Non-crossing two-rowed arrays and summations for Schur functions

(Summary)

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Abstract. In the first part of this paper (sections 1,2) we give combinatorial proofs for determinantal formulas for sums of Schur functions "in a strip" that were originally obtained by Gessel, respectively Goulden, using algebraic methods. The combinatorial analysis involves certain families of two-rowed arrays, asymmetric variations of Sagan and Stanley's skew Knuth-correspondence, and variations of one of Burge's correspondences. In the third section we specialize the parameters in these determinants to compute norm generating functions for tableaux in a strip. In case we can get rid of the determinant we obtain multifold summations that are basic hypergeometric series for A_r and C_r respectively. In some cases these sums can be evaluated. Thus in particular, an alternative proof for refinements of the Bender-Knuth and MacMahon (ex-)Conjectures, which were first obtained in another paper by the author, is provided. Although there are some parallels with the original proof, perhaps this proof is easier accessible. Finally, in section 4, we record further applications of our methods to the enumeration of paths with respect to weighted turns.

1. Generating functions for non-crossing two-rowed arrays. We consider two-rowed arrays $P = (p \mid q)$ of the form

$$\begin{matrix} p_{-a} & p_{-a+1} & \dots & p_{-1} & p_1 & \dots & p_k & & & & \\ & & & & q_1 & \dots & q_k & q_{-1} & \dots & q_{-b+1} & q_{-b} \end{matrix}, \tag{1.1}$$

where a, k, b are some nonnegative integers and where the entries p_i, q_i are positive integers such that both rows of the array are weakly increasing. (To be precise, if $k = 0$, i.e. the "middle part" of the array is empty, for a $c \le \min\{a, b\}$ we also allow the entries $p_{-1}, \dots, p_{-c}$ and $q_{-1}, \dots, q_{-c}$ to be "empty".) We say that P is *of the type* (a, b) and *of the shape* (a, k, b). If both rows of P are strictly increasing then we call P a *strict* two-rowed array. Given an array $P_1 = (p^{(1)} \mid q^{(1)})$ of the shape (a_1, k_1, b_1) and an array $P_2 = (p^{(2)} \mid q^{(2)})$ of the shape

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(a_2,k_2,b_2), we say that P_1 *dominates* (resp. *strictly dominates*) P_2 if the following three conditions hold:

(D1) $a_1 \le a_2$ and $p_l^{(1)} \le p_l^{(2)}$ (resp. $p_l^{(1)} < p_l^{(2)}$) for all $l = -1, -2, \dots, -\min\{a_1, a_2\}$. (By convention, these inequalities are also violated if $p_l^{(1)}$ should be an "empty" entry.)

(D2) $b_1 \le b_2$ and $q_l^{(1)} \ge q_l^{(2)}$ (resp. $q_l^{(1)} > q_l^{(2)}$) for all $l = -1, -2, \dots, -\min\{b_1, b_2\}$. (By convention, these inequalities are also violated if $q_l^{(1)}$ should be an "empty" entry.)

(D3) For each m, $1 \le m \le k_2$, there is an l, $1 \le l \le k_1$, such that $p_l^{(1)} \le p_m^{(2)}$ and $q_l^{(1)} \ge q_m^{(2)}$ (resp. $p_l^{(1)} < p_m^{(2)}$ and $q_l^{(1)} > q_m^{(2)}$).

Let $\alpha = (\alpha_r, \alpha_{r-1}, \dots, \alpha_1)$ and $\beta = (\beta_r, \beta_{r-1}, \dots, \beta_1)$ be r-tupels of nonnegative integers. A family $\mathcal{P} = (P_1, \dots, P_r)$ of two-rowed arrays is called *of the type* (α, β) if P_i is of the type (α_i, β_i), $i = 1, 2, \dots, r$. $\mathcal{P}$ is called *strict* if all P_i, $i = 1, \dots, r$, are strict. The family $\mathcal{P}$ is called *non-crossing* (resp. *strictly non-crossing*) if P_i dominates (resp. strictly dominates) P_{i+1}, $i = 1, \dots, r-1$.

Our objects of interest are non-crossing strict families of two-rowed arrays (see the example in section 2) and strictly non-crossing (ordinary) two-rowed arrays. If α and β are partitions, the most convenient way to look at a non-crossing strict family $\mathcal{P} = (P_1, \dots, P_r) = ((p^{(1)} \mid q^{(1)}), \dots, (p^{(r)} \mid q^{(r)}))$ of the type (α, β) is by rephrasing the three conditions (D1)–(D3) of dominance in terms of the following three properties:

(NS1) The *front part*, the array $(p_{-i}^{(r-j+1)})_{i,j \ge 1}$ is a column-strict plane partition of shape α'/ν.

(NS2) The *tail*, the array $(q_{-i}^{(r-j+1)})_{i,j \ge 1}$ is a tableau of shape β'/ν. ν is the same partition as in (NS1).

(NS3) The *middle part* of $\mathcal{P}$, interpreted as multiset $\{(p_j^{(i)}, q_j^{(i)}) : 1 \le i \le r,\ 1 \le j \le k_i\}$, can be viewed as a family of r pairwise non-crossing lattice paths in $\mathbb{Z}^2$ consisting of unit horizontal and vertical steps in the positive direction, all of which starting at $(0,0)$ and ending with an infinite horizontal part. (Thus the final points of the paths might be considered to be of the form (∞, y_i), for some nonnegative integers y_i.) This is seen as follows: For fixed i consider the points $(p_j^{(i)}, q_j^{(i)})$, $j = 1, \dots, k_i$. Now to these points apply Viennot's [31] light and shadow procedure (with the sun being located in the North-West). For each i this yields a lattice path of the described type. The condition (D3) (ordinary dominance) simply says that these r paths are pairwise non-crossing (from here we are lead to call the arrays under consideration "non-crossing"), the first path lying "avove" the second, the second "above" the third, etc.

A similar interpretation holds for strictly non-crossing (ordinary) families $\mathcal{P} = (P_1, \dots, P_r)$. This time the conditions (D1)–(D3) can be rephrased in the following way.

(SN1) The *front part*, the array $(p_{-j}^{(r-i+1)})_{i,j \ge 1}$ is a column-strict plane partition of shape α/ν. (ν is allowed to differ from 0 only if the middle part (see (SN3)) of $\mathcal{P}$ is empty.)

(SN2) The *tail*, the array $(q_{-j}^{(r-i+1)})_{i,j \ge 1}$ is a tableau of shape β/ν. ν is the same partition as in (SN1). (ν is allowed to differ from 0 only if the middle part (see (SN3)) of $\mathcal{P}$ is empty.)

(SN3) The *middle part* of $\mathcal{P}$, interpreted as multiset $\{(p_j^{(i)}, q_j^{(i)}) : 1 \le i \le r,\ 1 \le j \le k_i\}$, can be viewed as a family of r pairwise non-touching path-like objects, which were called *pathoids* by Kulkarni [21]. This is done in the same way as before. We omit the details.

Finally let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ be two infinite sequences of indeterminates. We define the *weight* $w_{\mathbf{x},\mathbf{y}}(P)$ of an array P of the form (1.1) to be the product $\prod x_\varepsilon \prod y_\eta$ where ε runs over all elements of the first row of P and η runs over all elements of the second row of P. The weight $w_{\mathbf{x},\mathbf{y}}(\mathcal{P})$ of a family $\mathcal{P} = (P_1, \dots, P_r)$ is defined to be the product over the weights of all arrays of $\mathcal{P}$, $\prod_{i=1}^r w_{\mathbf{x},\mathbf{y}}(P_i)$.

Now we are in the position to formulate the main theorems of this section.

Theorem 1. *Let α, β be partitions. The generating function $\sum w_{\mathbf{x},\mathbf{y}}(\mathcal{P})$ for non-crossing strict families $\mathcal{P}$ of the type (α, β) is*

$$\det_{1\le s,t\le r} (f_{\alpha_s+s-\beta_t-t}(\mathbf{x}, \mathbf{y})), \tag{1.2}$$

where $f_m(\mathbf{x}, \mathbf{y}) = \sum_k e_{k+m}(\mathbf{x})e_k(\mathbf{y})$ with $e_n(\mathbf{z})$ being the elementary symmetric function of order n in the variables $z_1, z_2, \dots$.

The generating function $\sum w_{\mathbf{x},\mathbf{x}}(\mathcal{P})$ for non-crossing strict families $\mathcal{P}$ of the type (α, β) with the additional property $p_j^{(i)} \ge q_j^{(i)}$, $i = 1, \dots, r$ and $j = 1, \dots, k_i$, is

$$\det_{1\le s,t\le r} (f_{\alpha_s+s-\beta_t-t}(\mathbf{x}, \mathbf{x}) - f_{\alpha_s+s+\beta_t+t}(\mathbf{x}, \mathbf{x})). \tag{1.3}$$

Theorem 2. *Let α, β be partitions. The generating function $\sum w_{\mathbf{x},\mathbf{y}}(\mathcal{P})$ for strictly non-crossing families $\mathcal{P}$ of the type (α, β) is*

$$\det_{1\le s,t\le r} (g_{\alpha_s+s-\beta_t-t}(\mathbf{x}, \mathbf{y})), \tag{1.4}$$

where $g_m(\mathbf{x}, \mathbf{y}) = \sum_k h_{k+m}(\mathbf{x})h_k(\mathbf{y})$ with $h_n(\mathbf{z})$ being the complete homogenous symmetric function of order n in the variables $z_1, z_2, \dots$.

The generating function $\sum w_{\mathbf{x},\mathbf{x}}(\mathcal{P})$ for strictly non-crossing families $\mathcal{P}$ of the type (α, β) with the additional property $p_j^{(i)} > q_j^{(i)}$, $i = 1, \dots, r$ and $j = 1, \dots, k_i$, is

$$\det_{1\le s,t\le r} (g_{\alpha_s+s-\beta_t-t}(\mathbf{x}, \mathbf{x}) - g_{\alpha_s+s+\beta_t+t}(\mathbf{x}, \mathbf{x})). \tag{1.5}$$

Sketch of Proof. We imitate the usual procedure with nonintersecting lattice paths ([12], see also [28, section 1]. For the proof of (1.2) we consider strict families $\mathcal{P} = (P_1, \dots, P_r)$ of two-rowed arrays where P_i is of the type $(a_{\sigma(i)} + \sigma(i) - i, \beta_i)$, $i = 1, 2, \dots, r$, for some permutation $\sigma \in \mathfrak{S}_r$. As with nonintersecting lattice paths we set up a weight-preserving involution on the crossing families of two-rowed arrays (where "crossing" means a violation against one of the conditions (D1)–(D3)), such that the corresponding permutations differ only by a transposition. After having observed that for $\sigma \ne \mathrm{id}$ there are no non-crossing strict families of the above type, the usual arguments lead to (1.2).

In order to prove (1.3) we consider strict families $\mathcal{P} = (P_1, \dots, P_r)$ of two-rowed arrays where P_i is of the type $(\eta_i(\alpha_{\sigma(i)} + \sigma(i)) - i, \beta_i)$, $i = 1, 2, \dots, r$, for some permutation $\sigma \in \mathfrak{S}_r$ and $\eta_i \in \{-1, 1\}$, $i = 1, \dots, r$. Also here we give a weight-preserving involution on the crossing families (where "crossings of the main diagonal", i.e. violations against $p_j^{(i)} \ge q_j^{(i)}$, have to be considered, too) such that either the corresponding permutations differ by a transposition and the corresponding η_i's are identical, or the corresponding permutations are identical and exactly one of the corresponding η_i's changes its sign.

The proofs for (1.4) and (1.5) are similar. Only at all places the roles of weak and strict order have to be exchanged.

The bijections of this section are inspired by ideas from [18]. $\square$

2. Summations for Schur functions "in a strip". In this section we relate Theorems 1 and 2 to summations for Schur functions. In the following given a partition μ the symbol μ' denotes the conjugate of μ, as usual. The number of odd parts of μ is denoted by oddrows(μ), the number of odd parts of μ' is denoted by oddcolumns(μ). (This terminology stems from visualizing partitions as Ferrer's boards.)

Theorem 3. *Let α, β be partitions.*

$$\sum_{\lambda,\lambda_1\le r} s_{\lambda/\beta'}(\mathbf{x})s_{\lambda/\alpha'}(\mathbf{y}) = \det_{1\le s,t\le r}\left(f_{\alpha_s+s-\beta_t-t}(\mathbf{x},\mathbf{y})\right) \tag{2.1}$$

$$\sum_{\lambda,\lambda_1\le c,\mathrm{oddrows}(\lambda)=p} s_\lambda(\mathbf{x}) = \begin{cases} \det\limits_{1\le s,t\le r}\big(f_{p\cdot\chi(s=r)+s-t}(\mathbf{x},\mathbf{x}) \\ \qquad\qquad -f_{p\cdot\chi(s=r)+s+t}(\mathbf{x},\mathbf{x})\big) & c=2r \\ e_p(\mathbf{x})\det\limits_{1\le s,t\le r}\left(f_{s-t}(\mathbf{x},\mathbf{x})-f_{s+t}(\mathbf{x},\mathbf{x})\right) & c=2r+1 \end{cases} \tag{2.2}$$

$$\sum_{\lambda,\ell(\lambda)\le r} s_{\lambda/\beta}(\mathbf{x})s_{\lambda/\alpha}(\mathbf{y}) = \det_{1\le s,t\le r}\left(g_{\alpha_s+s-\beta_t-t}(\mathbf{x},\mathbf{y})\right) \tag{2.3}$$

$$\sum_{\lambda,\ell(\lambda)\le c,\mathrm{oddcolumns}(\lambda)=p} s_\lambda(\mathbf{x}) = \begin{cases} \det\limits_{1\le s,t\le r}\big(g_{p\cdot\chi(s=r)+s-t}(\mathbf{x},\mathbf{x}) \\ \qquad\qquad -g_{p\cdot\chi(s=r)+s+t}(\mathbf{x},\mathbf{x})\big) & c=2r \\ h_p(\mathbf{x})\det\limits_{1\le s,t\le r}\left(g_{s-t}(\mathbf{x},\mathbf{x})-g_{s+t}(\mathbf{x},\mathbf{x})\right) & c=2r+1 \end{cases} \tag{2.4}$$

where χ is the usual truth function, $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise.

Remark. Identity (2.3) is due to Gessel [10, Theorem 16, cf. the paragraph just before Theorem 16], while (2.4) is due to Goulden [14, Theorems 2.4 and 2.6]. Clearly (2.1) and (2.2) follow from (2.3) and (2.4), respectively, by application of the homomorphism on symmetric functions that interchanges the roles of elementary and complete homogenous symmetric functions (cf. [23, pp. 14/15]). However, it is our goal to give *combinatorial* proofs for *all* of these identities.

Sketch of Proof. For a proof of (2.1) we set up a "part"-preserving bijection between non-crossing strict families $\mathcal{P}=(P_1,\dots,P_r)$ of the type (α,β) and pairs (π,τ) of a column-strict plane partition π of shape λ/β' and a tableaux τ of shape λ/α', with λ being a partition with $\lambda_1\le r$. (Of course "part"-preserving means that the multiset of the elements in $P_1,\dots,P_r$ is identically with the multiset of parts in π and τ.) This bijection is a variation of Sagan and Stanley's [27] skew Knuth correspondence. In the explanation of the bijection we refer to the Fomin–Roby [26, section 4.1; 8] description of the skew Knuth correpondence.

The bijection is best explained with a running example. Consider the family $\mathcal{P}^{(0)}=(P_1^{(0)},P_2^{(0)},P_3^{(0)})$ where

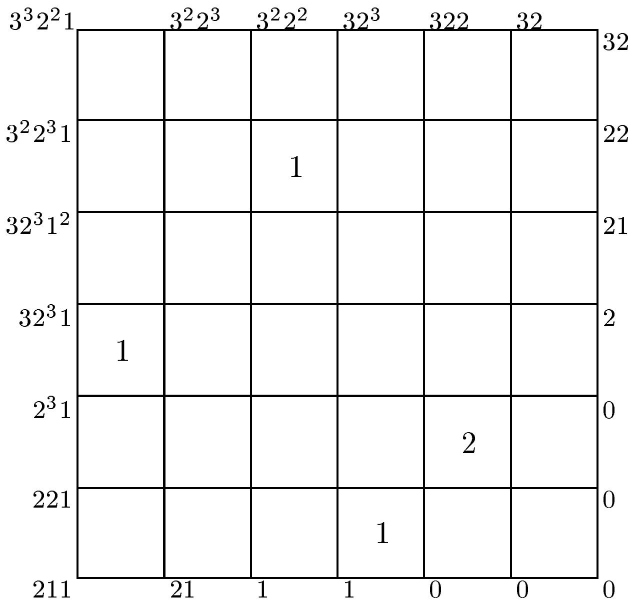
$$P_1^{(0)}=\begin{array}{lll}1&3&\\3&5&6\end{array},\quad P_2^{(0)}=\begin{array}{llll}2&4&5&\\&1&2&3\ 5\end{array},\quad P_3^{(0)}=\begin{array}{llll}1&2&4&5\\&&2&3\ 4\end{array}.$$

It is a non-crossing strict family of the type $(\alpha^{(0)},\beta^{(0)})=((3,1,0),(2,2,1))$. Now, given a non-crossing strict family interpret the front part as column-strict plane partition $\tilde\pi$ of shape

α'/ν and the tail as tableau $\tilde\tau$ of shape β'/ν as explained in (NS1) and (NS2), respectively, where ν is some partition. In our example the front part $\tilde\pi^{(0)}$ and the tail $\tilde\tau^{(0)}$ are as follows,

$$\tilde\pi^{(0)} = \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \quad \text{and} \quad \tilde\tau^{(0)} = \begin{array}{|c|c|c|}\hline 3 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}.$$

Next write $\tilde\pi$ and $\tilde\tau$ as multichains in Young's lattice, i.e. as sequences of partitions, the length of the sequences being determined by $m+1$, where m is the largest element in $\mathcal{P}$. For the tableau $\tilde\tau$ this is standard (cf. [23, pp. 4/5; 26, pp. 12/13]), for the column-strict plane partition $\tilde\pi$ the i-th partition in the sequence, i running from 0 to m, corresponds to the shape of the column-strict plane partition that results from $\tilde\pi$ by deleting the numbers $\{1,2,\dots,m-i\}$ in $\tilde\pi$. So in our example the sequence for $\tilde\pi^{(0)}$ is $0,0,0,1,1,21,211$, while the sequence for $\tilde\tau^{(0)}$ is $0,0,0,2,21,22,32$. (Here we use a short notation for partitions. For instance, 211 is short for $(2,1,1)$, 0 is short for the empty partition.)



$$\pi^{(0)} = \begin{array}{|c|c|c|}\hline & & \\ \hline & & 3 \\ \hline 5 & 5 & 1 \\ \hline 4 & 4 \\ \cline{1-2} 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}, \qquad \tau^{(0)} = \begin{array}{|c|c|c|}\hline & & 3 \\ \hline & 1 & 5 \\ \hline & 2 & 6 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

Figure 1

Now we are able to fill the appropriate Fomin–Roby picture. The lower border corresponds to $\tilde\pi$, the right border to $\tilde\tau$, the (s,t)-entry in the matrix corresponds to the number of pairs $(p_j^{(i)}, q_j^{(i)})$ in the middle part of $\mathcal{P}$ that equal (s,t). The left-hand diagram in Figure 1 shows the Fomin–Roby picture corresponding to our running example. The subdivision of rows and columns that contain "multiple entries" is done in direction South-East, (and not in direction North-East as in [26, section 4.1]). Figure 1a contains the "subdivided" Fomin–Roby picture that corresponds to the diagram in Figure 1. Also, in the diagram we are working upwards and to the left (and not upwards and to the right as in [26, section 4.1]). The rules for the algorithm are the same as in [26, Example 2.6.3]. From the upper border, in the same way as explained above, we read a column-strict plane partition π of shape λ/β', from the left border a tableau τ of shape λ/α', where λ is some partition. Examining the properties of this mapping it is not too difficult to see that thus we indeed obtain a part-preserving bijection between non-crossing strict families $\mathcal{P} = (P_1,\dots,P_r)$ of the type

(α,β) and pairs (π,τ) where π is a column-strict plane partition of shape λ/β' and τ is a tableaux of shape λ/α', with λ being a partition with $\lambda_1 \le r$. The pair $(\pi^{(0)},\tau^{(0)})$ resulting from our example is exhibited in Figure 1.

It is well-known that the skew Schur function $s_{\lambda/\beta'}$ can be either defined as generating function for tableaux of shape λ/β' or as generating function for column-strict plane partitions of shape λ/β'. (There is also a bijection basing on jeu de taquin which settles this equivalence of definitions for the Schur function.) Thus the above bijection by (1.2) proves (2.1).

$3^3 2^2 1$	$3^2 2^3 1$	$3^2 2^3$	$3^2 2^2 1$	$3^2 2^2$	32^3	$32^2 1$	32^2	321	32	32
$3^2 2^3 1$	$32^4 1$	32^4	$32^3 1$	32^3 ×	2^4	$2^3 1$	2^3	$2^2 1$	22	22
$32^4 1$	$2^5 1$	2^5	$2^4 1$	2^4	2^4	$2^3 1$	2^3	$2^2 1$	22	22
$32^3 1^2$	$2^4 1^2$	$2^4 1$	$2^3 1^2$	$2^3 1$	$2^3 1$	$2^2 1^2$	$2^2 1$	21^2	21	21
$32^3 1$ ×	$2^4 1$	2^4	$2^3 1$	2^3	2^3	$2^2 1$	22	21	2	2
$2^4 1$	$2^4 1$	2^4	$2^3 1$	2^3	2^3	$2^2 1$	22	21	2	2
$2^3 1^2$	$2^3 1^2$	$2^3 1$	$2^2 1^2$	$2^2 1$	$2^2 1$	21^2	21	11	1	1
$2^3 1$	$2^3 1$	2^3	$2^2 1$	22	22	21	2 ×	1	0	0
$2^2 1^2$	$2^2 1^2$	$2^2 1$	21^2	21	21	11	1	1 ×	0	0
$2^2 1$	$2^2 1$	22	21	2	2 ×	1	0	0	0	0
211	211	21	11	1	1	1	0	0	0	0

Figure 1a

For the $c=2r$ case of (2.2) we use a bijection due to Choi and Gouyou–Beauchamps [6, proof of Théorème 3] (cf. [18, Proposition 32] for a detailed description) between tableaux with p odd rows and with at most $2r$ columns, and pairs $(\mathcal{P},S)$, where $\mathcal{P}=(P_1,\dots,P_r)$ is a non-crossing strict family of r two-rowed arrays of the shape $(0,0)$, and S is a p-subset of $\{1,2,\dots,h-1\}$ where h is the smallest element of the first row of P_r. (Also here 0 denotes the empty partition. But in this context of course it means that it is coded by the r-tuple $(0,0,\dots,0)$.) Clearly S can be put at the beginning of the first row of P_r thus forming an array of the type $(p,0)$. Thus one obtains a bijection between tableaux with p odd rows and at most $2r$ columns and non-crossing strict families of the shape $((p,0,\dots,0),0)$. Use of (1.3) with this shape establishes the $c = 2r$ case of (2.2).

For the $c = 2r + 1$ case of (2.2) we use a bijection (cf. [18, proof of Theorem 21, last paragraph]) between tableaux with p odd rows and with at most $2r + 1$ columns and pairs $(\mathcal{P}, S)$ where $\mathcal{P}$ is a non-crossing strict family of r two-rowed arrays of the shape $(0, 0)$ and S is a p-subset of the positive integers. By (1.3) with $\alpha = \beta = 0$ this yields the $c = 2r + 1$ case of (2.2), where S produces $h_p(\mathbf{x})$ and $\mathcal{P}$ produces the determinant.

The bijections for establishing (2.2) base on one of Burge's [5, p. 22] variations of the Knuth correspondences [17].

The arguments for proving (2.3) and (2.4) are similar. $\square$

Also (1.3) and (1.5), for generic α, β, have interesting interpretations in terms of tableaux generating functions. For example, we can prove the following theorem.

Theorem 4. *The generating function $\sum_\tau w(\tau)$ over all oscillating semistandard tableaux $\tau = (\tau^{(i)})$, $\tau : \beta \to \alpha$ (cf. [30, 11, 26]), with at most r rows equals (1.5). The weight $w(\tau)$ is defined by $\prod x_i^{|\tau^{(2i-1)} - \tau^{(2i)}| + |\tau^{(2i-1)} - \tau^{(2i-2)}|}$.*

3. Norm generating functions for tableaux. Given a tableau τ we define the *norm*, $n(\tau)$, of τ to be the sum of all the entries of τ. In this section we list several results for norm generating functions that are obtained from (2.1)–(2.4) by specializing the indeterminates $\mathbf{x}$ and $\mathbf{y}$. Clearly, whenever we set $x_i = q^{M_i}$ and $y_i = q^{N_i}$ in (2.1)–(2.4) we obtain a determinant for the norm generating function for some family of tableaux (resp. pairs of tableaux). However, the cases of interest are only those where the determinant simplifies. For the simplification of the determinants we use special cases of the following lemma from [18, Lemma 34].

Lemma 5. *Let $X_1, X_2, \dots, X_r, A_2, A_3, \dots, A_r, C$ be indeterminates. If $p_0, p_1, \dots, p_{r-1}$ are Laurent polynomials with $\deg p_j \le j$ and $p_j(C/X) = p_j(X)$ for $j = 0, 1, \dots, r-1$, then*

$$\det_{1\le s,t\le r} \big((A_r + X_s)\cdots(A_{t+1} + X_s)(A_r + C/X_s)\cdots(A_{t+1} + C/X_s)\cdot p_{t-1}(X_s)\big)$$

$$= \prod_{1\le i<j\le r} (X_i - X_j)(1 - C/X_iX_j) \prod_{i=1}^{r} A_i^{i-1} \prod_{i=1}^{r} p_{i-1}(-A_i)\ , \qquad (3.1)$$

with the convention that empty products (like $(A_r + X_t)\cdots(A_{s+1} + X_t)$ for $s = r$) are equal to 1. (The indeterminate A_1, which occurs at the right-hand side of (5.5.1), in fact is superflous since it occurs in the argument of a constant polynomial.) A Laurent polynomial is a series $p(X) = \sum_{i=M}^{N} a_i x^i$, $M, N \in \mathbb{Z}$, $a_i \in \mathbb{R}$. Provided $a_N \ne 0$ the degree of p is defined by $\deg p := N$. $\square$

By this lemma we arrive at multifold sums. They are basically basic hypergeometric series for A_r respectively C_r. Sometimes even these series can be evaluated. These computations are very similar to those in [18].

From (2.1) we derive the following results.

Theorem 6. *Let α be a fixed partition. The generating function $\sum q^{n(\tau_1)+\tau_2}$ for pairs (τ_1, τ_2) of tableaux where τ_1 is of shape λ and τ_2 is of shape λ/α' with λ being some partition with $\lambda_1 \le r$, and where the parts of τ_1 are $\equiv m_1 \pmod a$ and between m_1 and $m_1 + (M_1 - 1)a$, and where the parts of τ_2 are $\equiv m_2 \pmod a$ and between m_2 and*

$m_2+(M_2-1)a$, *is*

$$\sum_{\lambda,\lambda_1\le r} s_\lambda(q^{m_1},q^{m_1+a},\dots,q^{m_1+(M_1-1)a})s_{\lambda/\alpha'}(q^{m_2},q^{m_2+a},\dots,q^{m_2+(M_2-1)a})=$$

$$=q^{a\sum_{i=1}^r\left(\binom{\alpha_i}{2}+m_1\alpha_i/a\right)}\sum_{k_1,\dots,k_r\ge0}\prod_{i=1}^r q^{ak_i(k_i+\alpha_i+\mu-1)}\begin{bmatrix}-\mu+1\\k_i\end{bmatrix}_{q^a}$$
$$\frac{(q^{a(M_2+\mu+\alpha_i+k_i+i-1)};q^a)_{M_1-\alpha_i-k_i}}{(q^a;q^a)_{M_1+r-\alpha_i-k_i-i}}\prod_{1\le i<j\le r}[\alpha_j+k_j+j-\alpha_i-k_i-i]_{q^a},\tag{3.2}$$

where $\mu:=(m_1+m_2)/a$, $(A;q)_n:=(A;q)_\infty/(Aq^n;q)_\infty$ *with* $(A;q)_\infty=\prod_{i=0}^\infty(1-Aq^i)$, *and* $[n]_q:=(1-q^n)$, *by definition. An alternative expression for the right-hand side of (3.2) is*

$$q^{a\sum_{i=1}^r\left(\binom{\alpha_i}{2}+m_1\alpha_i/a\right)}\sum_{k_1,\dots,k_r\ge0}\prod_{i=1}^r q^{ak_i(k_i+\alpha_i+\mu-1)}\begin{bmatrix}M_2\\k_i\end{bmatrix}_{q^a}$$
$$\frac{[M_1+i-1]_{q^a}!}{[M_1+r-\alpha_i-k_i-i]_{q^a}!\,[\alpha_i+k_i+i-1]_{q^a}!}\prod_{1\le i<j\le r}[\alpha_j+k_j+j-\alpha_i-k_i-i]_{q^a}\ .\tag{3.3}$$

If $m_1+m_2=a$ *the identities (3.2) (respectively (3.3)) reduce to*

$$\sum_{\lambda,\lambda_1\le r} s_\lambda(q^{m_1},q^{m_1+a},\dots,q^{m_1+(M_1-1)a})s_{\lambda/\alpha'}(q^{a-m_1},q^{2a-m_1},\dots,q^{M_2a-m_1})$$
$$=q^{a\sum_{i=1}^r\left(\binom{\alpha_i}{2}+m_1\alpha_i/a\right)}\prod_{i=1}^r\frac{[M_1+M_2+i-1]_{q^a}!}{[M_2+\alpha_i+i-1]_{q^a}!\,[M_1+r-\alpha_i-i]_{q^a}!}\prod_{1\le i<j\le r}[\alpha_j+j-\alpha_i-i]_{q^a}.\tag{3.4}$$

In particular, for $\alpha=0$ *this can be written as*

$$\sum_{\lambda,\lambda_1\le r} s_\lambda(q^{m_1},q^{m_1+a},\dots,q^{m_1+(M_1-1)a})s_\lambda(q^{a-m_1},q^{2a-m_1},\dots,q^{M_2a-m_1})$$
$$=\prod_{1\le i\le M_1,1\le j\le M_2}\frac{[r+i+j-1]_{q^a}}{[i+j-1]_{q^a}}.\tag{3.5}$$

Remark. The special cases $a=1$, $m_1=m_2=1$, $M_1=M_2$ of (3.2) and $a=2$, $m_1=1$, $M_1=M_2$ of (3.5) first appeared in [18, Theorems 20, 9].

Sketch of Proof. In (2.1) set $x_i=q^{m_1+(i-1)a}$ for $i=1,\dots,M_1$, $x_i=0$ for $i>M_1$, and $y_i=q^{m_2+(i-1)a}$ for $i=1,\dots,M_2$, $y_i=0$ for $i>M_2$. With these specializations the elementary symmetric functions reduce to q-binomial coefficients times some power of q (cf. [23, p. 19, Ex. 3]). Next to each of the entries (which are q-binomial summations) of the determinant one of Heine's ${}_2\phi_1$-transformations [9, Appendix (III.2)] is applied. In the resulting determinant we use the linearity in the rows to take out the summations, thus arriving at a multifold sum of determinants. The determinants are evaluated by taking some factors out of the determinants and then applying Lemma 5 with $C\to0$, $X_s=-q^{-a(k_s+\alpha_s+s)}$, $A_t=q^{-a(M_1+t)}$, $p_{i-1}(X)=\prod_{j=2}^i(q^{a(M_2+\mu-j)}+X)$. This gives (3.2) after some simplification. The expression in (3.3) results from the following A_r analogue of one of Heine's ${}_2\phi_1$-transformations [9, Appendix (III.2)], newly discovered by Gustafson [16],

$$\sum_{k_1,\dots,k_r\ge0}\left(\prod_{i=1}^r q^{k_i(1-i)}Z^{k_i}\frac{(A)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(CX_i)_{k_i}}\right)\prod_{1\le i<j\le r}\frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}=$$

$$= \prod_{i=1}^{r} \frac{(Cq^{i-r}/B)_\infty\,(BZX_i)_\infty}{(Zq^{i-r})_\infty\,(CX_i)_\infty}$$

$$\sum_{k_1,\dots,k_r\ge 0} \left(\prod_{i=1}^{r} q^{k_i(1-i)} \left(\frac{C}{B}\right)^{k_i} \frac{(ABZ/C)_{k_i}\,(BX_i)_{k_i}}{(q)_{k_i}\,(BZX_i)_{k_i}}\right) \prod_{1\le i<j\le r} \frac{1-\frac{X_j}{X_i}q^{k_j-k_i}}{1-\frac{X_j}{X_i}}.$$

If $m_1+m_2=a$, i.e. $\mu=1$, due the q-binomial coefficient the series in (3.2) only consists of the term for $k_1=\dots=k_r=0$. This immediately gives (3.4). For $\alpha=0$ the product can be written in the form (3.5). $\square$

Now we turn to implications of (2.2).

Theorem 7. *The generating function $\sum q^{n(\tau)}$ for tableaux τ with p odd rows, with at most c columns, and with parts $\equiv m \pmod a$ and between m and $m+(M-1)a$, for $c=2r$ is*

$$\sum_{\lambda,\lambda_1\le 2r,\mathrm{oddrows}(\lambda)=p} s_\lambda(q^m,q^{m+a},\dots,q^{m+(M-1)a})$$

$$= q^{a\binom{p}{2}+mp} \sum_{k_1,\dots,k_r\ge 0} \prod_{i=1}^{r} q^{ak_i(k_i-1+p\cdot\chi(i=r)+\mu)} \begin{bmatrix} -\mu+1 \\ k_i \end{bmatrix}_{q^a}$$

$$\frac{(q^{a(M+r+p\cdot\chi(i=r)+k_i+i+\mu)};q^a)_{M-r-p\cdot\chi(i=r)-k_i+i-1}}{(q^a;q^a)_{M+r-p\cdot\chi(i=r)-k_i-i}}$$

$$\prod_{1\le i<j\le r} [p\cdot\chi(j=r)+k_j+j-p\cdot\chi(i=r)-k_i-i]_{q^a}$$

$$\prod_{1\le i\le j\le r} [p\cdot\chi(j=r)+k_j+j+p\cdot\chi(i=r)+k_i+i+\mu-1]_{q^a}, \qquad (3.6a)$$

where $\mu=2m/a$, and for $c=2r+1$

$$\sum_{\lambda,\lambda_1\le 2r+1,\mathrm{oddrows}(\lambda)=p} s_\lambda(q^m,q^{m+a},\dots,q^{m+(M-1)a})$$

$$= q^{a\binom{p}{2}+mp} \begin{bmatrix} M \\ p \end{bmatrix}_{q^a} \sum_{k_1,\dots,k_r\ge 0} \prod_{i=1}^{r} q^{ak_i(k_i-1+\mu)} \begin{bmatrix} -\mu+1 \\ k_i \end{bmatrix}_{q^a} \frac{(q^{a(M+r+k_i+i+\mu)};q^a)_{M-r-k_i+i-1}}{(q^a;q^a)_{M+r-k_i-i}}$$

$$\prod_{1\le i<j\le r} [k_j+j-k_i-i]_{q^a} \prod_{1\le i\le j\le r} [k_j+j+k_i+i+\mu-1]_{q^a}, \qquad (3.6b)$$

In case $2m=a$ the identities (3.6a,b) reduce to

$$\sum_{\lambda,\lambda_1\le 2r,\mathrm{oddrows}(\lambda)=p} s_\lambda(q,q^3,\dots,q^{2M-1}) = q^{p^2} \frac{[2r+2p]_{q^2}\,[r]_{q^2}}{[2r+p]_{q^2}\,[r+p]_{q^2}} \begin{bmatrix} M \\ p \end{bmatrix}_{q^2} \frac{\begin{bmatrix} M+2r \\ M \end{bmatrix}_{q^2}}{\begin{bmatrix} M+2r+p \\ M \end{bmatrix}_{q^2}}$$

$$\times \prod_{i=1}^{M} \frac{[r+i]_{q^2}}{[i]_{q^2}} \prod_{1\le i<j\le M} \frac{[2r+i+j]_{q^2}}{[i+j]_{q^2}} \qquad (3.7a)$$

and

$$\sum_{\lambda,\lambda_1\le 2r+1,\text{oddrows}(\lambda)=p} s_\lambda(q,q^3,\dots,q^{2M-1}) = q^{p^2}\begin{bmatrix} M\\ p\end{bmatrix}_{q^2}\prod_{i=1}^{M}\frac{[r+i]_{q^2}}{[i]_{q^2}}\prod_{1\le i<j\le M}\frac{[2r+i+j]_{q^2}}{[i+j]_{q^2}} \tag{3.7b}$$

where without loss of generality we let $a=2$.

In case $m=a$ *the identities (3.6a,b) simplify to*

$$\sum_{\lambda,\lambda_1\le 2r,\text{oddrows}(\lambda)=p} s_\lambda(q,q^2,\dots,q^{M})$$

$$= q^{\binom{p+1}{2}}\frac{[2r]_q}{[2r+p]_q}\begin{bmatrix} M\\ p\end{bmatrix}_q \frac{\begin{bmatrix} M+2r\\ M\end{bmatrix}_q}{\begin{bmatrix} M+2r+p\\ M\end{bmatrix}_q}\prod_{1\le i\le j\le M}\frac{[2r+i+j]_q}{[i+j]_q} \tag{3.8a}$$

and

$$\sum_{\lambda,\lambda_1\le 2r+1,\text{oddrows}(\lambda)=p} s_\lambda(q,q^2,\dots,q^{M}) = q^{\binom{p+1}{2}}\begin{bmatrix} M\\ p\end{bmatrix}_q\prod_{1\le i\le j\le M}\frac{[2r+i+j]_q}{[i+j]_q}, \tag{3.8b}$$

where without loss of generality we let $a=1$.

Sketch of Proof. To prove (3.6a) we do the same as before in the proof of (3.2). The only difference is that now Lemma 5 with $C=1$, $X_s=q^{a(p\cdot\chi(s=r)+k_s+s+\mu/2-1/2)}$, $A_t=-q^{a(-M-\mu/2+1/2-t)}$, $p_{i-1}(X) := \Big(\prod_{j=1}^{i}(1/A_j+1/X)\prod_{j=-i+2}^{0}(A_j+X)-\prod_{j=1}^{i}(1/A_j+X)\prod_{j=-i+2}^{0}(A_j+1/X)\Big)\Big/\left(X-\frac{1}{X}\right)$ has to be applied. Obviously, the $p=0$ case of (3.6a) immediately yields (3.6b). For $m=2a$, i.e. $\mu=2$, due to the q-binomial coefficient the sums in (3.6a,b) reduce to the respective terms for $k_1=\dots=k_r=0$. This leads to (3.7a,b). For $m=a$, i.e. $\mu=1$, the sums in (3.6a,b) can be evaluated by a special case of Gustafson's [15, Theorem 5.1] C_r ${}_6\psi_6$ summation (see also [22; 18, subsection 5.6]). After some simplification one arrives at (3.8a,b). $\square$

Remark. Identity (3.7) is a refinement of the MacMahon (ex-)Conjecture which was first proved in another paper of the author [18, Theorem 11]. (The MacMahon Conjecture was proved by Andrews [4], Macdonald [23, pp. 51/52, Ex. 16,17], and Proctor [25, Proposition 7.3].) Likewise, identity (3.8) is a refinement of the Bender–Knuth (ex-)Conjecture which the author also first proved in the same paper [18, Theorem 21]. (The Bender–Knuth Conjecture was proved by Andrews [3], Gordon [13], Macdonald [23, pp. 51–53, Ex. 16,18], and Proctor [25, Proposition 7.2].) As was remarked in [18] by summing the expressions in (3.7) respectively (3.8) with respect to p we get new proofs of the MacMahon respectively the Bender–Knuth Conjecture. We formulate the MacMahon Conjecture in an equivalent form. The original formulation is in terms of symmetric plane partitions.

Corollary 8. *The generating function for tableaux with at most* c *columns, and with only odd parts which lie between* 1 *and* $2M-1$, *is given by*

$$\sum_{\lambda,\lambda_1\le c} s_\lambda(q,q^3,\dots,q^{2M-1}) = \prod_{i=1}^{M}\frac{[c+2i-1]_q}{[2i-1]_q}\prod_{1\le i<j\le M}\frac{[c+i+j-1]_{q^2}}{[i+j-1]_{q^2}}\,. \tag{MacMahon Conjecture}$$

The generating function for tableaux with at most c columns, and with parts which lie between 1 and M, is given by

$$\prod_{1\le i\le j\le M} \frac{[c+i+j-1]_q}{[i+j-1]_q}\,. \qquad \text{(Bender–Knuth Conjecture)}$$

Sketch of Proof. In order to obtain the MacMahon Conjecture the sum over all p of the expressions in (3.7a) is evaluated by a special case of the very well-poised ${}_6\phi_5$-summation (cf. [9, Appendix (II.21)]) while the sum over all p of the expressions in (3.7b) is evaluated by the q-binomial summation (cf. [2, (3.3.7)]). Similarly, in order to obtain the Bender–Knuth Conjecture the sum over all p of the expressions in (3.8a) is evaluated by the q-Kummer summation (cf. [9, Appendix (II.9)]), while the sum over all p of the expressions in (3.8b) is again evaluated by the q-binomial summation. $\square$

Finally we turn to the implications of (2.3) and (2.4). In the same special cases as before we can get rid of the determinants thus arriving at A_r- respectively C_r-type sums. But unfortunately here it does not seem to be possible to find any nontrivial cases in which these sums can be evaluated to result into closed forms. But it should be noted that these sums are *finite*.

Theorem 9. *Let α be a fixed partition. The generating function $\sum q^{n(\tau_1)+\tau_2}$ for pairs (τ_1,τ_2) of tableaux where τ_1 is of shape λ and τ_2 is of shape λ/α with λ being some partition with $\ell(\lambda)\le r$, and where the parts of τ_1 are $\equiv m_1 \pmod a$ and between m_1 and $m_1+(M_1-1)a$, and where the parts of τ_2 are $\equiv m_2 \pmod a$ and between m_2 and $m_2+(M_2-1)a$, for $M_1\ge r-\alpha_1$ is*

$$\sum_{\lambda,\ell(\lambda)\le r} s_\lambda(q^{m_1},q^{m_1+a},\dots,q^{m_1+(M_1-1)a})s_{\lambda/\alpha}(q^{m_2},q^{m_2+a},\dots,q^{m_2+(M_2-1)a})$$
$$= \frac{q^{m_1\sum_{i=1}^r\alpha_i}}{(q^a;q^a)_{M_1-1}^r\,(q^{m_1+m_2};q^a)_{M_2}^r}$$
$$\sum_{k_1,\dots,k_r\ge 0}\prod_{i=1}^r q^{ak_i(M_1+\alpha_i)}\frac{(q^{a(1-M_1)};q^a)_{k_i}\,(q^{m_1+m_2};q^a)_{k_i}}{(q^a;q^a)_{k_i}\,(q^{aM_2+m_1+m_2};q^a)_{k_i}}\prod_{1\le i<j\le r}(1-q^{a(k_j-k_i)}). \tag{3.9}$$

Remark. The restriction $M_1\ge r-\alpha_1$ actually is not a serious restriction. Obviously the tableaux τ_1 have at most M_1 rows. Therefore, if $M_1<r$ we may replace r by M_1 and then apply (3.9).

Sketch of Proof. We proceed in analogy with the proof of Theorem 6. In (2.3) set $x_i=q^{m_1+(i-1)a}$ for $i=1,\dots,M_1$, $x_i=0$ for $i>M_1$, and $y_i=q^{m_2+(i-1)a}$ for $i=1,\dots,M_2$, $y_i=0$ for $i>M_2$. With these specializations the complete homogenous symmetric functions reduce to q-binomial coefficients times some power of q (cf. [23, p. 19, Ex. 3]). This time a limiting case of another one of Heine's ${}_2\phi_1$-transformations [9, Appendix (III.1)] is applied. In the resulting determinant we use the linearity in the rows to take out the summations thus arriving at a multifold sum of determinants. Upon taking some factors out of these determinants, they reduce to Vandermonde determinants and are therefore easily evaluated. $\square$

Theorem 10. *The generating function $\sum q^{n(\tau)}$ for tableaux τ with p odd columns, with at most c rows, and with parts $\equiv m \pmod a$ and between m and $m+(M-1)a$, for $c=2r$ and $M \ge r$ is*

$$\sum_{\lambda,\ell(\lambda)\le 2r,\text{oddcolumns}(\lambda)=p} s_\lambda(q^m, q^{m+a}, \dots, q^{m+(M-1)a})$$
$$= \frac{q^{mp}}{(q^a;q^a)_{M-1}^r\,(q^{2m};q^a)_M^r} \sum_{k_1,\dots,k_r\ge 0} \prod_{i=1}^r q^{ak_i(M+p\cdot\chi(i=r))} \frac{(q^{a(1-M)};q^a)_{k_i}\,(q^{2m};q^a)_{k_i}}{(q^a;q^a)_{k_i}\,(q^{aM+2m})_{k_i}}$$
$$\times \prod_{1\le i<j\le r}(1-q^{a(k_j-k_i)}) \prod_{1\le i\le j\le r}(1-q^{a(k_i+k_j)+2m}), \quad (3.10a)$$

and for $c=2r+1$ and $M\ge r$

$$\sum_{\lambda,\ell(\lambda)\le 2r+1,\text{oddcolumns}(\lambda)=p} s_\lambda(q^m, q^{m+a}, \dots, q^{m+(M-1)a})$$
$$= \frac{q^{mp}}{(q^a;q^a)_{M-1}^r\,(q^{2m};q^a)_M^r} \begin{bmatrix} M+p-1 \\ p \end{bmatrix}_{q^a} \sum_{k_1,\dots,k_r\ge 0} \prod_{i=1}^r q^{ak_iM} \frac{(q^{a(1-M)};q^a)_{k_i}\,(q^{2m};q^a)_{k_i}}{(q^a;q^a)_{k_i}\,(q^{aM+2m})_{k_i}}$$
$$\times \prod_{1\le i<j\le r}(1-q^{a(k_j-k_i)}) \prod_{1\le i\le j\le r}(1-q^{a(k_i+k_j)+2m}). \quad (3.11a)$$

Remark. Again the restriction $M\ge r$ is not serious since if $M<r$ we may replace r by $\lceil \frac{M}{2}\rceil$.

Sketch of Proof. We do the same as before in the proof of Theorem 9. Here the determinants which have to be computed are

$$\det_{1\le s,t\le r}(q^{-at(m+k_s)} - q^{at(m+k_s)}).$$

This is done by using Lemma 5 with $C=1$, $X_s = q^{-a(m+k_s)}$, $A_t\to 0$, $p_{i-1}(X) = (X^i - 1/X^i)/(X-1/X)$ (see also [24, p. 10, (I)]). $\square$

4. "Flagged" non-crossing two-rowed arrays and enumeration of nonintersecting lattice paths with given number of turns. Theorems 1 and 2 can be generalized to "flagged" families of non-crossing two-rowed arrays by using the same proof. By "flagged" we mean that the entries in the i-th two-rowed array P_i of a family $\mathcal{P}$ are bounded by different lower and upper bounds. (This resembles the definition for flagged Schur functions (cf. e.g. [32]) in terms of tableaux with different lower and upper bounds on the entries in each row of the tableaux.) We do not have the space to write down the results. Instead, we record three interesting applications of these results in the enumeration of nonintersecting lattice paths with a given number of turns. These results are related to the computation of Hilbert polynomials of determinantal ideals (cf. [20, 7]). Similar results can be derived for the path-like objects that were called pathoids in [21]. For the interested reader we remark that enumeration results for pairs of nonintersecting lattice paths with a given number of EN-corners (see the definition below) of the "upper" path and a given number of NE-corners (see the definition below) of the "lower" path are given in [29, 19].

As in [18] a point X of a lattice path is called *NE-corner*, if X is the end point of a step in North direction and at the same time starting point of a step in East direction. Analogously, X is called *EN-corner*, if X is the end point of a step in East direction and at the same time starting point of a step in North direction.

Theorem 11. *Let $\mathcal{A}_i = (A_1^{(i)}, A_2^{(i)})$ and $\mathcal{E}_i = (E_1^{(i)}, E_2^{(i)})$ be lattice points satisfying $A_1^{(1)} \le A_1^{(2)} \le \cdots \le A_1^{(r)}$, $A_2^{(1)} > A_2^{(2)} > \cdots > A_2^{(r)}$, and $E_1^{(1)} < E_1^{(2)} < \cdots < E_1^{(r)}$, $E_2^{(1)} \ge E_2^{(2)} \ge \cdots \ge E_2^{(r)}$. The number of all families $\mathcal{P} = (P_1, \ldots, P_r)$ of nonintersecting lattice paths $P_i : \mathcal{A}_i \to \mathcal{E}_i$, such that the paths of $\mathcal{P}$ altogether contain exactly K NE-corners, is*

$$\sum_{k_1+\cdots+k_r=K} \det_{1\le s,t\le r} \left(\binom{E_1^{(t)} - A_1^{(s)} + s - t}{k_s + s - t} \binom{E_2^{(t)} - A_2^{(s)} - s + t}{k_s} \right). \qquad \square \tag{4.1}$$

Remark. A special case of this result is of relevance in the computation of Hilbert polynomials of determinantal ideals. In fact, Kulkarni [20, Main Theorem 5] derived this special case ($r = p$, $K = E$, $\mathcal{A}_i = (0, a_{p-i+1})$, $\mathcal{E}_i = (m(2) - b_{p-i+1}, m(1)))$ from Abhyankar's formula [1, (20.14.4), p. 484], while Conca and Herzog [7] used it to give an alternative proof of Abhyankar's formula. A special case of the "pathoid" analogue of Theorem 11 is also related to Abhyankar's formula (cf. [21, Main Theorem 7]).

Theorem 12. *Let $\mathcal{A}_i = (A_1^{(i)}, A_2^{(i)})$ and $\mathcal{E}_i = (E_1^{(i)}, E_2^{(i)})$ be lattice points satisfying $A_1^{(1)} \le A_1^{(2)} \le \cdots \le A_1^{(r)}$, $A_2^{(1)} > A_2^{(2)} > \cdots > A_2^{(r)}$, $E_1^{(1)} < E_1^{(2)} < \cdots < E_1^{(r)}$, $E_2^{(1)} \ge E_2^{(2)} \ge \cdots \ge E_2^{(r)}$, and $A_1^{(i)} \ge A_2^{(i)}$, $E_1^{(i)} \ge E_2^{(i)}$, $i = 1, \ldots, r$. The number of all families $\mathcal{P} = (P_1, \ldots, P_r)$ of nonintersecting lattice paths $P_i : \mathcal{A}_i \to \mathcal{E}_i$, which do not cross the line $x = y$, and where the paths of $\mathcal{P}$ altogether contain exactly K NE-corners, is*

$$\sum_{k_1+\cdots+k_r=K} \det_{1\le s,t\le r} \left(\binom{E_1^{(t)} - A_1^{(s)} + s - t}{k_s + s - t} \binom{E_2^{(t)} - A_2^{(s)} - s + t}{k_s} - \binom{E_1^{(t)} - A_2^{(s)} - s - t + 1}{k_s - t} \binom{E_2^{(t)} - A_1^{(s)} + s + t - 1}{k_s + s} \right). \qquad \square \tag{4.2}$$

Theorem 13. *Let $\mathcal{A}_i = (A_1^{(i)}, A_2^{(i)})$ and $\mathcal{E}_i = (E_1^{(i)}, E_2^{(i)})$ be lattice points satisfying $A_1^{(1)} < A_1^{(2)} < \cdots < A_1^{(r)}$, $A_2^{(1)} \ge A_2^{(2)} \ge \cdots \ge A_2^{(r)}$, $E_1^{(1)} \le E_1^{(2)} \le \cdots \le E_1^{(r)}$, $E_2^{(1)} > E_2^{(2)} > \cdots > E_2^{(r)}$, and $A_1^{(i)} \ge A_2^{(i)}$, $E_1^{(i)} \ge E_2^{(i)}$, $i = 1, \ldots, r$. The number of all families $\mathcal{P} = (P_1, \ldots, P_r)$ of nonintersecting lattice paths $P_i : \mathcal{A}_i \to \mathcal{E}_i$, $P_i : \mathcal{A}_i \to \mathcal{E}_i$, which do not cross the line $x = y$, and where the paths of $\mathcal{P}$ altogether contain exactly K EN-corners, is*

$$\sum_{k_1+\cdots+k_r=K} \det_{1\le s,t\le r} \left(\binom{E_1^{(t)} - A_1^{(s)} + s - t}{k_s + s - t} \binom{E_2^{(t)} - A_2^{(s)} - s + t}{k_s} - \binom{E_1^{(t)} - A_2^{(s)} - s - t + 3}{k_s - t + 1} \binom{E_2^{(t)} - A_1^{(s)} + s + t - 3}{k_s + s - 1} \right). \qquad \square \tag{4.3}$$

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