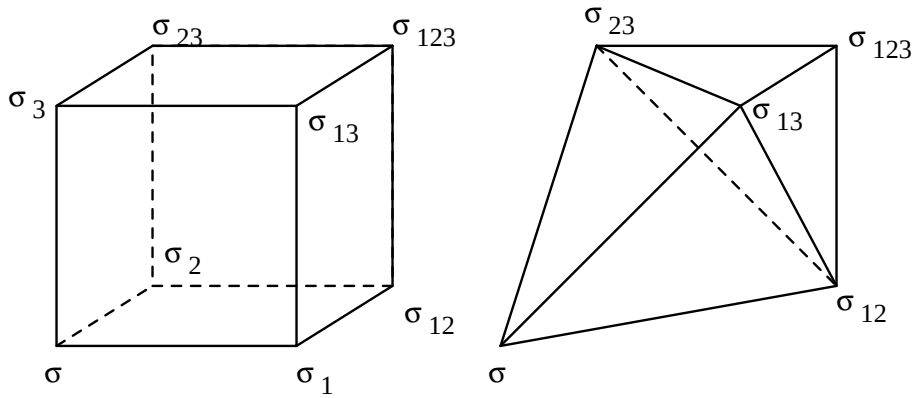

FIGURE 3. $F(\sigma, \mathcal{C})$ and $F'(\sigma, \mathcal{C})$ for Example 2.

LEMMA 4.1. *$\mathcal{I}_d$ is an induced face of T_{3d}.*

Proof: Let $\mathcal{C} = \{(1,2,3),(4,5,6),\ldots,(n-2,n-1,n)\}$. Then for any $S \subseteq [d]$, $\sigma\mathcal{C}(S) = (a_1,\ldots,a_n) \in \mathcal{T}_n$ where

$$a_{3i-2}, a_{3i-1}, a_{3i} = \begin{cases} 3i-2, 3i, 3i-1 & \text{if } i \in S \\ 3i-2, 3i-1, 3i & \text{otherwise} \end{cases}$$

Thus $F'(\sigma,\mathcal{C}) = F(\sigma,\mathcal{C}) = \mathcal{I}_d$. □

In a similar way, we show that we can remove a single vertex from a d-cube and obtain an induced face of T_{3d}.

LEMMA 4.2. *For any nonzero vertex $S \in \mathcal{I}_d$, $\mathcal{I}_d \setminus \{S\}$ is an induced face of T_{3d}*

Proof: We begin with $S = [d]$. Let

$$\mathcal{C} = \{C_1, C_2, \ldots, C_d\} = \{(1,3,5),(4,6,8),\ldots,(n-2,n,2)\}$$

(i.e. $C_i = (3i-2, 3i, 3i+2)$ for $i < d$ and $C_d = (n-2,n,2)$). We show

$$F'(\sigma,\mathcal{C}) = \mathcal{I}_d \setminus [d].$$

Since

$$\sigma\mathcal{C}[d] = (1,4,7,\ldots,n-2)(3,6,9,\ldots,n)(n-1,n-4,\ldots,2) \notin \mathcal{T}_n$$

this implies that $[d] \notin F'(\sigma,\mathcal{C})$. That all the other vertices of the cube $F(\sigma,\mathcal{C})$ are also vertices of T_n amounts to showing that if R is a proper subset of $[d]$, then $\sigma\mathcal{C}(R)$ is an n-cycle. This is the content of [**2**, Proposition 3.2].

The result for general S follows by conjugation. If $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ is as defined above, let $\overline{S} = [d] \backslash S$ and $\sigma\mathcal{C}(\overline{S}) = (a_1,\ldots,a_n)$. Set $\pi(i) = a_i$ for $i = 1,\ldots,n$ and define $\overline{C}_i$ by

$$\overline{C}_i = \begin{cases} \pi^{-1} C_i \pi & \text{if } i \in S; \\ \pi^{-1} C_i^{-1} \pi & \text{if } i \notin S. \end{cases}$$

If $\overline{\mathcal{C}} := \{\overline{C}_1, \ldots, \overline{C}_d\}$, we get $F'(\sigma,\overline{\mathcal{C}}) = \mathcal{I}_d \setminus \{S\}$. □

Example 3: Let $v = (1,0,0,0), \sigma = (1,\ldots,12)$. We find $\overline{\mathcal{C}} = \{\overline{C}_1, \overline{C}_2, \overline{C}_3, \overline{C}_4\}$ such that $F'(\sigma,\overline{\mathcal{C}}) = \mathcal{I}_4 \setminus \{v\}$. By Lemma 4.2, if

$$\mathcal{C} = \{C_1, C_2, C_3, C_4\} = \{(1,3,5),(4,6,8),(7,9,11),(10,12,2)\}$$

then $F'(\sigma,\mathcal{C}) = \mathcal{I}_4 \setminus \{(1,1,1,1)\}$.

Let $S = \{1\}$, the support of v, and $\overline{S} = \{2,3,4\}$,

$$\begin{aligned} \sigma\mathcal{C}(\overline{S}) &= \sigma C_2 C_3 C_4 \\ &= (1,2,11,8,5,6,9,12,3,4,7,10) =: (a_1, a_2, \ldots, a_{12}) \end{aligned}$$

with $a_1 = 1$. Setting $\pi(i) = a_i$ for $i = 1, \ldots, 12$ we get
$\pi = (3, 11, 7, 9)(4, 8, 12, 10)$ and $\sigma = \pi^{-1}\sigma\mathcal{C}(\overline{S})\pi$.
Then

$$\begin{aligned}
\overline{C}_1 &= \pi^{-1}C_1\pi = (1, 9, 5) \\
\overline{C}_2 &= \pi^{-1}C_2^{-1}\pi = (4, 6, 10) \\
\overline{C}_3 &= \pi^{-1}C_3^{-1}\pi = (3, 7, 11) \\
\overline{C}_4 &= \pi^{-1}C_4^{-1}\pi = (2, 8, 12)
\end{aligned}$$

Hence

$$\overline{\mathcal{C}} = \{(1, 9, 5), (4, 6, 10), (3, 7, 11), (2, 8, 12)\}$$

In the same way, one can show that if

$$\overline{\mathcal{C}} = \{(2, 4, 12), (1, 5, 7), (6, 8, 10), (3, 9, 11)\},$$

then $F'(\sigma, \overline{\mathcal{C}}) = \mathcal{I}_4 \setminus \{(0, 1, 1, 1)\}$. □

Now we tackle the general case of removing any set of vertices. Let $P \subset \mathbf{R}^d$ be a 0-1 polytope. Assume without loss of generality that $0 \in P$ and let $V := \{S_1, \ldots, S_k\}$ be the set of vertices of $\mathcal{I}_d$ that are not in P (i.e. $P = \mathcal{I}_d \setminus V$). By Lemma 4.2, we can find $\mathcal{C}_i = \{C_{i1}, \ldots, C_{id}\}$ such that $F'(\sigma, \mathcal{C}_i) = \mathcal{I}_d \setminus S_i$ for $i = 1, \ldots, k$, where $n = 3d$, $\sigma = (1, \ldots, n)$ and each C_{ij} is a 3-cycle. The idea is to concatenate the cycles $C_{1j}, \ldots, C_{kj}$ for $j = 1, \ldots, d$ to eliminate $S_1, \ldots, S_k$. For this, the cycles must be defined on distinct sets. So, assume that $\mathcal{C}_i$ is defined on $\{(i-1)n + 1, \ldots, in\}$ and let $\sigma_i = ((i-1)n + 1, \ldots, in)$ so that $F'(\sigma_i, \mathcal{C}_i) = \mathcal{I}_d \setminus S_i$ for $i = 1, \ldots, k$ (i.e. $\sigma_i\mathcal{C}_i(S)$ is an n-cycle for all $S \subseteq [d]$ except when $S = S_i$).

LEMMA 4.3. *There is an integer N and an ordered set of disjoint cycles $\mathcal{Y}' = \{Y_1', \ldots, Y_d'\} \subset \mathcal{S}_N$ such that if $\sigma' = (1, \ldots, N)$ then $F'(\sigma', \mathcal{Y}') = P$.*

Proof (sketch): Define the permutations $Y_j = C_{1j}C_{2j}\cdots C_{kj} \in \mathcal{S}_{kn}$ for $j = 1, \ldots, d$ and let $\mathcal{Y} = \{Y_1, \ldots, Y_d\}$, and $N = kn + (k+1)d$. The following array illustrates how Y_j is defined.

$$\begin{array}{cccccc}
\mathcal{C}_1 & C_{11} & \cdots & \boxed{C_{1j}} & \cdots & C_{1d} \qquad F'(\sigma_1, \mathcal{C}_1) = \mathcal{I}_d \setminus S_1 \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathcal{C}_k & C_{k1} & \cdots & \boxed{C_{kj}} & \cdots & C_{kd} \qquad F'(\sigma_k, \mathcal{C}_k) = \mathcal{I}_d \setminus S_k \\
 & & & Y_j & &
\end{array}$$

We first show that if $S \subseteq [d]$ then

$$\sigma'\mathcal{Y}(S) \in \mathcal{T}_N \Leftrightarrow \sigma_r\mathcal{C}_r(S) \text{ is an n-cycle } \textit{for } r = 1, \ldots, k. \tag{7}$$

It follows from (7) that $\sigma'\mathcal{Y}(S) \in \mathcal{T}_N$ if and only if $S \notin V$, i.e., if and only if S is a vertex of P. We now modify the permutations $Y_1, \ldots, Y_d$ to define *cycles* $Y_1', \ldots, Y_d'$ such that $\sigma'\mathcal{Y}(S) \in \mathcal{T}_N \Leftrightarrow \sigma'\mathcal{Y}'(S) \in \mathcal{T}_N$ for all $S \subseteq [d]$. To do this

we lace together the cycles in each Y_j by introducing $(k+1)$ new elements to construct the cycle Y_j' (the example below illustrates how this is done). Thus a total of $(k+1)d$ new elements are added to the original $kn = 3kd$, giving $N = (4k+1)d$. □

The following is a direct consequence of Lemma 4.3.

THEOREM 4.1. *If $P \subset \boldsymbol{R}^d$ is a 0-1 polytope with $2^d - k$ vertices, $k > 1$, then P appears as an induced face of T_n for $n \geq (4k+1)d$.* □

COROLLARY 4.1. *Every 0-1 d-polytope is a face of T_n for $d \sim \log n$.* □

Example 4: Let $P = \mathcal{I}_4 \setminus \{v_1, v_2\}$ where $v_1 = (1,0,0,0)$, $v_2 = (0,1,1,1)$.

In Example 3, we found $\mathcal{C}_1$ and $\mathcal{C}_2$ such that if $\sigma = (1, \ldots, 12)$ then $F'(\sigma, \mathcal{C}_1) = \mathcal{I}_4 \setminus \{v_1\}$ and $F'(\sigma, \mathcal{C}_2) = \mathcal{I}_4 \setminus \{v_2\}$. Now we add 12 to every number in $\mathcal{C}_2$ to get the permutations in different sets. Let

$$\begin{aligned} \sigma_1 &= (1, \ldots, 12)\ ,\ \mathcal{C}_1 = \{(1,9,5),(4,6,10),(3,7,11),(2,8,12)\} \\ \sigma_2 &= (13, \ldots, 24)\ ,\ \mathcal{C}_2 = \{(14,16,24),(13,17,19),(18,20,22),(15,21,23)\} \end{aligned}$$

so that

$$F'(\sigma_1, \mathcal{C}_1) = \mathcal{I}_4 \setminus \{v_1\} \text{ and } F'(\sigma_2, \mathcal{C}_2) = \mathcal{I}_4 \setminus \{v_2\}.$$

Now define

$$\begin{aligned} Y_1 &= (1,9,5)(14,16,24)\ ,\ Y_1' = (\underline{27}, 14, 16, 24, \underline{26}, 1, 9, 5, \underline{25}) \\ Y_2 &= (4,6,10)(13,17,19)\ ,\ Y_2' = (\underline{30}, 13, 17, 19, \underline{29}, 4, 6, 10, \underline{28}) \\ Y_3 &= (3,7,11)(18,20,22)\ ,\ Y_3' = (\underline{33}, 18, 20, 22, \underline{32}, 3, 7, 11, \underline{31}) \\ Y_4 &= (2,8,12)(15,21,23)\ ,\ Y_4' = (\underline{36}, 15, 21, 23, \underline{35}, 2, 8, 12, \underline{34}) \end{aligned}$$

where we have underlined the new elements in each Y_i'. Let $\sigma' = (1, \ldots, 36)$ and $\mathcal{Y}' = \{Y_1', Y_2', Y_3', Y_4'\}$. Then for example

$$\begin{aligned} \sigma' Y_3 = (&1,2,3,8,9,10,\underline{11,4},5,6,7,12,13,\ldots,18,21,\underline{22,19},20, \\ &23,24,\ldots,\underline{31,32,33,34},35,36) \in \mathcal{T}_{36} \end{aligned}$$

while

$$\begin{aligned} \sigma' Y_3' = (&1,2,3,8,9,10,\underline{11,32,4},5,6,7,12,13,\ldots,18,21,\underline{22,33,19},20, \\ &23,24,\ldots,\underline{31,34},35,36) \in \mathcal{T}_{36} \end{aligned}$$

and

$$\sigma' Y_2 Y_3 Y_4 = (14, 15, \underline{22, 19}) \cdots \notin \mathcal{T}_{36}$$

while

$$\sigma' Y_2' Y_3' Y_4' = (14, 15, \underline{22, 33, 19, 30}) \cdots \notin \mathcal{T}_{36}$$

where we have underlined the places where the permutations differ. Similarly multiplying σ' with all subsets of cycles of $\mathcal{Y}'$ we get $F'(\sigma', \mathcal{Y}') = P$. □

We note here that the above result is not valid for B_n since B_n has the property that every pair of its vertices is contained in a cubical face. For instance, the bipyramid over a triangle (of Example 2) cannot be a face of B_n. Also note that this universality is not a property of all polytopes corresponding to hard combinatorial problems. For example, the polytope associated with the quadratic assignment problem cannot have every 0-1 polytope among its faces since it is *neighborly*, i.e., every pair of its vertices are adjacent. This is so since, choosing a hyperplane containing any pair of vertices of B_n, and no others, its normal can be used (by forming a tensor product with itself) to define a supporting hyperplane to the desired edge of the quadratic assignment polytope. See [**1**, 2.2.3] for some background. (This fact and its proof was pointed out to us by A.I. Barvinok.)

A natural question to ask is whether the bound for n can be improved, that is, for a given d, is there $n \sim d^k$ such that all 0-1 polytopes in $\mathbf{R}^d$ appear as faces of T_n? The answer is *no* since there are 2^{2^d} 0-1 polytopes in $\mathbf{R}^d$, while T_n cannot have that many faces, when $n \sim d^k$. The answer is not known if we were to content to ask that all *combinatorially distinct* 0-1 polytopes are represented among the faces. See [**2**] for details.

Another problem is to find, if possible, a simple combinatorial description of the face lattice of T_n as was done for B_n, P_n and the various multipermutohedra $P(M)$. It seems unlikely that there is such a description, given that the problem of determining adjacency of two vertices of T_n is NP-complete [**12**].

Finally, we can generalize T_n in the following manner. A *partition* of n is a sequence of positive integers $\lambda := (\lambda_1, \ldots, \lambda_k)$ with $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. If $\pi \in \mathcal{S}_n$ is a product of k disjoint cycles of lengths $l_1, \ldots, l_k$ (we include cycles of length 1 as well) in non-increasing order then $(l_1, \ldots, l_k)$ is a partition of n. We call $(l_1, \ldots, l_k)$ the *cycle type* of π. Given the partition λ we define $\mathcal{S}_\lambda$ to be the set of all permutations in $\mathcal{S}_n$ of cycle type λ, that is, $\mathcal{S}_\lambda$ represents a conjugacy class in $\mathcal{S}_n$. We define

$$T_\lambda := \text{conv}\{X_\sigma | \sigma \in \mathcal{S}_\lambda\}.$$

Note that $T_n = T_{(n)}$ in this notation. We ask which choices of conjugacy classes $\lambda(n)$ (one class for each n) produce polytopes $Q_n = P_{\lambda(n)}$ having the property of Corollary 4.1, that is, having any 0-1 polytope as a face of Q_n for n large enough. We conjecture that this is so if and only if the sequence $\{\lambda_1(n)\}$ consisting of the largest part of $\lambda(n)$ is unbounded. The sufficiency of this condition follows from Corollary 4.1. Necessity is known for a few special cases.

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