On the Hadamard quotient of rational power series: an application*

M. Anselmo
Dipartimento di Informatica ed Applicazioni
Università di Salerno
84081 Baronissi (SA)

A. Bertoni
Dipartimento di Informatica ed Applicazioni
Università di Milano
20135 Milano

(Extended abstract)

Abstract

This work deals with the class $\mathcal{H}ad$ of Hadamard quotients of rational power series. Particular interest is devoted to power series with coefficients in $Q \ll z \gg$. Some considerations about derivation and some decidability results are given. It is then shown that the event defined by a 2PFA, the average time and its variance are all power series in $\mathcal{H}ad$, by using the derivation in $\mathcal{H}ad$.

1 Introduction

It is well-known that the Hadamard product of two rational power series is still a rational power series. The situation is more complicated when the Hadamard quotient is considered. The problem of stating whether the Hadamard quotient of two rational power series is rational has been solved only in a particular case: the Pisot's conjecture has been shown true in [?, ?] (see also [?]) after many years of researches.

*This work was partially supported by ESPRIT project ASMICS and project 40% MURST.

The Hadamard algebra of power series on a one-letter alphabet has been studied by B. Benzaghou in his thesys [?]. He studied some of its sub-algebras; in particular, the algebra obtained from rational power series and its rationality preserving operations. Other works related with the Hadamard quotient are [?, ?, ?, ?]. The most of these works deal with power series on one-letter alphabets and consider some analytical problems such as some partial solutions to the Pisot's conjecture or its possible extensions.

Here we consider problems of a different kind. We investigate the class $\mathcal{H}ad(K, \Sigma)$ which is the closure of rational power series on an arbitrary finite alphabet by the Hadamard inverse, as introduced in [?]. Some decidability results on $\mathcal{H}ad(K, \Sigma)$ are shown. Our interest is then devoted to power series with coefficients in $Q \ll z \gg$. We consider the derivative series of a series in $\mathcal{H}ad(Q \ll z \gg, \Sigma)$ and show that it is still in $\mathcal{H}ad(Q \ll z \gg, \Sigma)$.

An application is given to transient two-way probabilistic finite automata (2PFA). A 2PFA is a machine governed by a probabilistic finite-state control that can scan an input string moving the reading head both directions. The definition of 2PFA was introduced by Kuklin in [?]. Some fundamental works on 2PFA's are [?, ?].

We show that the event defined by a 2PFA, the average time of a 2PFA and its variance are power series in $\mathcal{H}ad(Q, \Sigma)$. This result is obtained by extending a construction in [?] concerning only the events defined by 2PFA's and by applying the results about derivation in $\mathcal{H}ad(Q \ll z \gg, \Sigma)$.

2 Definitions

Given a finite alphabet Σ, let $< \Sigma^*, ., \varepsilon >$ be the free monoid generated by it. Given a semiring K, the class $K \ll \Sigma \gg$ of *formal power series* in non-commuting variables in Σ and coefficients in K is the set of functions $s : \Sigma^* \to K$. As usual, the value of s on $w \in \Sigma^*$ is denoted by (s, w) and referred to as the *coefficient* of the series. The power series is written as a formal sum $s = \sum_{w \in \Sigma^*}(s, w)w$. The family of K-*rational* power series on Σ is the smallest family containing the polynomials and closed with respect to sum, Cauchy product and *.

The *Hadamard product* of s, $t \in K \ll \Sigma \gg$ is defined as $s \otimes t = \sum_{w \in \Sigma^*}(s, w)(t, w)w$. It is well-known that the family of K-*rational* power series is closed under Hadamard product when K is a commutative semiring. The identity element for $\otimes$ is the power series $1 = \sum_{w \in \Sigma^*} w$.

The *Hadamard quotient* s/t of s by t is the Hadamard product of s by the Hadamard inverse of t, whenever it exists. In fact, the Hadamard inverse of s exists iff $(s, w) \neq$

$0\ \forall w \in \Sigma^*$; in this case it is the series $1/s = \sum_{w\in\Sigma^*} 1/(s,w)w$.

Let $Rat(\Sigma, K)$ be the class of K-rational power series with respect to $+$, $\otimes$. Remark that the class $Rat(\Sigma, K)$ is not closed under the Hadamard inverse. For example, the Hadamard inverse of $s = 1 + \sigma^*\sigma\sigma^* = 1 + \Sigma n\sigma^n$ exists but it is not in $Rat(\Sigma, K)$.

Definition 1 *$\mathcal{H}ad(K, \Sigma)$ is the smallest family containing $Rat(K, \Sigma)$ and closed with respect to the Hadamard inverse.*

The class $\mathcal{H}ad(K, \Sigma)$ is characterized as follows.

Fact 1 *$\mathcal{H}ad(K, \Sigma) = \{\varphi/\psi$ such that $\varphi,\ \psi \in Rat(K, \Sigma)$ and $\psi(w) \neq 0\ \forall w \in \Sigma^*\}$.*

3 Some results on $\mathcal{H}ad$

Some decidability results on $\mathcal{H}ad(K, \Sigma)$ can be easily shown starting from those on $Rat(K, \Sigma)$, (see [?]). In particular:

(i) it is undecidable whether or not a series in $\mathcal{H}ad(K, \Sigma)$ admits its Hadamard inverse;

(ii) it is decidable whether or not two series in $\mathcal{H}ad(K, \Sigma)$ are equivalent (i.e. have same coefficients);

(iii) the coefficients of a given series in $\mathcal{H}ad(K, \Sigma)$ are computable.

Let us now consider the class $\mathcal{H}ad(Q \ll z \gg, \Sigma)$ of power series with coefficients in $Q \ll z \gg$.

Definition 2 *Let $s \in \mathcal{H}ad(Q \ll z \gg, \Sigma)$.*

The derivative of s is $s' = \sum_{w\in\Sigma^*} d/dz(s,w)w$.

Fact 2 *$\mathcal{H}ad(Q \ll z \gg, \Sigma)$ is closed with respect to derivation.*

Proof (outline) We have the following identities: $(s+t)' = s' + t'$, $(st)' = s't + st'$, and $(s^*)' = s^*s's^*$. This shows that $Rat(Q \ll z \gg, \Sigma)$ is closed w.r.t. derivation. The goal follows from the consideration that $(\varphi/\psi)' = (\varphi' \otimes \psi - \varphi \otimes \psi')/(\psi \otimes \psi)$, where "/" means the Hadamard quotient.$\diamond$

4 An application to 2PFA's

A *two-way probabilistic finite automaton* (2PFA) consists of a probabilistic finite-state control and an input tape which is scanned by a single two-way head, that is, the head can move both left and right.

More formally, a 2PFA $\mathcal{A}$ over an alphabet Σ is a 4-tuple $\mathcal{A} =< Q, \pi, \{A(\sigma, L), A(\sigma, R)/\sigma \in \Sigma\}, \eta >$, where $Q = \{q_1, q_2, \ldots, q_m\}$ is a finite set of states; π is a $1 \times m$ stochastic vector representing the initial probabilities; $A(\sigma, L)$ and $A(\sigma, R)$ represent the transition probabilities and are $m \times m$ matrices with nonnegative real entries such that $A(\sigma, L) + A(\sigma, R)$ is stochastic $\forall \sigma \in \Sigma$; η is a $m \times 1$ vector with 0 and 1 entries, interpretable as characteristic vector of the final states.

A *computation* on input w begins with the head scanning the leftmost symbol of w in state q_i with probability η_i and runs according to the transition probabilities given by matrices $A(\_,\_)$. In a computation it is forbidden to move off the left end of the input. The *event* defined by $\mathcal{A}$ is $P_{\mathcal{A}} : \Sigma^* \to [0, 1]$ where $P_{\mathcal{A}}(w)$ is the probability that the automaton in a computation on $w \in \Sigma^*$ moves off the right end entering a final state.

Given $w \in \Sigma^*$ and $1 \leq t \leq \mid w \mid$, state q is said (t, w)- *absorbing* if the automaton beginning in state q scanning the t-th symbol of w does not admit any computation moving off some end with a non zero probability but there are an infinite number of computations reentering state q scanning the t-th symbol of w. If no state is absorbing, the automaton is said *transient*.

Let $p_n(w)$ be the probability that w is accepted by $\mathcal{A}$ in n steps.

Theorem 1 *Given a 2PFA $\mathcal{A}$ on Σ the power series*

$$f_{\mathcal{A}} = \textstyle\sum_{w \in \Sigma^*} \sum_{n \in N} p_n(w) z^n w$$

belongs to $\mathcal{H}ad(Q \ll z \gg, \Sigma)$.

Proof (outline) For all $w = a_1 a_2 \cdots a_n \in \Sigma^*$ consider the following transition diagram. The states are all couples (i, s) where $0 \leq i \leq n + 1$ and s is a state of the given 2PFA. From state (i, s) to state (j, r) there is a transition labelled pz, where z is a variable, if p is the probability that $\mathcal{A}$ scanning the i-th symbol of w in state s enters state r scanning the j-th symbol (in fact $|j - i| = 1$).

Let $M_w(z)$ be the transition matrix of this diagram; it has entries in $Q \ll z \gg$. We have $\sum_{n \in N} p_n(w) z^n = (M_w(z)^*)_{1,n+1}$. We need now some more considerations than in the case of [?]. Firstly, observe that matrix $M_w(z)^* = (Id - M - w(z))^{-1}$ and the inverse can be computed as the quotient between some cofactors and the determinant, as by Cauchy formula, because the entries of $Id - M_w(z)$ are only of the form az for some $a \in Q$ or 1 and they thus commute. Secondly, matrix $M_w(z)^*$ always exists because $Id - M_w(z)$ has always an invertible determinant.

The main observation is now that the power series asssociating with a word w the determinant of the so constructed matrix $Id - M_w(z)$ is a power series in $\mathcal{H}ad(Q \ll z \gg$

$,\Sigma)$; and similarly for the cofactors. This can be proved extending the proof given in [?].$\diamond$

Let $AVT(w) = \sum_{n\in N} np_n(w)$ be the average time to compute w and $VAR(w) = \sum_{n\in N} n^2 p_n(w) - (\sum_{n\in N} np_n(w))^2$ be its variance.

Corollary 1 *The event defined by a transient 2PFA, the average time and its variance are all in $\mathcal{H}ad(Q,\Sigma)$.*

Proof (outline) Let $h_1 : Q \ll z \gg \to Q$ be the evaluation in 1, that is the homomorphism mapping z to 1. If $s \in \mathcal{H}ad(Q \ll z \gg, \Sigma)$ then $h_1 s \in \mathcal{H}ad(Q,\Sigma)$, because h_1 is a semiring homomorphism. The goal now follows from Fact 2 and from:

(i) $P_{\mathcal{A}} = h_1 \cdot f_{\mathcal{A}}$;

(ii) $AVT = h_1 \cdot (f_{\mathcal{A}})'$;

(iii) $VAR = h_1 \cdot (f_{\mathcal{A}})' + h_1 \cdot (f_{\mathcal{A}})'' - (h_1 \cdot f'_{\mathcal{A}})^2$.

Note that only with transient 2PFA's we are sure that the evaluation h_1 of $f_{\mathcal{A}}$ and its derivatives is defined, that is it is not a quotient with a 0 denominator.$\diamond$

Corollary 2 *It is decidable whether two transient 2PFA's define the same event and whether a string is accepted by a given 2PFA with probability greater than a given value.*

Remark that Corollaries 1 and 2 extend to the case of nontransient 2PFA's, but this case needs a more complex proof. The proof that events defined by transient and nontransient 2PFA's are in $\mathcal{H}ad(Q,\Sigma)$ is shown in [?].

References

[1] M. Anselmo and A. Bertoni, Two-way Probabilistic Automata and Rational Power series, *Proc. IV Conv. It. Inform. Teor.*, L'Aquila 28/30 Ottobre 1992, World Scientific, 9-23

[2] B. Benzaghou, Algèbres de Hadamard, *Bull. Soc. Math. France*, 98 (1970) 209-252

[3] J. Berstel, Sur les pôles et le quotient de Hadamard de séries N-rationnelles, *C. R. Acad. Sc. Paris* 272, série A (1971), 1079-1081

[4] J. P. Bézivin, Quotient de fonctions entières et quotients de Hadamard de séries formelles, *Ann. Inst. Fourier*, Grenoble, 39 (3) (1989), 737-752

[5] C. Dwork and L. Stockmeyer, A time complexity gap for two-way probabilistic finite-state automata, *SIAM J. of Comput.* 19 (6), (1990) 1011-1023

[6] R. Freivalds, Probabilistic Two-way Machines, in *Proc. Int. Symp. on Math. Found. of Comp. Sc.*, LNCS 118, (1981) 333-45

[7] K. Gérardin, Quotient de Hadamard de series rationnelles en variables non commutatives, *Proc. Ecole de Printemps d'Inform. Théor.* 23/27 may 1977, 119-134

[8] Yu. I. Kuklin, Two-way Probabilistic Automata, *Avtomatika i vycislitelnaja tekhnika*, 5 (1973), 35-36 (Russian)

[9] G. Pathiaux, Algèbres de Hadamard de fractions rationnelles, *C. R. Acad. Sc. Paris*, 267 (1968) série A, 977-979

[10] Y. Pourchet, Solution du problème arithmétique du quotient de Hadamard de deux fractions rationnelles, *C. R. Acad. Sc. Paris* 288, série A (1979), 1055-1057

[11] R. Rumely, Notes on Van der Poorten's proof of the Hadamard quotient theorem, *Séminaire de Th. de Nombres de Paris*

[12] A. J. Van der Poorten, Solution de la conjecture de Pisot sur le quotient de Hadamard de deux fractions rationnelles, *C. R. Acad. Sc. Paris* 306, série I (1988), 97-102