# Juggling and applications to $q$-analogues

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**Abstract**

We consider juggling patterns where the juggler can only catch and throw one ball at a time, and patterns where the juggler can handle many balls at the same time. Using a crossing statistic, we obtain explicit $q$-enumeration formulas. Our techniques give a natural interpretation of the $q$-Stirling numbers of the second kind and a bijective proof of an identity of Carlitz. Also, juggling patterns enable us to easily compute the Poincaré series of the affine Weyl group $\widetilde{A}_{d-1}$.

**Résumé**

Nous considérons les configurations de jonglerie dans lesquelles le jongleur ne peut attraper ou lancer qu'une seule balle à la fois, ainsi que les configurations où le jongleur peut manipuler plusieurs balles à la fois. Utilisant une statistique de croisements, nous obtenons des formules explicites de $q$-énumération. Nos techniques fournissent des interprétations naturelles pour les $q$-nombres de Stirling de deuxième espèce ainsi qu'une preuve bijective d'une identité de Carlitz. Les configurations de jonglerie nous permettent aussi de calculer la série de Poincaré du groupe de Weyl affine $\widetilde{A}_{d-1}$.

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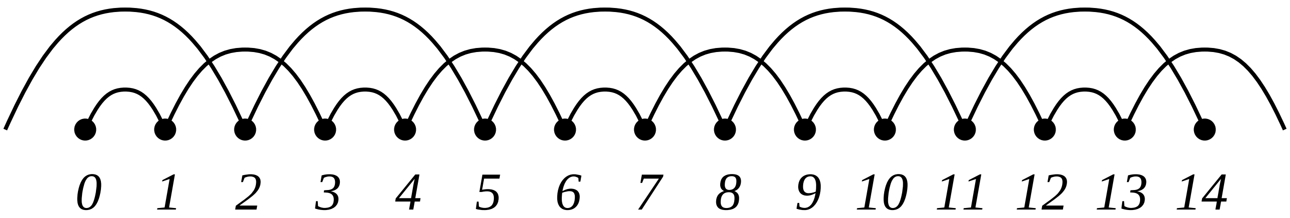


Figure 1: A juggling pattern with $d = 3$, $\mathrm{x} = (0, 1, 2)$, and $\mathbf{a} = (1, 2, 3)$.

# 1 Introduction

Consider the pattern in Figure 1. We can think of this picture as the pattern that juggling balls describe as they are juggled. The horizontal axis is the time axis. At each integer time point one ball is caught and then thrown. At time points $0, 3, 6, \dots$ each ball is thrown high enough so that it lands one time unit later. Similarly, at time points $1, 4, 7, \dots$ each ball is thrown so that it lands two time units later, while at time points $2, 5, 8, \dots$ each ball will land three time units later. Thus this pattern is periodic with period $d = 3$. In this pattern there are two balls since the arcs describe two infinite paths.

In this paper we will enumerate periodic patterns like the one just described. Figure 1 shows a pattern where the juggler can only catch and throw one ball at a time. We will also consider patterns where the juggler has the ability to catch and throw many balls at a time. See Figure 2 for an example of such a pattern. Among jugglers this is called *multiplex*. We say that a juggling pattern is *simple* if the juggler can only catch and throw one ball at a time.

We denote the pattern in Figure 1 by the vectors $\mathbf{x} = (0, 1, 2)$ and $\mathbf{a} = (1, 2, 3)$. The fact that there is one 0 in the vector $\mathbf{x}$ means that at times $0 \bmod d$ the juggler catches and throws one ball. If there were three 1's appearing in the vector $\mathbf{x}$, this would mean that the juggler catches and throws three balls at times $1 \bmod d$. The entries of the vector $\mathbf{a}$ indicate how far each ball is thrown, that is, when it will return to the juggler's hand. Thus at time periods $x_i \bmod d$ the juggler throws a ball $a_i$ time units. The pattern in Figure 2 is represented by $d = 2$, $\mathbf{x} = (0, 0, 1)$, and $\mathbf{a} = (1, 4, 1)$.

Buhler, Eisenbud, Graham, and Wright proved that the number of simple juggling patterns of period $d$ and at most $n$ balls is equal to $n^d$ [1]. Their proof uses the fact that the number of permutations with $k$ excedances is equal to the Eulerian number $A(n, k+1)$ [11, Proposition 1.3.12]. Stanley bijectified their proof. Using a completely different approach, we simultaneously generalize the $n^d$ result in two ways. We include juggling patterns with multiplex and give $q$-analogues of these results.

Between time points 1 and 2 in Figure 1, the paths of the two balls cross. We call this a *crossing*. Since the pattern is periodic, similar crossings appear between 4 and 5, 7 and 8, etc. There is one more crossing, namely between time points 2 and 3. Thus we say this pattern has two crossings. Define the *weight* of a juggling pattern to be $q$ to the power of the number of crossings of the pattern. The $q$-analogue of the $n^d$ result is the following:

**Theorem 1** *The sum of the weight of simple juggling sequences, with period $d$ and at most $n$ balls, is equal to*

$$(1 + q + \cdots + q^{n-1})^d.$$

As a corollary to this theorem, we are able to easily compute the Poincaré series of the affine Weyl group $\widetilde{A}_{d-1}$.

The results for multiplex include a product of Gaussian coefficients, which is presented in Theorem 2. While studying the multiplex case, we came across a natural interpretation of $S[n, k]$,

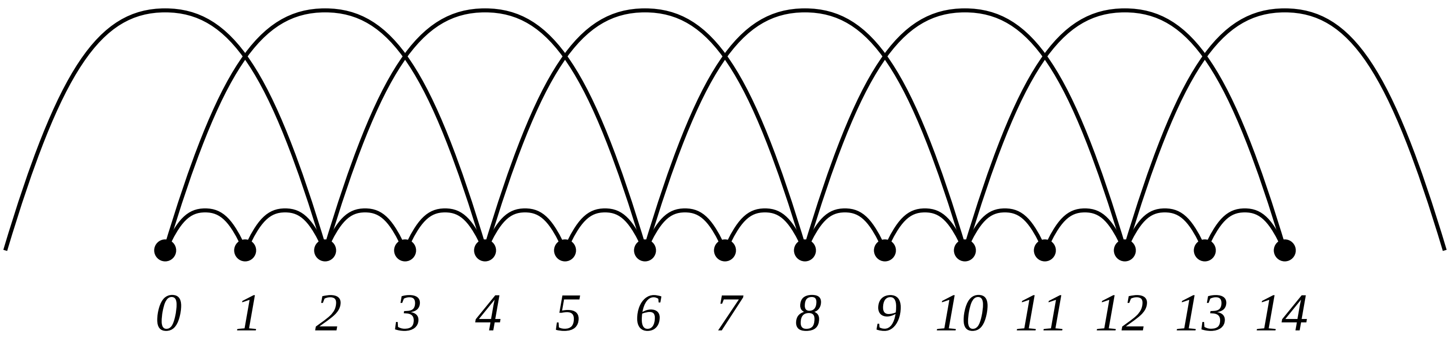


Figure 2: A juggling pattern with $d = 2$, $\mathrm{x} = (0, 0, 1)$, and $\mathbf{a} = (1, 4, 1)$.

the $q$-Stirling numbers of the second kind, using intertwining numbers of blocks. This method can easily be shown to be equivalent to Garsia and Remmel's [4] idea of obtaining the $q$-Stirling numbers from rook placements. We give a bijective proof of an identity of Carlitz [3] involving $S[n, k]$ by contracting simple juggling graphs.

Observe for a multiplex juggling pattern that at each time point the number of balls the juggler catches is equal to the number of balls he throws at that time point. In Section 5 we enumerate patterns without this property; see Theorem 4. For this generalization of juggling patterns, we use two vectors $\mathrm{x}$ and $\mathbf{y}$ to describe such patterns. The vector $\mathrm{x}$ describes how many balls are thrown at each time point, while the vector $\mathbf{y}$ describes the number of balls caught. The proof of this theorem is easily bijectified, and as a special case this gives a bijection for Theorem 2.

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## 2 Definitions

We say that two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ are *similar* if there exists a permutation $\pi \in S_m$ such that $u_i = v_{\pi(i)}$ for all $i = 1, 2, \ldots, m$. We write $\mathbf{u} \sim \mathbf{v}$ when $\mathbf{u}$ and $\mathbf{v}$ are similar. Also, we will use the following two notations $\mathbf{0}_m = (\underbrace{0, 0, \ldots, 0}_{m})$ and $\mathbf{1}_m = (\underbrace{1, 1, \ldots, 1}_{m})$.

**Definition 1** *A* juggling triple $(d, \mathrm{x}, \mathbf{a})$ *consists of a positive integer* $d$*, a vector* $\mathrm{x} = (x_1, x_2, \ldots, x_m)$ *of integers and a vector* $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ *of positive integers, such that the following two conditions hold:*

1. $0 \leq x_i \leq d - 1$ *for all* $i = 1, 2, \ldots, m$.
2. $(\mathbf{a} + \mathrm{x}) \bmod d \sim \mathrm{x}$, *where the* mod $d$ *applies component-wise.*

*We call* $d$ *the* period, $\mathrm{x}$ *the* base vector, *and* $\mathbf{a}$ *the* juggling sequence.

To a juggling triple $(d, \mathrm{x}, \mathbf{a})$ we associate the following directed multigraph $G$ on the integers $\mathbb{Z}$. The vertex set of the graph is $\mathbb{Z}$ and the directed edge set is given by

$$E(G) = \{(x_i + k \cdot d, x_i + a_i + k \cdot d) \ : \ 1 \leq i \leq m, \ k \in \mathbb{Z}\}.$$

Observe that all the edges are directed increasingly with respect to time. Moreover, the condition $(\mathbf{a} + \mathrm{x}) \bmod d \sim \mathrm{x}$ implies that for every vertex its outdegree is equal to its indegree. Hence we can decompose the graph into a finite number of edge-disjoint paths that are increasing. We call the number of edge-disjoint paths the *number of balls* of the juggling triple $(d, \mathrm{x}, \mathbf{a})$. We denote this number by $ball(d, \mathrm{x}, \mathbf{a})$, or by $ball(\mathbf{a})$ when the time period and base vector are clear by the context. It is easy to show that:

**Lemma 1** *The number of balls of the juggling triple* $(d, \mathrm{x}, \mathbf{a})$ *is given by* $\frac{1}{d} \cdot (a_1 + a_2 + \cdots + a_m)$.

Let $\alpha_j$ be equal to the outdegree at vertex $j$ in the associated graph. That is, for $0 \leq j \leq d-1$, $\alpha_j$ is the cardinality of the set $\{i \ : \ x_i = j\}$. We say that a juggling triple is a *simple* juggling triple if $m = d$ and the base vector $\mathrm{x}$ is given by $\mathrm{x} = (0, 1, \ldots, d-1)$. This implies that $\alpha_0 = \alpha_1 = \cdots = \alpha_{d-1} = 1$. Every vertex in the associated graph of a simple juggling sequence has outdegree and indegree one.

In the directed graph $G$ we define a *crossing* to be a pair of two edges $(x, y)$ and $(u, v)$ such that $x < u < y < v$. We say that two crossings $(x_1, y_1)$ and $(u_1, v_1)$, and $(x_2, y_2)$ and $(u_2, v_2)$ are *equivalent* if there exist an integer $k$ such that

$$x_1 = x_2 + k \cdot d, \quad y_1 = y_2 + k \cdot d, \quad u_1 = u_2 + k \cdot d, \quad \text{and} \quad v_1 = v_2 + k \cdot d.$$

Define the number of *external crossings* to be the number of classes of equivalent crossings of the graph.

An *internal crossing* of a juggling triple $(d, \mathrm{x}, \mathbf{a})$ is a pair $(i, j)$ such that $1 \leq i < j \leq m$, $x_i = x_j$, and $a_i > a_j$. For example, the juggling triple $(2, (0, 0, 1), (1, 4, 1))$ that appears in Figure 2 has no internal crossings, whereas the juggling triple $(2, (0, 0, 1), (4, 1, 1))$ has one internal crossing. Observe that these two juggling triples have the same associated graph. No internal crossings occur for a simple juggling triple since all the entries of the base vector are different. The *number of crossings* of a juggling triple $(d, \mathrm{x}, \mathbf{a})$ is the sum of the number of external and internal crossings. We denote the number of crossing of a juggling triple by $cross(d, \mathrm{x}, \mathbf{a})$, or by $cross(\mathbf{a})$ when there is no confusion. We define the *weight* of a juggling triple $(d, \mathrm{x}, \mathbf{a})$ to be $q$ to the power of the number of crossings, that is, $q^{cross(d, \mathrm{x}, \mathbf{a})}$.

Following the convention for $q$-analogues, we define $[n] = 1 + q + \cdots + q^{n-1}$ and $[n]! = [1] \cdot [2] \cdots [n]$. The *Gaussian coefficient*, or *q-binomial coefficient*, is given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]! \cdot [n-m]!}.$$

# 3 Simple and multiplex juggling

We will consider simple and multiplex juggling patterns. We begin by presenting the proof of Theorem 1 by giving an explicit bijection $\Phi$. Note that the base vector for a simple juggling pattern is $(0, 1, \ldots, d-1)$, which we denote by $\delta_d$.

Define the map $\Phi$ from simple juggling triples to $\mathbb{N}^d$ by $\Phi(d, \delta_d, \mathbf{a}) = (\phi_1, \phi_2, \ldots, \phi_d)$, where

$$\phi_i = |\{(u, v) \in E(G) \;:\; i-1 < u < i-1+a_i < v\}|.$$

That is, $\phi_i$ counts the number of directed edges of the associated graph that crosses the edge $(i-1, i-1+a_i)$ from the "inside." Directly we have that *cross* $(d, \delta_d, \mathbf{a}) = \phi_1 + \phi_2 + \cdots + \phi_d$.

It is easy to obtain the statement

$$\mathit{ball}\,(d, \delta_d, \mathbf{a}) = \max(\phi_1, \phi_2, \ldots, \phi_d) + 1.$$

Hence $\Phi$ is a bijection between simple juggling triples of period $d$ having at most $n$ balls and the set $\{0, 1, \ldots, n-1\}^d$. Now the theorem follows since

$$\sum_{0 \le \phi_1 \le n-1} \sum_{0 \le \phi_2 \le n-1} \cdots \sum_{0 \le \phi_d \le n-1} q^{\phi_1 + \phi_2 + \cdots + \phi_d} = [n]^d.$$

Recall that for multiplex juggling $\alpha_j$ is defined to be the outdegree at vertex $j$. We can now state an analogous theorem for multiplex juggling.

**Theorem 2** *The sum of the weight of juggling triples, with period $d$, base vector $\mathbf{x}$, and at most $n$ balls, is equal to*

$$\begin{bmatrix} n \\ \alpha_0 \end{bmatrix} \cdot \begin{bmatrix} n \\ \alpha_1 \end{bmatrix} \cdots \begin{bmatrix} n \\ \alpha_{d-1} \end{bmatrix}.$$

We will use this result when the period $d$ is 1 to prove Carlitz's identity in Section 6. Observe that Theorem 2 implies Theorem 1 in the case when $\alpha_0 = \alpha_1 = \cdots = \alpha_{d-1} = 1$. We omit the proof of this theorem, since it will follow from Theorem 4.

## 4 The Affine Weyl group $\widetilde{A}_{d-1}$

We will now consider the affine Weyl group $\widetilde{A}_{d-1}$. For more detailed accounts, see [5, 9].

**Definition 2** *Let $\widetilde{A}_{d-1}$ be the group of bijections $\sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$ under composition, where the bijections satisfy the following two conditions:*

1. $\sigma(i + d) = \sigma(i) + d$ *for all* $i$,
2. $\sum_{i=1}^{d} (\sigma(i) - i) = 0$.

This combinatorial description of $\widetilde{A}_{d-1}$ is due to Lusztig. $\widetilde{A}_{d-1}$ is a Coxeter group, and when $d \geq 2$ it is generated by the simple reflections $s_0, s_1, \ldots, s_{d-1}$, where

$$s_i(k) = \begin{cases} k+1 & \text{if } k \equiv i \bmod d \\ k-1 & \text{if } k \equiv i+1 \bmod d \\ k & \text{if } k \not\equiv i, i+1 \bmod d. \end{cases}$$

An element $\sigma \in \widetilde{A}_{d-1}$ may be written as a product of simple reflections. Define the *length* $l(\sigma)$ of the element $\sigma$ as the smallest integer $r$ such that one can write $\sigma$ as a product of $r$ simple reflections. Observe that $\widetilde{A}_0$ is the one element group.

**Theorem 3** *Let $\sigma$ be an element in $\widetilde{A}_{d-1}$ and $n$ a positive integer such that $n > \max(i - \sigma(i))$ for all $i = 1, 2, \ldots, d$. Form the sequence $\mathbf{a} = (a_1, a_2, \ldots, a_d)$, where $a_i = \sigma(i) - i + n$. Then $(d, \delta_d, \mathbf{a})$ is a juggling triple with ball$(d, \delta_d, \mathbf{a}) = n$ and cross$(d, \delta_d, \mathbf{a}) = (n-1) \cdot d - l(\sigma)$.*

By Theorems 1 and 3 we obtain:

**Corollary 1** *The Poincaré series of the group $\widetilde{A}_{d-1}$ is given by:*

$$\sum_{\sigma \in \widetilde{A}_{d-1}} q^{l(\sigma)} = \frac{1-q^d}{(1-q)^d}.$$

## 5 A generalization of juggling

We now consider patterns where the number of balls caught at a particular time point does not necessarily equal the number of balls thrown at that same time point. Similar to a juggling triple we define a juggling quadruple.

**Definition 3** *A* juggling quadruple $(d, \mathrm{x}, \mathrm{y}, \mathbf{a})$ *consists of a positive integer $d$, two vectors $\mathrm{x} = (x_1, x_2, \ldots, x_m)$ and $\mathrm{y} = (y_1, y_2, \ldots, y_m)$ of integers, and a vector $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ of positive integers, such that the following two conditions hold:*

1. $0 \leq x_i, y_i \leq d-1$ *for all* $i = 1, 2, \ldots, m$.
2. $(\mathbf{a} + \mathrm{x}) \bmod d \sim \mathrm{y}$, *where the* mod $d$ *applies component-wise.*

*We call $d$ the* period, x *the* throw vector, y *the* catch vector, *and* $\mathbf{a}$ *the* juggling sequence.

Observe that when $\mathrm{x} = \mathrm{y}$, this is equivalent to a juggling triple.

The number of balls of a juggling quadruple is not well-defined. Instead will consider the sum $\sum_{i=1}^{m} a_i$. Observe that $\sum_{i=1}^{m} a_i \equiv \sum_{i=1}^{m}(y_i - x_i) \bmod d$.

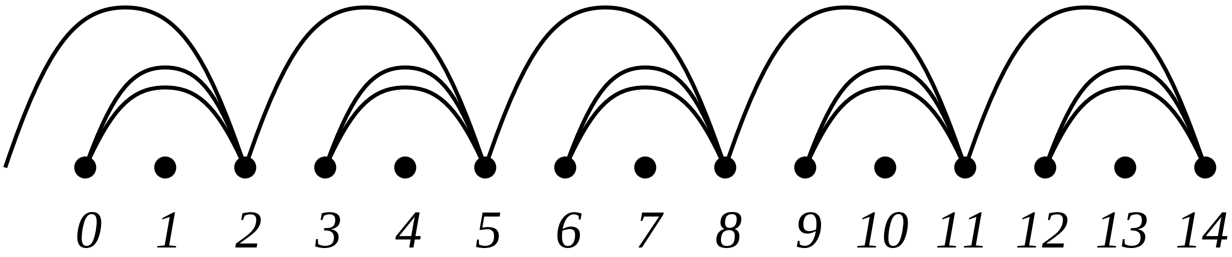


Figure 3: A juggling pattern with $d = 3$, $\mathrm{x} = (0, 0, 2)$, $\mathrm{y} = (2, 2, 2)$, and $\mathbf{a} = (2, 2, 3)$.

As before, to a juggling quadruple $(d, \mathrm{x}, \mathrm{y}, \mathbf{a})$ we associate the following directed multigraph $G$ on the integers $\mathbb{Z}$. The vertex set of the graph is $\mathbb{Z}$ and the directed edge set is given by

$$E(G) = \{(x_i + k \cdot d, x_i + a_i + k \cdot d) \ : \ 1 \leq i \leq m, \ k \in \mathbb{Z}\}.$$

Let $\alpha_j$ be equal to the outdegree at vertex $j$ in the associated graph, and $\beta_j$ equal to the indegree at vertex $j$. That is, for $0 \leq j \leq d-1$, $\alpha_j$ is the cardinality of the set $\{i \ : \ x_i = j\}$, and similarly $\beta_j = |\{i \ : \ y_i = j\}|$. As for juggling triples, we define external and internal crossings in the same manner. We say that the weight of a juggling quadruple is equal to $q$ to the number of crossings.

Let $V$ be the linear space $\left\{(w_0, w_1, \ldots, w_{d-1}) \in \mathbb{R}^d \ : \ w_0 + w_1 + \cdots + w_{d-1} = 0\right\}$. Define the linear map $L : V \longrightarrow V$ by $L(\mathbf{e}_i - \mathbf{e}_{i-1}) = \mathbf{1}_d - d \cdot \mathbf{e}_i$ for $i = 0, 1, \ldots, d-1$ and expand the definition by linearity. For a vector $\mathbf{w} \in V$, define $E_i(\mathbf{w})$ to be the $i$th coordinate of the vector $\mathbf{w}$.

To make notation easier, for two vectors $\mathbf{n}$ and $\mathbf{k}$ we write

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} n_0 \\ k_0 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ k_1 \end{bmatrix} \cdots \begin{bmatrix} n_{d-1} \\ k_{d-1} \end{bmatrix},$$

where $\mathbf{n} = (n_0, n_1, \ldots, n_{d-1})$ and $\mathbf{k} = (k_0, k_1, \ldots, k_{d-1})$.

**Theorem 4** *The sum of the weight of juggling quadruples $(d, \mathrm{x}, \mathrm{y}, \mathbf{a})$ having period $d$, throw vector $\mathrm{x}$, catch vector $\mathrm{y}$, and $\sum_{i=1}^{d} a_i \leq N$, where $N \equiv \sum_{i=1}^{m} (y_i - x_i) \bmod d$, is equal to*

$$\begin{bmatrix} \frac{1}{d} \cdot (N \cdot \mathbf{1}_d + L(\alpha - \beta)) \\ \beta \end{bmatrix}.$$

In the case when $\mathrm{x} = \mathrm{y}$, Theorem 4 implies Theorem 2.

As for Theorem 1 there is a bijective proof of Theorem 4. The bijection is between the set of juggling quadruples having period $d$, throw vector $\mathrm{x}$, catch vector $\mathrm{y}$, and $\sum_{i=1}^{d} a_i \leq N$, and the set of lists of $d$ multisets $M_0, M_1, \ldots, M_{d-1}$, such that the entries of $M_j$ are integers between 0 and $E_j\left(\frac{1}{d} \cdot (N \cdot \mathbf{1}_d + L(\alpha - \beta))\right) + \beta_j$. The bijection is as follows. Let $\Psi(d, \mathrm{x}, \mathrm{y}, \mathbf{a}) = (\psi_1, \psi_2, \ldots, \psi_m)$, where

$$\psi_i = |\{(u, v) \in E(G) \ : \ x_i < u < x_i + a_i < v\}| + |\{j \ : \ 1 \leq j < i, \ x_i = x_j, \ a_i < a_j\}|.$$

Define the multiset $M_j$, where $0 \leq j \leq d-1$ by $M_j = \{\psi_i \ : \ a_i + x_i \equiv j \bmod d\}$. Observe that the cardinality of $M_j$ is $\beta_j$. Since we may view multisets as weakly increasing sequences and we have the following interpretation of the Gaussian coefficient:

$$\sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n-k} q^{i_1+i_2+\cdots+i_k} = \begin{bmatrix} n \\ k \end{bmatrix},$$

this completes the proof of Theorem 4.

## 6 $q$-Stirling numbers of the second kind

The $q$-Stirling numbers have been well-studied in the literature. See for example [4, 6, 7, 8, 12]. Let $\Pi_k[n]$ denote the set of all partitions of $\{1, 2, \ldots, n\}$ into $k$ blocks. For two integers $i$ and $j$ define the interval $int\,(i,j)$ to be the set

$$int\,(i,j) = \{n \in \mathbb{Z} \ : \ \min(i,j) < n < \max(i,j)\}.$$

Observe that the interval is symmetric in $i$ and $j$, that is, $int\,(i,j) = int\,(j,i)$

**Definition 4** *For two disjoint nonempty subsets $B$, $C$ of $\{1, 2, \ldots, n\}$, define the* intertwining number *$\iota(B, C)$ to be the cardinality of the set $\{(b, c) \in B \times C \ : \ int\,(b,c) \cap (B \cup C) = \emptyset\}$. The intertwining number is independent of order, that is, $\iota(B, C) = \iota(C, B)$. For a partition $\pi = \{B_1, B_2, \ldots, B_k\}$ of the set $\{1, 2, \ldots, n\}$ define the intertwining number $\iota(\pi)$ to be*

$$\iota(\pi) = \sum_{1 \leq i < j \leq k} \iota(B_i, B_j).$$

*Since the intertwining number of two blocks is independent of their order, the intertwining number of a partition does not depend upon how the blocks are ordered.*

As an example, consider the partition $\pi = \{\{1,3,6\}, \{2,4\}, \{5\}\}$ in Figure 4. The intertwining number of the two blocks $\{1,3,6\}$ and $\{2,4\}$ is 4, which is equal to the number of crossings between the solid line and the dashed line. Also the intertwining number of $\pi$ is equal to 7, which is the total number of crossings in Figure 4.

**Definition 5** *The q-Stirling numbers of the second kind, $S[n,k]$, are defined by*

$$S[n,k] = \sum_{\pi} q^{\iota(\pi)} \qquad (n \geq 1 \text{ and } k \geq 1),$$

*where the sum ranges over all partitions $\pi$ with $k$ blocks, that is, $\Pi_k[n]$. When $n = 0$ or $k = 0$, define $S[n,k] = \delta_{n,k}$.*

Observe that the intertwining number of two disjoint blocks is greater than or equal to 1. Hence the intertwining number of a partition $\pi$ is greater than or equal to $\binom{k}{2}$, where $k$ is the number of blocks of $\pi$. This implies the Stirling number $S[n,k]$ is divisible by $q^{\binom{k}{2}}$.

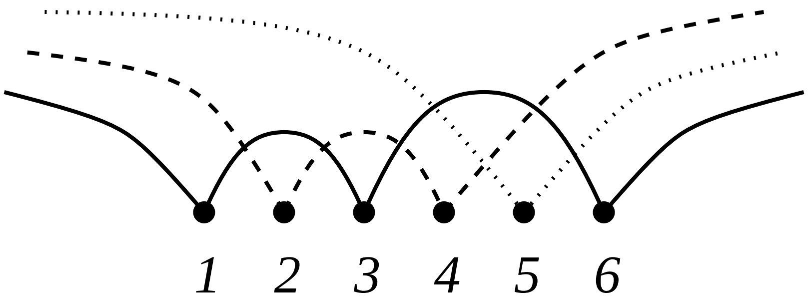


Figure 4: Computation of the intertwining number of the partition $\pi = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\}$. $\iota(\pi) = \iota(\{1, 3, 6\}, \{2, 4\}) + \iota(\{1, 3, 6\}, \{5\}) + \iota(\{2, 4\}, \{5\}) = 4 + 2 + 1 = 7$.

This definition is equivalent to the definition by Garsia and Remmel [4]. There is a natural bijection between partitions of $\{1, 2, \ldots, n\}$ into $k$ blocks, and rook placements of $n - k$ rooks on the triangular board of shape $(0, 1, \ldots, n-1)$. (This bijection is given after Corollary 2.4.2 in [11].) It is easy to see that the intertwining number of a partition is equal to the Garsia-Remmel statistic of the corresponding rook placement. Thus, our combinatorial approach to $q$-Stirling numbers of the second kind differs from that of Sagan [8] and Wachs and White [12].

By conditioning on which block of a partition of the set $\{1, 2, \ldots, n\}$ the element $n$ lies, we can easily derive the following recurrence:

**Lemma 2** *The $q$-Stirling numbers of the second kind satisfy*

$$S[n, k] = q^{k-1} \cdot S[n-1, k-1] + [k] \cdot S[n-1, k],$$

*where $n, k \geq 1$.*

The following identity is due to Carlitz [3]. It is a $q$-analogue of a well-known identity for Stirling numbers of the second kind. (See for example [11].) Milne proved this $q$-identity by using finite operator techniques on restricted growth functions [7]. See also de Médicis and Leroux [6] for a combinatorial proof.

**Theorem 5 (Carlitz [3])**

$$[n]^d = \sum_{m=0}^{d} S[d, m] \cdot [m]! \cdot \begin{bmatrix} n \\ m \end{bmatrix}.$$

**Proof:** The idea of the proof is to study simple juggling graphs of period $d$. We contract $d$ consecutive vertices of the graph to form a multiplex juggling graph of period 1. Carlitz's identity will follow by keeping track of what happens to the crossings in the graph under contraction.

Let $(d, \delta_d, \mathbf{a})$ be a simple juggling triple. Observe that $(d, \delta_d, \mathbf{a})$ does not have any internal crossings. By Theorem 1, we know that the sum of the weight of such juggling triples with at most $n$ balls is $[n]^d$. Contract the vertices $k \cdot d, k \cdot d + 1, \ldots, (k+1) \cdot d - 1$ of the associated graph $G$ into a new vertex $k$. We then obtain a graph associated with a juggling triple $(1, \mathbf{0}_m, \mathbf{c})$. Observe that $1 \le m \le d$, since some arcs will be contracted to arcs of length 0. Thus we remove them. Formally, this contraction is described by letting $b_i = \left\lfloor \frac{a_i+i-1}{d} \right\rfloor$, and removing the zero entries from the sequence $(b_1, b_2, \ldots, b_d)$ to produce the juggling sequence $\mathbf{c} = (c_1, c_2, \ldots, c_m)$. Note that $ball(d, \delta_d, \mathbf{a}) = ball(1, \mathbf{0}_m, \mathbf{c})$.

Observe that $m$ edge-disjoint paths partition the vertex set $\{0, 1, \ldots, d-1\}$ into $m$ disjoint blocks. Thus this is a partition $\pi$ with $m$ blocks. Moreover, the intertwining number of $\pi$ is the number of crossings that occur between time points 0 and $d-1$.

Now we see what happens to a crossing $(x, y)$, $(u, v)$ when the graph $G$ is contracted. Four cases occur. First, if the vertices $y$ and $u$ are contracted together, then the crossing is counted by the $q$-Stirling number $S[d, m]$. In the three remaining cases, we may assume that $y$ and $u$ are not contracted together. If none of the vertices $x$, $y$, $u$, and $v$ are contracted together, then the crossing remains an external crossing of $(1, \mathbf{0}_m, \mathbf{c})$, and thus is counted by $\left[ {n \atop m} \right]$. If $x$ and $u$ are contracted together, but not $y$ and $v$, then the crossing becomes an internal crossing of $(1, \mathbf{0}_m, \mathbf{c})$, and thus is counted by $\left[ {n \atop m} \right]$. Finally, if $y$ and $v$ are contracted together, then we may view this as an inversion of a permutation of $m$ elements. The weight of all such inversions is counted by the factor $[m]!$. $\square$

# References


[1] J. Buhler, D. Eisenbud, R. Graham, and C. Wright, Juggling Drops and Descents, *Amer. Math. Monthly* **101** (1994), 507–519.

[2] J. Buhler and R. Graham, A note on the binomial drop polynomial of a poset, *J. Combin. Theory Ser. A* **66** (1994), 321–326.

[3] L. Carlitz, $q$-Bernoulli numbers and polynomials, *Duke Math. J.* **15** (1948), 987–1000.

[4] A. M. Garsia and J. B. Remmel, $q$-Counting rook configurations and a formula of Frobenius, *J. Combin. Theory Ser. A* **41** (1986), 246–275.

[5] J. E. Humphreys, "Reflection Groups and Coxeter Groups," Cambridge University Press, 1990.

[6] A. de Médicis and P. Leroux, A unified combinatorial approach for $q$- (and $p, q$-) Stirling numbers, *J. Statistical Planning and Inference* **34** (1993), 89–105.

[7] S. C. Milne, A $q$-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials, *Trans. Amer. Math. Soc.* **245** (1978), 89–118.

[8] B. E. Sagan, A maj statistic for partitions, *European J. Combin.* **12** (1991), 69–79.

[9] J. Y. Shi, "The Kazhdan-Lusztig Cells in Certain Affine Weyl Groups," Lecture Notes in Math., Vol. 1179, Springer-Verlag, Berlin, 1986.

[10] R. Simion, Combinatorial statistics on non-crossing partitions, *J. Combin. Theory Ser. A* **66** (1994), 270–301.

[11] R. P. Stanley, "Enumerative Combinatorics, Vol. I," Wadsworth and Brooks/Cole, Pacific Grove, 1986.

[12] M. Wachs and D. White, $p, q$-Stirling numbers and set partition statistics, *J. Combin. Theory Ser. A* **56** (1991), 27–46.