FPSAC 2011, Reykjavík, Iceland DMTCS proc. **AO**, 2011, 191–198

Shortest path poset of Bruhat intervals

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Abstract. Let $[u, v]$ be a Bruhat interval and $B(u, v)$ be its corresponding Bruhat graph. The combinatorial and topological structure of the longest u-v paths of $B(u, v)$ has been extensively studied and is well-known. Nevertheless, not much is known of the remaining paths. Here we describe combinatorial properties of the shortest u-v paths of $B(u, v)$. We also derive the non-negativity of some coefficients of the complete $\mathbf{cd}$-index of $[u, v]$.

Résumé. Soit $[u, v]$ un intervalle de Bruhat et $B(u, v)$ le graphe de Bruhat associé. La structure combinatoire et topologique des plus longs chemins de u à v dans $B(u, v)$ est bien comprise, mais on sait peu de chose des autres chemins. Nous décrivons ici les propriétés combinatoires des plus courts de chemins de u à v. Nous prouvons aussi que certains coefficients du $\mathbf{cd}$-indice complet de $[u, v]$ sont positifs.

Keywords: Bruhat interval, shortest-path poset, complete $\mathbf{cd}$-index

1 Introduction

While the paths of the Bruhat graph $B(u, v)$ of the Bruhat interval $[u, v]$ only depend on the isomorphism type of $[u, v]$ (see (Dye91)), all of the u-v paths of $B(u, v)$ are needed to compute the $\widetilde{R}$-polynomial, as well as the complete $\mathbf{cd}$-index of $[u, v]$. Unfortunately, the structure of $B(u, v)$ is not easy to understand. Thus we focus on the shortest paths of $B(u, v)$, since their combinatorial structure is more manageable. In particular, they form a Hasse diagram of a poset, which we denote by $SP(u, v)$.

The order of the paper is as follows: In Section 2 we summarize the basic properties of $SP(u, v)$, and describe their structure two specific cases: (i) if W is finite, with $u = e$ and $v = w_0^W$ (longest-length element of W) and (ii) if the number of rising chains (under a reflection order) is one. In Section 2.3 we provide an algorithm that allows us to separate the chains in $SP(u, v)$ into subposets, each of which has properties resembling properties of $[u, v]$. In Section 3 we derive consequences of the work done to the complete $\mathbf{cd}$-index.

1.1 Basic definitions

Let (W, S) be a Coxeter system, and let $T \stackrel{\text{def}}{=} T(W) = \{wsw^{-1} : s \in S, w \in W\}$ be the set of *reflections* of (W, S). The *Bruhat graph* of (W, S), denoted by $B(W, S)$ or simply $B(W)$, is the directed graph with vertex set W, and a directed edge $w_1 \to w_2$ between $w_1, w_2 \in W$ if $\ell(w_1) < \ell(w_2)$ and there exists $t \in T$ with $tw_1 = w_2$. Here ℓ denotes the *length function* of (W, S). The edges of $B(W)$ are labeled

[†]The author was supported in part by NSF grant DMS-0555268.

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by reflections; for instance the edge $w_1 \to w_2$ is labeled with t. The Bruhat graph of an interval $[u, v]$, denoted by $B(u, v)$, is the subgraph of $B(W)$ obtained by only considering the elements of $[u, v]$. A *path* in the Bruhat graph $B(u, v)$, will always mean a *directed* path from u to v. As it is the custom, we will label these paths by listing the edges that are used. Furthermore, we denote the set of paths of length k in $B(u, v)$ by $B_k(u, v)$.

A *reflection order* $<_T$ is a total order of T so that $r <_T rtr <_T rtrtr <_T \ldots <_T trt <_T t$ or $t <_T trt <_T trtrt <_T \ldots <_T rtr <_T r$ for each Coxeter system $(\langle r, t\rangle, \{r, t\})$ where $r, t \in T$. Let $\Delta = (t_1, t_2, \ldots, t_k)$ be a path in $B(u, v)$, and define the *descent set* of Δ by $D(\Delta) = \{j : t_{j+1} <_T t_j\} \subset [k-1]$. If $D(\Delta) = \emptyset$, we say that Δ is rising.

Let $w(\Delta) = x_1 x_2 \cdots x_{k-1}$, where $x_i = \mathbf{a}$ if $t_i < t_{i+1}$, and $x_i = \mathbf{b}$, otherwise. In other words, set x_i to $\mathbf{a}$ if $i \notin D(\Delta)$ and to $\mathbf{b}$ if $i \in D(\Delta)$. Billera and Brenti (BB) showed that $\sum_{\Delta \in B(u,v)} w(\Delta)$ becomes a polynomial in the variables $\mathbf{c}$ and $\mathbf{d}$, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the *complete* $\mathbf{cd}$-*index* of $[u, v]$, and it is denoted by $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. Notice that the complete $\mathbf{cd}$-index of $[u, v]$ is an encoding of the distribution of the descent sets of each path Δ in the Bruhat graph of $[u, v]$, and thus seems to depend on $<_T$. However, it can be shown that this is not the case. For details on the complete $\mathbf{cd}$-index, see (BB).

As an example, consider S_3 with generators $s_1 = (1 \;\; 2)$ and $s_2 = (2 \;\; 3)$. Then $t_1 = s_1 <_T t_2 = s_1 s_2 s_1 <_T t_3 = s_2$ is a reflection ordering. The paths of length 3 are: $(t_1, t_2, t_3), (t_1, t_3, t_1), (t_3, t_1, t_3)$, and (t_3, t_2, t_1), that encode to $\mathbf{a^2} + \mathbf{ab} + \mathbf{ba} + \mathbf{b^2} = \mathbf{c^2}$. There is one path of length 1, namely t_2, which encodes simply to 1. So $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \mathbf{c^2} + 1$.

Given a monomial $m \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$, we denote the coefficient of m in $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ by $[m]_{u,v}$. Notice that $[\mathbf{c}^n]_{u,v}$ is the number of rising paths in $B_{n+1}(u, v)$.

2 Shortest path poset

We begin with some basic properties of $SP(u, v)$.

Proposition 2.1 *Let $[u, v]$ be a Bruhat interval, then the undirected edges of the shortest paths of $B(u, v)$ form the Hasse diagram of a poset.*

We point out that in general the edges of paths in $B_k(u, v)$ need not form a Hasse diagram of a poset. Indeed, it is possible to have elements $u \leq x_0 < x_1 < x_2 < x_3 \leq v$ so that $x_0 \to x_1 \to x_2 \to x_3$ and $x_0 \to x_3$ are all in $B(u, v)$.

We call the poset of Proposition 2.1 , the *shortest path poset* of $[u, v]$, and we denote it by $SP(u, v)$. Furthermore, the edges of the Hasse diagram of $SP(u, v)$ inherit the labels of the corresponding edges in $B(u, v)$. In particular, we say that a maximal chain C in $SP(u, v)$ is *rising* if the path corresponding to C in $B(u, v)$ is rising.

Proposition 2.2 *$SP(u, v)$ is a graded poset, and for $x \in SP(u, v)$, the rank of x is the length of the shortest u-x path in $B(u, x)$.*

To illustrate the definition consider B_2 and $SP(e, \underline{1}\,\underline{2})$ as depicted in Figure 1. Notice that the rank of $SP(e, \underline{1}\,\underline{2})$ is 2, the length of the shortest paths in $B(B_2)$.

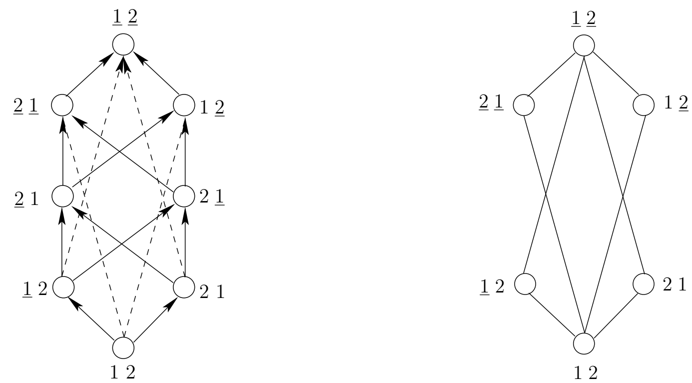

Fig. 1: $B(B_2)$ and $SP(B_2)$.

2.1 Finite Coxeter groups

For any finite Coxeter group W, there is a word w_0^W of maximal length. It is a well-known fact that $\ell(w_0^W) = |T|$. For any $w \in W$, one can write $t_1 t_2 \cdots t_n = w$ for some $t_1, t_2, \ldots, t_n \in T$. If n is minimal, then we say that w is *T-reduced*, and that the *absolute length* of w is n. We write $\ell_T(w) = n$.

Notice that for $w \in W$, if $\ell_T(w) = m$, then $t_1 t_2 \cdots t_m = w$ for some reflections in T, but this *does not* mean that $(t_1, t_2, \ldots, t_m)$ is a (directed) path in $B(e, w)$. Nevertheless, for finite W and $w = w_0^W$, $(t_1, t_2, \ldots, t_m)$ and any of its permutations $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(m)})$, $\tau \in A_{m-1}$, are paths in $B(W)$ (see Theorem 2.3 below).

Let $SP(W)$ denote the poset $SP(e, w_0^W)$. The combinatorial structure of $SP(W)$ was described in (Bla09). For the sake of completeness, we include the main results therein.

Theorem 2.3 *Let W be a finite Coxeter group and $\ell_0 = \ell_T(w_0^W)$, the absolute length of the longest element of W. Then $SP(W)$ is isomorphic to the union of Boolean posets of rank ℓ_0. Each copy of $B(\ell_0)$ share at least e and w_0^W*

We summarize the number of Boolean posets that form $SP(W)$ and the rank of $SP(W)$ for each finite Coxeter group in Table 1.

where

$$b_n = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!} \prod_{i=0}^{j-1} \binom{n-2i}{2}.$$

and

$$d_m = \frac{1}{\lfloor \frac{m}{2} \rfloor !} \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-2i}{2}$$

where $m = n$ if n is even, and $m = n-1$ if n is odd.

We point out that the union of the Boolean posets could share more elements than e and w_0^W. For instance, consider $SP(B_3)$ below.

Tab. 1: Finite coxeter groups W, rank$(SP(W))$, and the number of Boolean posets in $SP(W)$

W	rank$(SP(W))$	α_W = # of Boolean posets in $SP(W)$
A_{n-1}	$\lfloor \frac{n-1}{2} \rfloor$	1
B_n	n	b_n
D_n	n if n is even; $n-1$ if n is odd	d_n
$I_2(m)$	2 if m is even; 1 if m is odd	$\frac{m}{2}$ if m is even; 1 if m is odd
F_4	4	24
H_3	3	5
H_4	4	75
E_6	4	3
E_7	7	135
E_8	8	2025

While some elements other than e and $w_0^{B_3}$ are shared by more than one Boolean poset, each maximal chain belongs to a *unique* Boolean poset.

2.2 One rising chain

Since $[u, v]$ is *EL-shellable* (see (BW82) and (Dye93)), then $[u, v]$ has a unique maximal chain that is rising. So it is reasonable to study the structure of $SP(u, v)$ under the assumption that there is a unique rising chain. Even though this seems to be a strong assumption, there are several examples of Bruhat intervals where $SP(u, v)$ has a unique rising chain; for instance, $[21435, 53241]$.

An important tool in our study are the $\widetilde{R}$*-polynomials*, defined below.

Definition 2.4 ($\widetilde{R}$-polynomials) *Let $s \in S$ so that $\ell(vs) < \ell(v)$. Then define $\widetilde{R}_{u,v}(\alpha)$ by*

$$\widetilde{R}_{u,v}(\alpha) = \begin{cases} \widetilde{R}_{us,vs}(\alpha) & \text{if } \ell(us) < \ell(u), \\ \widetilde{R}_{us,vs}(\alpha) + \alpha \widetilde{R}_{u,vs}(\alpha) & \text{if } \ell(us) > \ell(vs). \end{cases}$$

Dyer (Dye01) provided an interpretation of $\widetilde{R}_{u,v}(\alpha)$ in terms of the number of rising paths of $B(u, v)$. Namely,

$$\widetilde{R}_{u,v}(\alpha) = \sum_{\substack{\Delta \in B(u,v) \\ D(\Delta)=\emptyset}} \alpha^{\ell(\Delta)}.$$

With this interpretation in mind, we have

Proposition 2.5 $\widetilde{R}_{u,y}(\alpha)\widetilde{R}_{y,v}(\alpha) \leq \widetilde{R}_{u,v}(\alpha)$.

We point out, in passing, that the above proposition generalizes Theorem 5.4, Corollary 5.5 and Theorem 5.6 in (Bre97) .

Proposition 2.5 yields the following theorem.

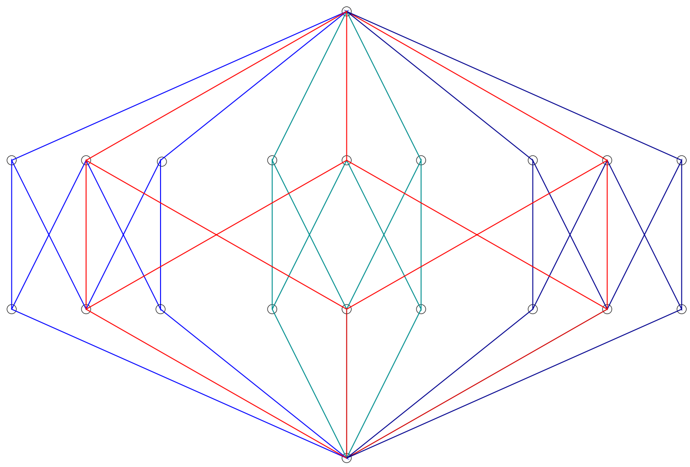

Fig. 2: $SP(B_3)$ has 4 copies of $B(3)$. Notice these copies intersect, but each maximal chain is in a unique Boolean poset.

Theorem 2.6 *If $SP(u, v)$ has a unique rising chain, then*
(a) $SP(u, v)$ is EL-shellable.
(b) $SP(u, v)$ is thin*, i.e., every subinterval of length two of $SP(u, v)$ has four elements.*

These topological properties will have consequences on the complete $\mathbf{cd}$-index, and it will be discussed in Section 3.

2.3 Flip algorithm

Let $k + 1 \stackrel{\text{def}}{=} \text{rank}(SP(u, v))$. An important distinction between $[u, v]$ and $SP(u, v)$ is that $[u, v]$ has a unique maximal, rising chain whereas $SP(u, v)$ could have more than one. So we propose an algorithm that splits the chains of $SP(u, v)$ into $[\mathbf{c}^k]_{u,v}$ posets P_i, $i = 1, \ldots, [\mathbf{c}^k]_{u,v}$. The structure of each P_i is easier to understand than $SP(u, v)$. So far we have been shown that the P_i have properties that resemble those of $[u, v]$.

We now follow (BB) to define the *flip* of $\Gamma \in B_2(u, v)$. Let (t_1, t_2) and (r_1, r_2) be in $B_2(u, v)$. We say that $(t_1, t_2) \leq_{lex} (r_1, r_2)$ if $t_1 <_T r_1$ or if $t_1 = r_1$ and $t_2 <_T r_2$, or $t_2 = r_2$. The existence of the complete $\mathbf{cd}$-index implies that there are as many paths with empty descent set in $B_2(u, v)$ as those with descent set $\{1\}$. Order all the paths in $B_2(u, v)$ lexicographically and let

$$r(\Gamma) = |\{\Delta \in B_2(u, v) : D(\Delta) = D(\Gamma), \Delta \leq_{lex} \Gamma\}|.$$

Definition 2.7 *With everything as above, we define the* flip *of Γ is the $r(\Gamma)$-th Bruhat path in $\{\Delta \in B_2(u, v) \mid D(\Delta) \neq D(\Gamma)\}$ ordered by $\leq_{lex}$. We denote this path by flip(Γ).*

Given $\Delta = (t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_k) \in B_k(u, v)$, we denote the path $(t_1, t_2, \ldots, t'_i, t'_{i+1}, \ldots, t_k)$, where $flip(t_i, t_{i+1}) = (t'_i, t'_{i+1})$, by $\text{FLIP}_i(\Delta)$. We are now ready to describe our algorithm.

The pseudocode of FLIP is given in Algorithm 1. In a few words, FLIP returns a (directed) graph G whose vertices are the maximal chains of $SP(u, v)$ and (C, C') is an edge if $\text{FLIP}_j(C) = C'$, where

Algorithm 1 FLIP$(SP(u,v))$

$G := (V, E)$, with V is the set of chains of $B(SP(u,v))$ and $E := \emptyset$.
$T := V$
for C a maximal chain of $SP(u,v)$ **do**
  if $D(C) \neq \emptyset$ **then**
    $i := \min D(C)$
    $C' :=$ FLIP$_i(C)$
    Add edge (C, C') to E.
  end if
end for
return G

$j = \min\{D(C)\}$. Notice that G has $[\mathbf{c}^k]_{u,v}$ connected components, say $G_1, G_2, \ldots, G_{[\mathbf{c}^k]_{u,v}}$. We define P_i to be the poset $SP(u,v)$ with all the chains (represented by vertices) *not* in G_i removed.

Let us illustrate FLIP with the following example. Notice that the chains in $SP(u,v)$ are represented by the labels assigned to the corresponding edges in the $B(u,v)$.

Example 1 *Consider the 10 elements of* $B_3(1234, 4312)$. *Then the output of* FLIP *is depicted below. In the first column we have the two components of* G, *and in the right column the posets* P_i *corresponding to each component.*

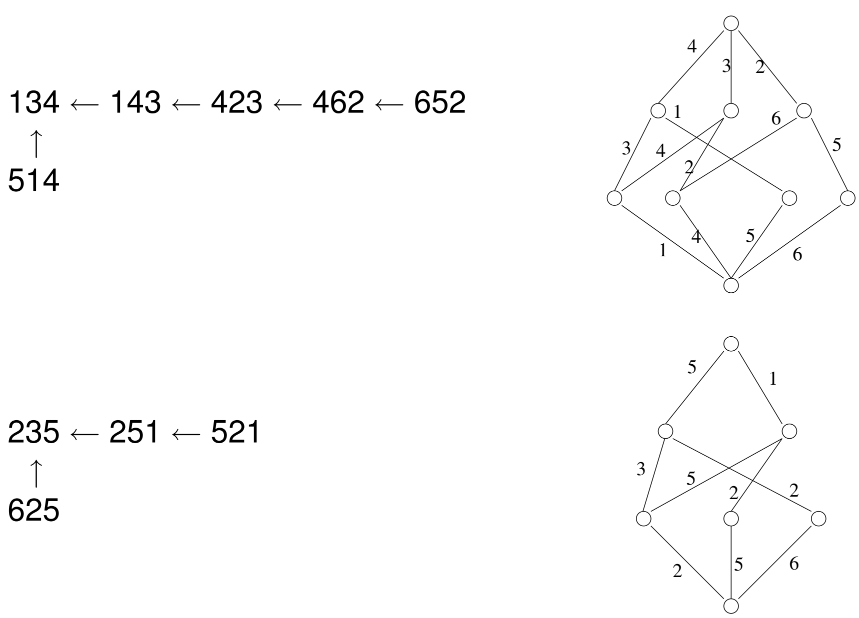

Fig. 3: On the left, we find the output of FLIP: two connected components. On the right the corresponding posets are depicted.

Each P_i satisfies properties resembling those of Bruhat intervals. Concretely, we have

Proposition 2.8 *(a) P_i is graded.*
(b) Every subinterval of P_i has at most one rising chain.
(c) Every subinterval of length two of P_i has at most two coatoms.

Bruhat intervals satisfy the properties above once we replace "at most" with "exactly".

2.4 FLIP *applied to A_n, B_n and D_n*

When applied to A_{n-1}, the output of FLIP is a unique graph G and the corresponding poset P is simply $SP(A_{n-1})$. Furthermore, one can choose a reflection order for the reflections of B_n (see (Bla11)) so that FLIP outputs b_n copies of $B(n)$ (see Table 1). For instance, FLIP($SP(B_3)$) separates $SP(B_3)$ into four copies of $B(3)$ (see Figure 2, where the four copies are drawn with different colors). The same holds, mutatis mutandis, for D_n.

So in these cases, FLIP produces the expected results: it divides $SP(W)$ into α_W subposets $P_1, \ldots, P_{\alpha_W}$ (where α_W is given in Table 1), and each P_i is a Boolean poset.

3 Connections to the complete $\mathbf{cd}$-index

In (Bla09), it is shown that the lowest-degree terms of $\widetilde{\psi}_{e,w_0^W}(\mathbf{c},\mathbf{d})$ are non-negative. Thus we have the theorem below.

Theorem 3.1 *If W is a finite Coxeter group, then he lowest degree terms of $\widetilde{\psi}_{e,w_0^W}(\mathbf{c},\mathbf{d})$ are nonnegative.*

In fact, these terms can be computed quite easily (see (Bla09) for details).

Now under the assumption of Theorem 2.6, $SP(u,v)$ is EL-shellable and thin. Thus Theorem 3.1.12 in (Wac07) yields the following proposition.

Proposition 3.2 *If $SP(u,v)$ has a unique rising chain, then it is a Gorenstein* poset.*

Now as a consequence of (Kar06, Theorem 4.10), we have the following theorem.

Theorem 3.3 *If $SP(u,v)$ has a unique rising chain, then the lowest degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ are non-negative.*

Moreover, in the case rank$(SP(u,v)) = 2$, the posets P_i described before Example 1 contribute a non-negative quantity to the lowest degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$. We hope to extend this result to rank$(SP(u,v)) = 3$.

Acknowledgements

I want to thank Louis Billera for many helpful discussions, as well as kind advice and encouragement throughout the years. I also want to thank Matthew Dyer for his hospitality during my visit to Notre Dame and his willingness to discuss his work.

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