POWERS OF STAIRCASE SCHUR FUNCTIONS AND SYMMETRIC ANALOGUES OF BESSEL POLYNOMIALS

Bernard LECLERC †

Abstract. We present several identities involving staircase Schur functions. These identities are then interpreted in terms of a sequence of orthogonal polynomials in one variable x, with coefficients in the ring of symmetric functions. By an appropriate specialization these polynomials reduce to Bessel polynomials. This leads to a new determinantal expression for Bessel polynomials and suggests that their combinatorics might be linked to Young tableaux or shifted Young tableaux.

1 Introduction

Schur functions S_I and skew Schur functions $S_{I/J}$ are indexed by partitions I or skew partitions I/J, which are visualized graphically by a diagram of boxes. Several families of Schur functions associated with special diagrams are known to satisfy particular identities. Thus, Lascoux and Pragacz provided a determinantal expression of Schur functions and skew Schur functions in terms of ribbon functions, which generalizes the classical decomposition into hook functions given by Giambelli [**LP**].

In this paper we are interested in Schur functions whose diagram is a staircase. Our starting point is the following formula for the square of a staircase Schur function

$$S_{12\ldots n}^2 = S_{12\ldots n+1/1} S_{12\ldots n-1} - S_{12\ldots n+1} S_{12\ldots n-1/1} \ ,$$

which first arose in the context of the Korteweg-de-Vries equation [**AM**]. It shows that the polynomials θ_d constructed by Adler and Moser to represent the rational solutions of the KdV equation are, up to normalization, nothing but staircase Schur functions *. In a recent article Goulden has provided a combinatorial proof of this identity [**Go**]. We shall present more general formulae, expressing any power of a staircase Schur function.

An interpretation of these formulae in terms of orthogonal polynomials is then given. Indeed, it is shown that the family of staircase Schur functions $S_{12\ldots n}(\mathbb{E}+x)$, $n \geq 0$, is a family of orthogonal polynomials in the variable x, which may be seen as a symmetric analogue of the family of Bessel polynomials. This provides a new determinantal expression for Bessel polynomials. It also shows that the powerful machinery of symmetric functions might be of some help in the study of this sequence of polynomials. Thus, using the fact that staircase S-functions are special cases of Schur Q-functions, Thibon and the author derived from a recent multiplication formula due to Worley the computation of the linearization coefficients [**LT**]which had been conjectured by Favreau [**Fa**].

---

† Institut Gaspard Monge, Université de Marne-la-Vallée, 2 rue de la Butte-Verte 93160 Noisy-le-Grand Cedex France

* We thank D. Svrtan for pointing out this fact to us and kindly providing references on this subject.

Finally, we show that our formulae are also linked to an old problem first studied by Borchardt [**Bo**] and Laguerre [**La**], and further investigated by numerous authors including Pólya [**Po**] and Foulkes [**Fo**]. The problem is to express any symmetric polynomial of n indeterminates as a rational function of the power sums of odd degree. Set $\mathbb{X} = \{x_1, \ldots, x_n\}$. The expression given by Polya and Foulkes for the k^{th} elementary symmetric polynomial $\Lambda_k(\mathbb{X})$ is the coefficient of $(-x)^{n-k}$ in the ratio $S_{12\ldots n}(\mathbb{X} - x)/S_{12\ldots n-1}(\mathbb{X})$. We obtain a similar expression for the complete symmetric polynomial $S_k(\mathbb{X})$.

2 Schur functions and minors identities

In this section we shall define our notations and recall some basic facts about Schur functions and minors identities. General references are [**LS**], [**Mc**] and [**Le**].

Let $\mathbb{E}$ denote a set of indeterminates or *alphabet*. The *complete symmetric functions* $S_i(\mathbb{E})$ and the *elementary symmetric functions* $\Lambda_i(\mathbb{E})$ are defined by means of the generating series

$$\sum_i S_i(\mathbb{E}) t^i = \prod_{e \in \mathbb{E}} (1 - te)^{-1} \ , \tag{1}$$

$$\sum_i \Lambda_i(\mathbb{E}) t^i = \prod_{e \in \mathbb{E}} (1 + te) \ . \tag{2}$$

When there is no danger of confusion we shall omit the mention of the alphabet. For $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, the *Schur function* S_I is defined by Jacobi–Trudi identity $S_I = \det[S_{i_l+l-k}]_{1\le k,l\le n}$. In other words, denoting by $\mathbf{S}$ the infinite Toeplitz matrix $[S_{j-i}]_{i,j\ge 0}$ where $S_k = 0$ for $k < 0$, S_I is the minor of $\mathbf{S}$ taken on the lines $0, 1, \ldots, n-1$ and the columns $i_1, i_2+1, \ldots, i_n+n-1$. More generally, given $J = (j_1, \ldots, j_n) \in \mathbb{N}^n$ one defines the *skew Schur function* $S_{I/J}$ as the minor of $\mathbf{S}$ taken on the lines $j_1, j_2+1, \ldots, j_n+n-1$ and the columns $i_1, i_2+1, \ldots, i_n+n-1$, *i. e.* $S_{I/J} = \det[S_{i_l+l-j_k-k}]_{1\le k,l\le n}$. Note that this definition makes sense even if the sequences I and J are not arranged in increasing order. However, by permutation of columns and rows one can always assume that I and J are *partitions* i.e. weakly *increasing* sequences of nonnegative integers. A partition I is represented by a *diagram* of boxes having i_1 boxes on its first row, i_2 boxes on its second row, etc.. $|I| = i_1 + \ldots + i_n$ is called the *weight* of I. The *skew diagram* associated with $S_{I/J}$ is the complement of the diagram J in the diagram I.

Schur functions may also be expressed in terms of elementary symmetric functions Λ_i. Indeed, defining by analogy the functions $\Lambda_{I/J}$ by $\Lambda_{I/J} = \det[\Lambda_{i_l+l-j_k-k}]_{1\le k,l\le n}$, one has the relation

$$\Lambda_{I/J} = S_{I^\sim/J^\sim}, \tag{3}$$

where $I^\sim$ denotes the *conjugate partition* of I i.e. the partition whose diagram is obtained by interchanging the rows and columns of the diagram of I.

Suppose $\mathbb{F}$ is a second alphabet and denote by $\mathbb{E}+\mathbb{F}$ the union of $\mathbb{E}$ and $\mathbb{F}$. It follows from (1), (2) that

$$S_n(\mathbb{E} + \mathbb{F}) = \sum_{0\le i\le n} S_i(\mathbb{E}) S_{n-i}(\mathbb{F}) \ , \tag{4}$$

$$\Lambda_n(\mathbb{E}+\mathbb{F}) = \sum_{0\le i\le n} \Lambda_i(\mathbb{E})\Lambda_{n-i}(\mathbb{F}) \ . \tag{5}$$

We shall also define the difference $\mathbb{E}-\mathbb{F}$ by setting

$$\sum_i S_i(\mathbb{E}-\mathbb{F})t^i = \frac{\prod_{f\in\mathbb{F}}(1-tf)}{\prod_{e\in\mathbb{E}}(1-te)} \ , \qquad \sum_i \Lambda_i(\mathbb{E}-\mathbb{F})t^i = \frac{\prod_{e\in\mathbb{E}}(1+te)}{\prod_{f\in\mathbb{F}}(1+tf)} \ ,$$

so that in particular

$$\Lambda_i(\mathbb{E}) = (-1)^i S_i(-\mathbb{E}) \ , \tag{6}$$

which is another way of expressing the duality between complete and elementary symmetric functions. This yields a compact expression for the polynomial whose set of zeros is the alphabet $\mathbb{A}$ of cardinality n

$$\prod_{a\in\mathbb{A}}(x-a) = \sum_i x^{n-i}(-1)^i\Lambda_i(\mathbb{A}) = \sum_i S_{n-i}(x)S_i(-\mathbb{A}) = S_n(x-\mathbb{A}) = (-1)^n\Lambda_n(\mathbb{A}-x) \ .$$

Taking different alphabets $\mathbb{E}_1,\ldots,\mathbb{E}_n$ in the columns of the determinant of Jacobi–Trudi formula, one gets a *multi-Schur function* $S_I(\mathbb{E}_1,\ldots,\ \mathbb{E}_n) = \det[S_{i_l+l-k}(\mathbb{E}_l)]_{1\le k,l\le n}$. An example of multi-Schur function considered in this paper is the n^{th} orthogonal polynomial associated with the linear functional μ of moments $\mu(x^i) = S_i(\mathbb{E})$. (Note that, since the S_i are algebraically independent, one may always define a formal alphabet $\mathbb{E}$ by assigning to each S_i a given value $S_i(\mathbb{E})$ in a commutative ring.) Indeed this polynomial $p_n(x)$ has, up to a constant factor, the expression

$$p_n(x) = S_{n^n0}(\mathbb{E},\mathbb{E},\ldots,\mathbb{E},x) = \begin{vmatrix} S_n(\mathbb{E}) & S_{n+1}(\mathbb{E}) & \ldots & S_{2n-1}(\mathbb{E}) & x^n \\ S_{n-1}(\mathbb{E}) & S_n(\mathbb{E}) & \ldots & S_{2n-2}(\mathbb{E}) & x^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ S_0(\mathbb{E}) & S_1(\mathbb{E}) & \ldots & S_{n-1}(\mathbb{E}) & 1 \end{vmatrix} \ , \tag{7}$$

(see **[Sz]** p.27). By subtraction of lines in the determinant, this polynomial may also be written as an ordinary Schur function on the alphabet $\mathbb{E}-x$. Indeed, one has $p_n(x) = S_{n^n}(\mathbb{E}-x)$. This is a consequence of the following important lemma.

LEMMA 2.1 *Let m, n be two integers $m\le n$, $\mathbb{F}$ an alphabet of cardinality m, and $I=(i_1,\ldots,i_n)$ a partition. Then*

$$S_I(\mathbb{E}_1,\ldots,\mathbb{E}_n) = \begin{vmatrix} S_{i_1}(\mathbb{E}_1-\mathbb{F}) & \ldots & S_{i_n+n-1}(\mathbb{E}_n-\mathbb{F}) \\ \vdots & & \vdots \\ S_{i_1-n+m+1}(\mathbb{E}_1-\mathbb{F}) & \ldots & S_{i_n+m}(\mathbb{E}_n-\mathbb{F}) \\ S_{i_1-n+m}(\mathbb{E}_1) & \ldots & S_{i_n+m-1}(\mathbb{E}_n) \\ \vdots & & \vdots \\ S_{i_1-n+1}(\mathbb{E}_1) & \ldots & S_{i_n}(\mathbb{E}_n) \end{vmatrix} \ .$$

Proof. By (4), we have $S_k(\mathbb{E}_j-\mathbb{F}) = S_k(\mathbb{E}_j) + S_{k-1}(\mathbb{E}_j)S_1(-\mathbb{F}) + \ldots + S_k(-\mathbb{F})$. But $S_k(-\mathbb{F}) = (-1)^k\Lambda_k(\mathbb{F}) = 0$ if $k>m$. Thus, the transformation consists in adding to each of the first $n-m$ rows a linear combination of the next m rows. $\square$

The Jacobi–Trudi formula expresses Schur functions as minors of the matrix $\mathbf{S}$ of complete symmetric functions S_i. This shows that many algebraic relations for Schur functions may be obtained by merely specializing the numerous identities satisfied by minors of a generic matrix. We have shown elsewhere [**Le**] that a great number of these identities are easily derived from a single one by Turnbull. We shall briefly recall this identity for the convenience of the reader. We first need some appropriate notations.

Let M be a $n\times\infty$ matrix. We shall be interested in identities satisfied by maximal minors of M. Let a,b,…,c be n column vectors of M. The maximal minor of M taken on these n columns will be denoted by either a bracket or a one–line tableau

$$[ab\dots c] = \boxed{a \quad b \quad \dots \quad c}\,.$$

A product of p minors of M will be designated by a $p\times n$ tableau

$$[ab\dots c].[de\dots f].[gh\dots i] = \begin{array}{|cccc|}\hline a & b & \dots & c \\ \hline d & e & \dots & f \\ \hline g & h & \dots & i \\ \hline \end{array}\,.$$

To denote certain alternating sums of products of minors, we shall use tableaux with boxes enclosing some particular vectors. They are defined in the following way.

Let T be a $p\times n$ tableau and A a subset of elements of T. Given a permutation σ of the elements of A, denote by $\sigma(T)$ the tableau in which the elements of A are permuted by σ. Now, boxing in T the elements of A, we get a new tableau τ which will represent the alternating sum of all the tableaux $\sigma(T)$, taking into account the fact that a permutation of elements of the same row gives a trivial action. More precisely, $\tau = \sum_\sigma sign(\sigma).\sigma(T)$, where σ runs through the cosets of the symmetric group $\mathfrak{S}(A)$ modulo the subgroup of the permutations which leave unchanged the rows of T.

Let i_k, $k=1,\dots p$ denote the number of elements of A lying in the k^{th} row of τ. The number of terms in this sum will be $(i_1+\dots+i_p)!/i_1!\dots i_p!$. In particular, if all the enclosed elements of τ lie in the same row, it reduces to the single tableau T.

To illustrate these notations, let us write a well–known minor identity (Plücker's relations) and then its transcription with tableaux.

Let M be the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & \dots \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & \dots \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & \dots \end{pmatrix}\,.$$

We have

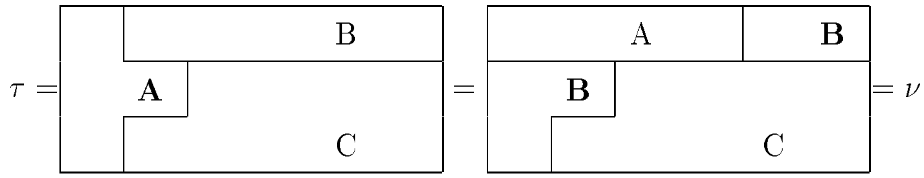
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} - \begin{vmatrix} e_1 & b_1 & c_1 \\ e_2 & b_2 & c_2 \\ e_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} d_1 & a_1 & f_1 \\ d_2 & a_2 & f_2 \\ d_3 & a_3 & f_3 \end{vmatrix}$$
$$- \begin{vmatrix} f_1 & b_1 & c_1 \\ f_2 & b_2 & c_2 \\ f_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} d_1 & e_1 & a_1 \\ d_2 & e_2 & a_2 \\ d_3 & e_3 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & e_1 & f_1 \\ a_2 & e_2 & f_2 \\ a_3 & e_3 & f_3 \end{vmatrix} . \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}\,.$$

Denoting the column vectors of M by a,b,c,d,e,f, this identity written using tableaux takes the following form

$$\left|\begin{array}{ccc} \boxed{a} & b & c \\ \hline d & \boxed{e} & \boxed{f} \end{array}\right| = \left|\begin{array}{ccc} a & e & f \\ \hline d & b & c \end{array}\right| .$$

It is readily deduced from Turnbull's identity, that we shall now state.

Let τ be a tableau and denote by A the set of its boxed elements. Let B be the set of elements of a given row, for example the first one, which do not belong to A. Finally let C denote the set of all other elements of τ. One can build another tableau υ equal to τ by merely exchanging the roles of A and B as shown by the following diagram:



More precisely, we have

THEOREM 2.1. *(Turnbull's identity,* **[Tu]**, p.209.)

Let k be the number of elements of A.

(i) *If $k > n$, $\tau = 0$.*

(ii) *If $k \leq n$, and υ is built by*

a. *exchanging each element of A which does not belong to the first row, with any element of B;*

b. *boxing the elements of B;*

c. *removing the boxes of the elements of A;*

then $\tau = \upsilon$.

For instance, taking $n = 6$, $k = 4$, $A = \{a,b,c,d\}$, $B = \{\alpha,\beta,\gamma,\delta,\varepsilon\}$, one has

$$\tau = \left|\begin{array}{cccccc} \boxed{a} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \hline \boxed{b} & \boxed{c} & f & g & h & i \\ \hline \boxed{d} & j & k & l & m & o \end{array}\right| = \left|\begin{array}{cccccc} a & b & c & d & \boxed{\delta} & \boxed{\varepsilon} \\ \hline \boxed{\alpha} & \boxed{\beta} & f & g & h & i \\ \hline \boxed{\gamma} & j & k & l & m & o \end{array}\right| = \upsilon .$$

We shall now deduce from this identity some formulae for the power of a staircase Schur function.

3 Powers of a staircase Schur function and symmetric analogues of Bessel polynomials

Let $\rho(k,m)$ denote the staircase partition $(m,\ 2m, \ldots,\ km)$. The n^{th} power of $S_{\rho(k,m)}$ admits the following expression:

THEOREM 3.1.

$$(S_{\rho(k,m)})^n = \sum_{I\subset\rho(n-1,m)} (-1)^{|\rho(n-1,m)|-|I|}\, S_{\rho(n+k-1,m)/I}\, \Delta_I\ ,$$

where Δ_I is the minor taken on the lines $i_1+1,\ i_2+2, \ldots,\ i_{n-1}+n-1$ of the following $(m+1)(n-1)\times(n-1)$ matrix:

$$M = \begin{pmatrix} S_{\rho(k-1,m)/1^m} & S_{\rho(k-1,m)/1^{2m+1}} & \ldots & S_{\rho(k-1,m)/1^{(n-1)m+n-2}} \\ S_{\rho(k-1,m)/1^{m-1}} & S_{\rho(k-1,m)/1^{2m}} & \ldots & S_{\rho(k-1,m)/1^{(n-1)m+n-3}} \\ \vdots & \vdots & & \vdots \\ S_{\rho(k-1,m)} & S_{\rho(k-1,m)/1^{m+1}} & \ldots & S_{\rho(k-1,m)/1^{(n-2)(m+1)}} \\ 0 & S_{\rho(k-1,m)/1^m} & \ldots & S_{\rho(k-1,m)/1^{(n-2)m+n-3}} \\ 0 & S_{\rho(k-1,m)/1^{m-1}} & \ldots & S_{\rho(k-1,m)/1^{(n-2)m+n-4}} \\ \vdots & \vdots & & \vdots \\ 0 & S_{\rho(k-1,m)} & \ldots & S_{\rho(k-1,m)/1^{(n-3)(m+1)}} \\ 0 & 0 & \ldots & S_{\rho(k-1,m)/1^{(n-3)m+n-4}} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & S_{\rho(k-1,m)} \end{pmatrix} .$$

Note that we allow the first parts of I to be zero. Before proving the theorem, let us illustrate it by a few examples.

EXAMPLE 3.2. We take $n=3$, $m=1$ and $k=4$. The theorem states that

$$(S_{1234})^3 = \sum_{I\subset 12} (-1)^{3-|I|}\, S_{123456/I}\, \Delta_I\ ,$$

where I runs over the set $\{00,\ 01,\ 02,\ 11,\ 12\}$ and Δ_I is the minor taken on the lines i_1+1, i_2+2 of the 4×2 matrix:

$$M = \begin{pmatrix} S_{123/1} & S_{123/1^3} \\ S_{123} & S_{123/1^2} \\ 0 & S_{123/1} \\ 0 & S_{123} \end{pmatrix} = \begin{pmatrix} S_{1234/1112} & S_{1234/1114} \\ S_{1234/1111} & S_{1234/1113} \\ S_{1234/1110} & S_{1234/1112} \\ S_{1234/111(-1)} & S_{1234/1111} \end{pmatrix} .$$

The second expression of M shows more clearly the regularity of its columns (we recall that for any partition J, the determinants $S_{J/1110}$ and $S_{J/111(-1)}$ are null having two identical rows). Expanding this sum and the determinants Δ_I yields the following relation:

$$\begin{aligned}(S_{1234})^3 = {} & S_{123456}\, S_{123}\, S_{123/111} - S_{123456}\, S_{123/1}\, S_{123/11} + S_{123456/1}\, S_{123/1}\, S_{123/1} \\ & - S_{123456/11}\, S_{123}\, S_{123/1} - S_{123456/2}\, S_{123}\, S_{123/1} + S_{123456/12}\, S_{123}\, S_{123}\ .\end{aligned}$$

EXAMPLE 3.3. We take $n=2$, so that theorem 3.1 reads

$$(S_{\rho(k,m)})^2 = \sum_{0\le i\le m} (-1)^{m-i}\, S_{\rho(k+1,m)/i}\, S_{\rho(k-1,m)/1^{m-i}}\ ,$$

which is the formula investigated by Goulden **[Go]** for the square of a staircase Schur function.

Proof of Theorem 3.1. We shall apply Turnbull's identity 2.1. Set $N = k + (n-1)(m+1)$ and consider the $N \times \infty$ matrix

$$M = \begin{pmatrix} S_0 & S_1 & S_2 & S_3 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & S_0 & S_1 & S_2 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & S_0 & S_1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} .$$

Let $0,\ 1,\ 2,\ 3, \dots$ denote the column vectors of the left part of M, and $1^*,\ 2^*,\ 3^*, \dots N^*$, denote the last N column vectors of M. For brevity we shall write $m_p = pm + p - 1$. The Schur function $S_{\rho(k,m)}$ may be expressed as a maximal minor of M in n different ways

$$\begin{aligned} S_{\rho(k,m)} &= [m_1\ m_2\ \dots\ m_k\ (k+1)^*\ (k+2)^*\ \dots\ N^*] \\ &= [0\ 1\ \dots\ (m_1-1)\ m_1\ m_2\ \dots\ m_{k+1}\ (k+m_1+1)^*\ (k+m_1+2)^*\ \dots\ N^*] \\ &\ \vdots \\ &= [0\ 1\ 2\ \dots\ (m_{n-1}-1)\ m_{n-1}\ m_n\ \dots\ m_{k+n-1}] , \end{aligned}$$

so that its n^{th} power is equal to the product of these n minors

$$(S_{\rho(k,m)})^n = \begin{array}{|l|} \hline m_1 \quad \dots \quad m_k \quad (k+1)^* \quad \dots \qquad\qquad\qquad\qquad\qquad \dots \quad N^* \\ \hline 0 \quad \dots \quad (m_1-1) \quad m_1 \quad \dots \quad m_{k+1} \quad (k+m_1+1)^* \quad \dots \quad N^* \\ \hline \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \hline 0 \quad \dots \qquad\qquad\qquad \dots \quad (m_{n-1}-1) \quad m_{n-1} \quad \dots \quad m_{k+n-1} \\ \hline \end{array} .$$

Now, recalling that a tableau having two equal letters on the same row is equal to zero, we may box the letters $m_1,\ m_2,\ \dots, m_k$ in the first row, the letter m_{k+1} in the second row, $\dots$, the letter m_{k+n-1} in the n^{th} row, and apply Turnbull's identity. This yields

$$\begin{aligned} (S_{\rho(k,m)})^n & \\ = & \begin{array}{|l|} \hline \boxed{m_1} \quad \dots \quad \boxed{m_k} \quad (k+1)^* \quad \dots \qquad\qquad\qquad\qquad\qquad \dots \quad N^* \\ \hline 0 \quad \dots \quad (m_1-1) \quad m_1 \quad \dots \quad \boxed{m_{k+1}} \quad (k+m_1+1)^* \quad \dots \quad N^* \\ \hline \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \hline 0 \quad \dots \qquad\qquad\qquad \dots \quad (m_{n-1}-1) \quad m_{n-1} \quad \dots \quad \boxed{m_{k+n-1}} \\ \hline \end{array} \\ = & \begin{array}{|l|} \hline m_1 \quad \dots \quad m_k \quad m_{k+1} \quad \dots \quad m_{k+n-1} \quad \boxed{(k+n)^*} \quad \dots \quad \boxed{N^*} \\ \hline 0 \quad \dots \quad (m_1-1) \quad m_1 \quad \dots \quad \boxed{(k+1)^*} \quad (k+m_1+1)^* \quad \dots \quad N^* \\ \hline \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \hline 0 \quad \dots \qquad\qquad \dots \quad (m_{n-1}-1) \quad m_{n-1} \quad \dots \quad \boxed{(k+n-1)^*} \\ \hline \end{array} . \end{aligned}$$

This last tableau represents a sum of products of Schur functions of the following type

$$\pm S_{\rho(n+k-1,m)/I}\ S_{\rho(k-1,m)/1^{j_1}} \cdots\ S_{\rho(k-1,m)/1^{j_{n-1}}}\ .$$

The coefficient of a given $S_{\rho(n+k-1,m)/I}$ in this sum is in fact a tableau having $n-1$ rows with only one letter boxed in each row, that is a determinant of order $n-1$. There is no difficulty in recognizing this determinant as the minor Δ_I defined above. □

The formula for the square of $S_{\rho(k,1)}$ may be presented as the computation of a determinant of Schur functions

$$\begin{vmatrix} S_{\rho(k-1,1)} & S_{\rho(k+1,1)} \\ S_{\rho(k-1,1)/1} & S_{\rho(k+1,1)/1} \end{vmatrix} = \left(S_{\rho(k,1)}\right)^2\ . \tag{8}$$

The same may be done for theorem 3.1, which has also a determinantal formulation. We shall write for brevity $\rho(k)$ in place of $\rho(k,1)$.

THEOREM 3.4.

$$\begin{vmatrix} S_{\rho(k-1)} & 0 & \cdots & 0 & \cdots & 0 & S_{\rho(k+1)} \\ S_{\rho(k-1)/1} & 0 & \cdots & 0 & \cdots & S_{\rho(k+2)} & S_{\rho(k+1)/1} \\ S_{\rho(k-1)/1^2} & S_{\rho(k,1)} & \cdots & 0 & \cdots & S_{\rho(k+2)/1} & S_{\rho(k+1)/1^2} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ S_{\rho(k-1)/1^{2n-3}} & S_{\rho(k-1)/1^{2n-5}} & \cdots & S_{\rho(k+n-1)/1^{n-1}} & \cdots & S_{\rho(k+2)/1^{2n-4}} & S_{\rho(k+1)/1^{2n-3}} \end{vmatrix}$$
$$= \left(S_{\rho(k)}\right)^n\ S_{\rho(k+1)}\ S_{\rho(k+2)} \cdots S_{\rho(k+n-2)}\ .$$

EXAMPLE 3.5. For $n=3$ and $k=4$ one has

$$\begin{vmatrix} S_{123} & 0 & 0 & S_{12345} \\ S_{123/1} & 0 & S_{123456} & S_{12345/1} \\ S_{123/1^2} & S_{123} & S_{123456/1} & S_{12345/1^2} \\ S_{123/1^3} & S_{123/1} & S_{123456/1^2} & S_{12345/1^3} \end{vmatrix} = \left(S_{1234}\right)^3\ S_{12345}\ .$$

EXAMPLE 3.6. For $n=4$ and $k=6$ one has

$$\begin{vmatrix} S_{12345} & 0 & 0 & 0 & 0 & S_{1234567} \\ S_{12345/1} & 0 & 0 & 0 & S_{12345678} & S_{1234567/1} \\ S_{12345/1^2} & S_{12345} & 0 & S_{123456789} & S_{12345678/1} & S_{1234567/1^2} \\ S_{12345/1^3} & S_{12345/1} & 0 & S_{123456789/1} & S_{12345678/1^2} & S_{1234567/1^3} \\ S_{12345/1^4} & S_{12345/1^2} & S_{12345} & S_{123456789/1^2} & S_{12345678/1^3} & S_{1234567/1^4} \\ S_{12345/1^5} & S_{12345/1^3} & S_{12345/1} & S_{123456789/1^3} & S_{12345678/1^4} & S_{1234567/1^5} \end{vmatrix}$$
$$= \left(S_{123456}\right)^4\ S_{1234567}\ S_{12345678}\ .$$

Proof. The proof of this last example will illustrate enough the principle of our computation. We consider the 12×30 matrix

$$M = \begin{pmatrix} S_0 & S_1 & S_2 & S_3 & \cdots & S_{17} & 1 & 0 & 0 & \cdots & 0 \\ 0 & S_0 & S_1 & S_2 & \cdots & S_{16} & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & S_0 & S_1 & \cdots & S_{15} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & S_6 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

whose column vectors are denoted $0,\ 1,\ 2, \ldots 17,\ 1^*,\ 2^*, \ldots\ 12^*$. The determinant D of example 3.6 is represented by the following tableau

$$D = \begin{array}{|cccccccccccc|}
\hline
0 & 1 & 3 & 5 & 7 & 9 & 11 & \boxed{7^*} & 9^* & 10^* & 11^* & 12^* \\
\hline
0 & 1 & 2 & 3 & 5 & 7 & 9 & 11 & 13 & \boxed{8^*} & 11^* & 12^* \\
\hline
0 & 1 & 2 & 3 & 4 & 5 & 7 & 9 & 11 & 13 & 15 & \boxed{9^*} \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & \boxed{10^*} & 11^* & 12^* \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & \boxed{11^*} & 12^* \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & 2 & \boxed{12^*} \\
\hline
\end{array} .$$

Now we transform this tableau by Turnbull's identity and note that all boxes may be removed because of the presence of identical letters

$$D = \begin{array}{|cccccccccccc|}
\hline
0 & 1 & 3 & 5 & 7 & 9 & 11 & \boxed{13} & 9^* & 10^* & 11^* & 12^* \\
\hline
0 & 1 & 2 & 3 & 5 & 7 & 9 & 11 & 13 & \boxed{15} & 11^* & 12^* \\
\hline
0 & 1 & 2 & 3 & 4 & 5 & 7 & 9 & 11 & 13 & 15 & \boxed{17} \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & \boxed{0} & 11^* & 12^* \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & \boxed{2} & 12^* \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 7^* & 8^* & 9^* & 10^* & 11^* & 12^* \\
\hline
\end{array}$$

$$= \begin{array}{|cccccccccccc|}
\hline
0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 9^* & 10^* & 11^* & 12^* \\
\hline
0 & 1 & 2 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 11^* & 12^* \\
\hline
0 & 1 & 2 & 3 & 4 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & 11^* & 12^* \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & 2 & 12^* \\
\hline
1 & 3 & 5 & 7 & 9 & 11 & 7^* & 8^* & 9^* & 10^* & 11^* & 12^* \\
\hline
\end{array}$$

$$= (S_{123456})^4\ S_{1234567} S_{12345678} \qquad \square$$

We shall end this section with an interpretation of theorem 3.4 in terms of orthogonal polynomials. First we shall give an immediate generalization of formula (8).

Proposition 3.7. *For $1 \le p \le k+1$ we have*

$$\begin{vmatrix} S_{\rho(k-1)} & S_{\rho(k+1)} \\ S_{\rho(k-1)/1^p} & S_{\rho(k+1)/1^p} \end{vmatrix} = S_{\rho(k)}\ S_{\rho(k)/1^{p-1}}\ ,$$

where $\rho(k)$ is short for the partition $12\ldots k$.

Proof. This is an instance of Plücker relations. We consider again the matrix M of the proof of theorem 3.1, taking $N = k+2$. The determinant to be computed is then represented by the $2 \times (k+2)$ tableau

$$(-1)^p \begin{array}{|ccccccc|}
\hline
0 & 1 & 3 & 5 & \ldots & 2k-1 & \boxed{N^*} \\
\hline
1 & 3 & 5 & 7 & \ldots & 2k+1 & \boxed{(N-p)^*} \\
\hline
\end{array} .$$

Applying Turnbull's identity, this tableau is transformed into

$$(-1)^p \left[\begin{array}{ccccccc} 0 & 1 & 3 & 5 & \dots & 2k-1 & \boxed{2k+1} \\ \hline \boxed{1} & \boxed{3} & \boxed{5} & \boxed{7} & \dots & N^* & (N-p)^* \end{array}\right]$$

$$= \ (-1)^p \left[\begin{array}{ccccccc} 0 & 1 & 3 & 5 & \dots & 2k-1 & 2k+1 \\ \hline 1 & 3 & 5 & 7 & \dots & N^* & (N-p)^* \end{array}\right] = S_{\rho(k)}\ S_{\rho(k)/1^{p-1}}\ . \qquad \square$$

¿From this proposition we deduce the following result:

THEOREM 3.8. *Let $\mathbb{E}$ be an alphabet and let us denote by π_k the polynomial of degree k in x with coefficients in the ring of symmetric functions of $\mathbb{E}$: $\pi_k(x) = S_{\rho(k)}(\mathbb{E}-x)$. Then π_k satisfies the three term recurrence relationship ($k \geq 1$)*

$$S_{\rho(k-1)}(\mathbb{E})\pi_{k+1}(x) + xS_{\rho(k)}(\mathbb{E})\pi_k(x) - S_{\rho(k+1)}(\mathbb{E})\pi_{k-1}(x) = 0\ .$$

In other words $\big(\pi_k(x)\big)_{k\geq 0}$ is a family of orthogonal polynomials.

Proof. The expansion of π_k is obtained by considering it as a multi-Schur function and expanding this determinant along its last column. Indeed, by lemma 2.1,

$$\pi_k(x) = S_{12\dots k0}(\mathbb{E},\dots,\mathbb{E},x) = \sum_{0\leq i\leq k} S_{12\dots k/1^i}(\mathbb{E})(-x)^i\ .$$

By linearity we deduce from proposition 3.7 that

$$\begin{vmatrix} S_{\rho(k-1)} & S_{\rho(k+1)} \\ \pi_{k-1}(x) & \pi_{k+1}(x) \end{vmatrix} = -xS_{\rho(k)}\ \pi_k(x)\ ,$$

which is the required relationship. $\square$

Thus, theorem 3.8 states that the elimination of x^0 between $\pi_{k-1}(x)$ and $\pi_{k+1}(x)$ produces a polynomial proportional to $x\pi_k(x)$. As a consequence the elimination of x^0, x^1, x^2 between $\pi_{k-1}(x)$, $\pi_{k+1}(x)$, $x\pi_{k+2}(x)$, $x^2\pi_{k-1}(x)$ will produce a polynomial proportional to $x^3\pi_k(x)$. This elimination is expressed by the following formula

$$\begin{vmatrix} S_{\rho(k-1)} & 0 & 0 & S_{\rho(k+1)} \\ S_{\rho(k-1)/1} & 0 & S_{\rho(k+2)} & S_{\rho(k+1)/1} \\ S_{\rho(k-1)/1^2} & S_{\rho(k-1)} & S_{\rho(k+2)/1} & S_{\rho(k+1)/1^2} \\ \pi_{(k-1)}(x) & x^2\pi_{k-1}(x) & -x\pi_{k+2}(x) & \pi_{k+1}(x) \end{vmatrix} = -(S_{\rho(k)})^2\ S_{\rho(k+1)}\ x^3\pi_k(x).$$

Picking up the coefficient of x^3 on each side we find again the case $n = 3$ of theorem 3.4. And this process may clearly be carried out for larger values of n.

The computation of the moments of the sequence of orthogonal polynomials $\big(\pi_k(x)\big)_{k\geq 0}$ will result from the following identity. We shall denote by $\rho(n) + m$ the partition $(1, 2, \dots, n-1, n+m)$.

THEOREM 3.9. *Let k, n be integers such that $1 \le k \le n$. There holds:*

$$\begin{vmatrix} S_{\rho(k)/1^{k-1}} & S_{\rho(k+1)/1^{k-1}} & \dots & S_{\rho(n)/1^{k-1}} \\ S_{\rho(k)/1^{k}} & S_{\rho(k+1)/1^{k}} & \dots & S_{\rho(n)/1^{k}} \\ \vdots & \vdots & & \vdots \\ S_{\rho(k)/1^{n-1}} & S_{\rho(k+1)/1^{n-1}} & \dots & S_{\rho(n)/1^{n-1}} \end{vmatrix} = S_{\rho(k-1)} P_{\rho(k)+(n-k+1)} S_{\rho(k+1)} S_{\rho(k+2)} \cdots S_{\rho(n-1)},$$

where $P_{\rho(k)+(n-k+1)} = \sum_{0\le 2i \le n-k+1} \Lambda_{\rho(k)+2i} S_{n-k+1-2i}$.

EXAMPLE 3.10. For $n = 6$, $k = 3$ one has

$$\begin{vmatrix} S_{123/1^2} & S_{1234/1^2} & S_{12345/1^2} & S_{123456/1^2} \\ S_{123/1^3} & S_{1234/1^3} & S_{12345/1^3} & S_{123456/1^3} \\ S_{123/1^4} & S_{1234/1^4} & S_{12345/1^4} & S_{123456/1^4} \\ S_{123/1^5} & S_{1234/1^5} & S_{12345/1^5} & S_{123456/1^5} \end{vmatrix} = S_{12} P_{127} S_{1234} S_{12345} \ ,$$

where $P_{127} = \Lambda_{123} S_4 + \Lambda_{125} S_2 + \Lambda_{127}$.

Proof. The proof is similar to the proof of theorem 3.4, the only difference being that one has to apply Turnbull's identity several times in order to obtain the required factorization.

COROLLARY 3.11. *The n^{th} moment of the sequence of orthogonal polynomials $\left(\pi_k(x)\right)_{k\ge 0}$ is the sum of all hook Schur functions of weight $n+1$: $\mu(x^n) = \sum_{0\le i\le n} S_{1^i(n+1-i)}(\mathbb{E})$.*

Proof. Let $q_k(x)$ denote the monic polynomial $q_k(x) = S_{\rho(k)}(\mathbb{E} - x)/(-1)^k S_{\rho(k-1)}(\mathbb{E})$. Theorem 3.9 may be seen as the resolution of the linear system in the unknown a_{nk}:

$$x^n = \sum_k a_{nk} q_k(x) \ . \tag{9}$$

The result is that $a_{nk} = P_{\rho(k+1)+n-k}(\mathbb{E})/S_{\rho(k+1)}$. Denote by $Sym(\mathbb{E})$ the ring of symmetric functions of the alphabet $\mathbb{E}$ and let μ be the functional on $Sym(\mathbb{E})[x]$ associated with the sequence of orthogonal polynomials $\left(\pi_k(x)\right)_{k\ge 0}$. In fact μ is defined up to a constant factor, so that we can add the condition $\mu(1) = S_1(\mathbb{E})$. Then, recalling that by definition $\mu(q_k(x)) = 0$, $k \ge 1$ and applying μ to the equality (9), one obtains that $\mu(x^n) = P_{n+1}(\mathbb{E}) = \sum_{0\le i\le n} S_{1^i(n+1-i)}(\mathbb{E})$, the last equality resulting easily from Pieri rule for the multiplication of a Schur function by an elementary symmetric function. ∎

Remarks. (i) It can be shown that the symmetric function

$$\sum_{0\le 2i\le n-k+1} \Lambda_{\rho(k)+2i} S_{n-k+1-2i}$$

is nothing but the Schur P-function $P_{\rho(k)+n-k+1}$, hence the notation used above. More generally it has been shown in [**LLT**] that there exists for every Schur P-function a similar quadratic expansion in terms of ordinary Schur S-functions.

(ii) It is easily checked that $S_{\rho(n)/1^m} = S_{\rho(n)/m}$. It follows that the polynomials $\phi_k(x) = S_{\rho(k)}(\mathbb{E}+x)$ are equal to the polynomials $\pi_k(-x)$, and therefore also satisfy a three term recurrence relationship:

$$S_{\rho(k-1)}(\mathbb{E})\phi_{k+1}(x) - xS_{\rho(k)}(\mathbb{E})\phi_k(x) - S_{\rho(k+1)}(\mathbb{E})\phi_{k-1}(x) = 0 \ .$$

They are orthogonal for the moments $\nu(x^n) = (-1)^n \sum_{0\le i\le n} S_{1^i(n+1-i)}(\mathbb{E})$.

We must now emphasize on the differences between the two Schur function expressions of orthogonal polynomials hitherto encountered. As recalled in section 2, the family of orthogonal polynomials associated with a given functional μ may *always* be represented up to a constant factor by the sequence of rectangle Schur functions $S_{n^n}(\mathbb{E}-x)$, $n\ge 0$, where $\mathbb{E}$ is the alphabet formally defined by $S_k(\mathbb{E}) = \mu(x^k)$, $k\ge 0$. In contrast Schur functions of the type $S_{12\ldots n}(\mathbb{E}-x)$, $n\ge 0$ or $S_{12\ldots n}(\mathbb{E}+x)$, $n\ge 0$ represent only particular families of orthogonal polynomials. Indeed, the recurrence relationship of theorem 3.8 is not generic, for the coefficient of $\pi_n(x)$ in this relationship is of the type $\alpha_n x$ while the general form of this coefficient is $\alpha_n x + \beta_n$. The following example shows that these polynomials may be seen as symmetric analogues of Bessel polynomials.

Example 3.12.

Let $\mathcal{E}$ be the alphabet formally defined by $S_k(\mathcal{E}) = 1/k!$, $k\ge 0$. It is a classical result that for any partition I of weight n, there holds $S_I(\mathcal{E}) = f_I/n!$, where f_I is the number of standard Young tableaux of shape I, *i.e.* the dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by I. It follows that the moments associated to the sequence $\big(\phi_k(x)\big)_{k\ge 0}$ are equal to $\nu(x^n) = (-2)^n/(n+1)!$, that is, are equal to the moments of Bessel polynomials (see for example **[Ch]**). The corresponding polynomials (suitably normalized)

$$y_n(x) = S_{12\ldots n}(\mathcal{E}+x)/S_{12\ldots n}(\mathcal{E}), \quad n\ge 0, \tag{10}$$

are therefore the *Bessel polynomials.* The first ones are

$$\begin{aligned}
y_0(x) =& 1,\\
y_1(x) =& 1+x,\\
y_2(x) =& 1+3x+3x^2,\\
y_3(x) =& 1+6x+15x^2+15x^3,\\
y_4(x) =& 1+10x+45x^2+105x^3+105x^4.
\end{aligned}$$

Formula (10) provides the following determinantal expression

$$y_n(x) = \prod_{1\le i\le n}(2(n-i)+1)^i \begin{vmatrix} 1/1! & 1/3! & 1/5! & \ldots & 1/(2n-1)! & (-x)^n \\ 1/0! & 1/2! & 1/4! & \ldots & 1/(2n-2)! & (-x)^{n-1} \\ 0 & 1/1! & 1/3! & \ldots & 1/(2n-3)! & (-x)^{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & & & & 1/n! & -x \\ 0 & & & & 1/(n-1)! & 1 \end{vmatrix} .$$

There exists a fairly extensive literature devoted to these polynomials. For a detailed account up to 1978, the interested reader is referred to [**Gr**]. We shall also mention the recent work of Dulucq and Favreau who have presented a combinatorial model for these polynomials, based upon weighted involutions [**DF**].

Note that our point of view provides at once a q-analogue for Bessel polynomials, by merely replacing $\mathcal{E}$ by the alphabet $\mathcal{E}_q = \{1, q, q^2, \ldots\}$. More precisely, setting $v = x/(1-q)$ we define

$$y_n(x,q) = q^{\binom{n}{2}} S_{12\ldots n}(\mathcal{E}_q + v)/S_{12\ldots n}(\mathcal{E}_q).$$

The $y_n(x,q)$ are orthogonal polynomials in x, whose coefficients are polynomials in q. The first ones are

$$\begin{aligned} y_0(x,q) =& 1, \\ y_1(x,q) =& 1 + x, \\ y_2(x,q) =& q + (1+q+q^2)x + (1+q+q^2)x^2, \\ y_3(x,q) =& q^3 + q(1+q^2)(1+q+q^2)x + (1+q+q^2)(1+q+q^2+q^3+q^4)x^2 \\ & + (1+q+q^2)(1+q+q^2+q^3+q^4)x^3. \end{aligned}$$

These q-analogues are equal to those obtained by Dulucq by means of the combinatorial model of Dulucq and Favreau [**D**].

4 Expression of a symmetric polynomial in terms of the power sums of odd degree

Let $\mathbb{X}$ denote a finite set of variables $\mathbb{X} = \{x_1, \ldots, x_n\}$. It is well known that any symmetric polynomial $F(\mathbb{X})$ may be expressed as a polynomial function of the power sums $\psi_k(\mathbb{X}) = \sum_{x \in \mathbb{X}} x^k$, $k = 1,2 \ldots n$. It was shown in the last century that $F(\mathbb{X})$ may also be expressed as a rational function of $\psi_k(\mathbb{X})$, $k = 1,3,5, \ldots 2n-1$ ([**Bo**], [**La**]). Pólya [**Po**] and Foulkes [**Fo**], among others, have proposed explicit expressions for some particular symmetric polynomials $F(\mathbb{X})$. For instance

Proposition 4.1. *The k^{th} elementary symmetric function is equal to $\Lambda_k(\mathbb{X}) = S_{\rho(n)/1^{n-k}}(\mathbb{X})/S_{\rho(n-1)}(\mathbb{X})$, the right-hand side being a rational function of the power sums of odd degree.*

Proof. For sake of completeness we sketch the proof. It is known that $S_{\rho(n-1)}(\mathbb{X}) = \prod_{i<j}(x_i + x_j)$. Taking an additionnal variable z we have also

$$S_{\rho(n)}(\mathbb{X} + z) = \prod_{i<j}(x_i + x_j) \prod_i (x_i + z) = S_{\rho(n-1)}(\mathbb{X}) \sum_k z^{n-k} \Lambda_k(\mathbb{X}) \, .$$

Thus, comparing the coefficients of z^{n-k}, we get $\Lambda_k(\mathbb{X}) = S_{\rho(n)/n-k}(\mathbb{X})/S_{\rho(n-1)}(\mathbb{X}) = S_{\rho(n)/1^{n-k}}(\mathbb{X})/S_{\rho(n-1)}(\mathbb{X})$. Finally, we recall that the staircase Schur function $S_{\rho(n)}$ and all its derivatives $S_{\rho(n)/I}$ depend only on the odd power sums ψ_{2p+1}, that is, belong to the subring generated by $\psi_1, \psi_3, \psi_5, \ldots$. $\square$

Now theorem 3.9 provides at once a similar expression for the complete symmetric functions $S_k(\mathbb{X})$.

PROPOSITION 4.2. *The k^{th} complete symmetric function is equal to*

$$S_k(\mathbb{X}) = \frac{\sum_{0\le 2i\le k} \Lambda_{\rho(n-1)+2i}(\mathbb{X}) S_{k-2i}(\mathbb{X})}{S_{\rho(n-1)}(\mathbb{X})} ,$$

the right-hand side being a rational function of the power sums of odd degree.

Proof. Theorem 3.9 shows that the function $\sum_{0\le 2i\le k} \Lambda_{\rho(n-1)+2i} S_{k-2i}$ may be expressed in terms of Schur functions of the type $S_{\rho(m)/1^p}$, which all belong to $\mathbb{Z}[\psi_1, \psi_3, \psi_5, \ldots]$. On the other hand, since $\mathbb{X}$ is finite of cardinality n, $\Lambda_{\rho(n-1)+2i}(\mathbb{X}) = 0$ for all $i > 0$. □

REFERENCES

[AM] M. ADLER AND J. MOSER, *On a class of polynomials connected with the Korteweg-de-Vries equation*, Commun. math. Phys. 61, (1978), 1–30.

[Bo] C.W. BORCHARDT, *Über eine Eigenschaft der Potenzsummen ungerader Ordnung*, Monastber. Akad. Berlin 1857, 301–311.

[Ch] T.S. CHIHARA, *Introduction to Orthogonal Polynomials*, Gordon and Breach, New-York, 1978.

[D] S. DULUCQ, *Un q-analogue des polynômes de Bessel*, Actes du 25-ème Séminaire Lotharingien de Combinatoire, 1990.

[DF] S. DULUCQ AND L. FAVREAU, *Un modèle combinatoire pour les polynômes de Bessel*, Actes du 25-ème Séminaire Lotharingien de Combinatoire, 1990.

[Fa] L. FAVREAU, *Combinatoire des tableaux oscillants et des polynômes de Bessel*, Publ. L.A.C.I.M., U.Q.A.M., Montréal, 1991.

[Fo] H.O. FOULKES, *Theorems of Pólya and Kakeya on power-sums*, Math. Zeitschr. 65, 345–352, 1956.

[Go] I. P. GOULDEN, *Quadratic forms of skew Schur functions*, Europ. J. Combinatorics (1988) **9**, 161–168.

[Gr] E. GROSSWALD, *Bessel Polynomials*, Springer, 1978.

[La] E. LAGUERRE, *Sur un problème d'algèbre*, Bull. Soc. Math. France 5, 26–30, 1877.

[LLT] A. LASCOUX, B. LECLERC ET J.Y. THIBON, *Une nouvelle expression des fonctions P de Schur*, C. R. Acad. Sci. Paris, t. 316, Série I, 221-224, 1993.

[LT] B. LECLERC ET J.Y. THIBON, *Analogues symétriques des polynômes de Bessel*, C. R. Acad. Sci. Paris, t. 315, Série I, 527–530, 1992.

[LP] A. LASCOUX AND P. PRAGACZ, *Ribbon Schur functions*, Europ. J. Combinatorics (1988) **9**, 561–574.

[LS] A. Lascoux and M.P. Schützenberger, *Formulaire raisonné de fonctions symétriques*, Publ. Math. Univ. Paris 7, 1985.

[Le] B. Leclerc, *On identities satisfied by minors of a matrix*, Adv. in Math., 100, (1993), 101-132.

[Mc] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Math. Monographs, 1979.

[Po] G. Pólya, *Remarques sur un problème d'algèbre étudié par Laguerre*, J. Math. Pur. Appl. 31, 37–47, 1951.

[Sz] G. Szegö, *Orthogonal polynomials*, A.M.S. Colloquium Publications, Vol. 23, Providence, RI, 1975.

[Tu] H.W. Turnbull, *The irreducible concomitants of the quadratics in n variables*, Transac. Cambridge Philos. Soc., xxi pp. 197–240, 1909.