ON (SOME) FUNCTIONAL EQUATIONS ARISING IN ENUMERATIVE COMBINATORICS

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ABSTRACT. I shall try to convince the audience not to be shy with functional equation approaches in enumerative combinatorics. They not only solve (some) problems, but they often teach us a lot too. The proofs they provide for a specific problem might be less nice than more combinatorial proofs. But functional equation approaches sometimes give a unified description of apparently distinct problems, and the efforts we make to solve one specific functional equation often teach us what to do in a more generic case. The talk will be based on recent examples dealing with very classical combinatorial objects: lattice paths, maps, permutations.

RÉSUMÉ. Cet exposé se veut un plaidoyer en faveur de l'utilisation des équations fonctionnelles en combinatoire énumérative. Elles permettent tout d'abord de démontrer des résultats, ce qui n'est pas rien. On leur reproche souvent de fournir des preuves moins agréables et moins convaincantes que d'autres preuves dont la combinatoire est plus riche. Ce reproche me paraît justifié. En contrepartie, les efforts fournis pour résoudre une équation fonctionnelle donnée obligent à apprendre, voire à inventer, de nouvelles techniques, susceptibles de s'adapter ensuite à une grande variété de problèmes d'origine disparate. L'exposé sera construit à partir d'exemples récents portant sur des objets combinatoires classiques : cartes, chemins, permutations.

1. INTRODUCTION

When asked what I have been doing recently (mathematics-wise), I often hesitate between two answers: "counting things" or "playing with functional equations". The truth is, of course, that I have been counting things *by* playing with functional equations.

The solution of an enumerative problem often consists in two steps. The first one is purely combinatorial: it consists in examining carefully the structure of the objects under consideration. The objective is to discover some property (for instance, the existence of a recursive description for the objects) that can be translated into a mathematical, non-tautological information on the number a_n of objects of size n: a recurrence relation, a determinantal formula, a functional equation for the generating function $\sum_n a_n t^n$, etc. The second step is naturally to solve this recurrence relation/functional equation.

Let us take a very classical example: rooted Cayley trees (Figure 1). One soon realizes that such a tree is formed by a root to which a set of rooted trees is attached. This recursive description translates into the following functional equation for the exponential generating function $T(t)$ of trees:

$$T(t) = t\exp(T(t)).$$

The nth coefficient in $T(t)$ can be derived from this equation by the Lagrange inversion formula:

$$\begin{aligned}[t^n]T(t) &= \frac{1}{n}[t^{n-1}]\exp(nt)\\ &= \frac{n^{n-1}}{n!}\end{aligned}$$

so that the number of rooted trees of size n is n^{n-1}. This simple formula certainly calls for a more combinatorial explanation, which can be given in the form of a Prüfer code (see [18, p. 25]). Still, the first proof has allowed us to *discover* and *prove* the result, and has forced us to understand two important tools: the exponential formula and the Lagrange inversion formula, which have a much wider scope of applications than the Prüfer code, and allow a unified, systematic treatment of many combinatorial problems.

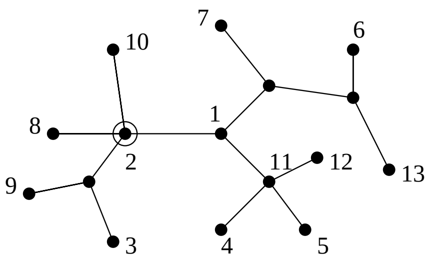

FIGURE 1. A rooted Cayley tree.

To quote H. Wilf, "bijective proofs give one a certain satisfying feeling that one really understands why a theorem is true" [19]. I believe one should not neglect proofs that lie at the other end of the scale: they allow us to attack problems of which we only have a very naive, elementary combinatorial understanding. They might yield strange functional equations; but solving these functional equations is likely to teach us some tools that will be useful in many other circumstances.

Of course, we shall not deal here with all types of functional equations that arise in combinatorial enumeration. The first (and most polished) part of the talk will be devoted to a unified treatment of a class of equations whose instances have popped up naturally in combinatorial enumeration since, at least, the sixties. These equations usually define bivariate generating functions. The second part will be devoted to various extensions of this class, and will essentially consist of examples, since no general results seem to be available for the moment.

2. A class of functional equations defining bivariate power series

2.1. **A simple specimen.** One of the simplest equation in this class is obtained by counting sub-diagonal North-East paths. We are mostly interested in their length. But, being naive combinatorialists, we adopt a simple step-by-step construction of these paths: this forces us to take into account an additional parameter, namely the distance of the endpoint of the path to the main diagonal (Figure 2). We thus obtain the following equation:

$$F(t,u) = 1 + t\,(u + 1/u)\,F(t,u) - \frac{t}{u}F(t,0), \tag{1}$$

which defines the generating function $F(t,u)$ for these paths, counted by their length (variable t) and their distance to the main diagonal (variable u).

Let us make two observations:

- This equation uniquely defines $F(t,u)$ as a power series in t with polynomial coefficients in u. But there are others, non-power-series solutions, for instance the rational function
$$F(t,u) = \frac{2tu-1}{2t(u-t(u^2+1))}.$$
Hence, any method for solving (1) must *use* the fact that $F(t,u)$ is a power series.
- It is not obvious to derive from (1) an equation in $F(t,0)$ only; we cannot set directly $u=0$, and if we first muliply by u, and then set $u=0$, we end up with
$$0 = tF(t,0) - tF(t,0),$$
a tautology.

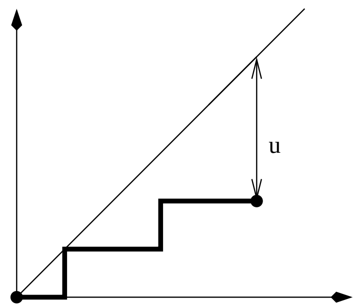

FIGURE 2. A sub-diagonal North-East path.

All the equations of the class we are going to study share the two above properties: they define uniquely all their unknowns as power series in t, and become tautological when we try to solve them in the most obvious way. Observe that (1) has another strange feature: it is linear (in $F(t,u)$ and in $F(t,0)$), but we strongly expect its solution to be algebraic, and to satisfy

$$F(t,0) = \frac{1-\sqrt{1-4t^2}}{2t^2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n},$$

since sub-diagonal walks ending on the main diagonal are well-known to be counted by Catalan numbers. Linearity is, however, not a generic property of the class of functional equations under consideration.

2.2. **The general case.** The generic form of our equations is

$$P(F(t,u), F_1(t), F_2(t), \ldots, F_k(t), t, u) = 0 \tag{2}$$

where P is a polynomial in $k+3$ variables with real coefficients. We assume that this equation defines uniquely all its "unknowns" as power series in t: the series $F_i(t)$ have real coefficients, while $F(t,u)$ has its coefficients in $\mathbb{R}[u]$. Eq. (1) belongs to this class, since it reads

$$(u - t(u^2+1))F(t,u) - u + tF_1(t) = 0.$$

Note that setting $u=0$ forces

$$F_1(t) = F(t,0).$$

Other recent examples include the enumeration of two-stack sortable permutations [20],

$$F(t,u) = \frac{1}{1-tu} + \frac{tu}{(1-u)^2}\left(F(t,u) - F_1(t)\right)\left(uF(t,u) - F_1(t)\right), \tag{3}$$

which, after multiplying by $(u-1)^2$ and setting $u=1$, forces

$$F_1(t) = F(t,1),$$

and the enumeration of 3-*constellations* (a special family of planar maps, see Figure 3):

$$\begin{aligned} F(t,u) = 1 + tuF(t,u)^3 \quad &+ \quad \frac{tu}{u-1}\left(F(t,u) - F_1(t)\right)\left(2F(t,u) + F_1(t)\right) \\ &+ \quad \frac{tu}{(u-1)^2}\left(F(t,u) - F_1(t) - (u-1)F_2(t)\right), \end{aligned}$$

which, once multiplied by $(1-u)^2$ and differentiated with respect to u, provides

$$F_1(t) = F(t,1) \quad \text{and} \quad F_2(t) = \frac{\partial F}{\partial u}(t,1).$$

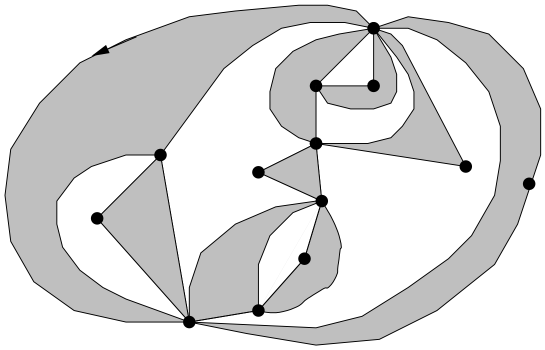

FIGURE 3. A rooted 3-constellation.

Actually, the enumeration of planar maps seems to be the first topic that led, in the 60's and 70's, to equations of the form (2); see for instance the section "Rooted planar maps and the quadratic method" in [13]. Most of these equations were (at most) quadratic in $F(t,u)$, with only one additional series $F_1(t)$. Several authors independently developed methods to solve those instances. The main two techniques are now known as the *kernel method* for the linear case [9, 12, 16, 17] and the *quadratic method* for the quadratic case, mainly with $k=1$ [8, 13, p. 138].

We shall present an approach for solving *any equation of the form* (2). This approach encompasses the previously developed techniques. We shall predict, if not prove, that the solution $F(t,u)$ is always an algebraic function of t and u. For instance, we shall obtain, for the paths of Eq. (1),

$$F(t,0) = 1 + t^2 F(t,0)^2,$$

for the two-stack sortable permutations of Eq. (3),

$$t^2F(t,1)^3 + t(2+3t)F(t,1)^2 + (3t^2 - 14t + 1)F(t,1) + 11t - 1 + t^2,$$

and for the 3-constellations of Eq. (4),

$$81t^2F(t,1)^3 - 9t(-2+9t)F(t,1)^2 + (1 - 66t + 27t^2)F(t,1) - 1 + 47t - 3t^2.$$

From these algebraic equations, one derives, in the first example,

$$F(t,0) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n},$$

in the second one,

$$F(t,1) = 1 + 2\sum_{n\geq 1} \frac{1}{(n+1)(2n+1)}\binom{3n}{n} t^n, \tag{4}$$

and in the third one,

$$F(t,1) = 1 + 2\sum_{n\geq 1} \frac{3^{n-1}}{(n+1)(2n+1)}\binom{3n}{n} t^n.$$

There is no doubt that these simple formulae call for more combinatorial explanations. Such explanations have actually been provided, at least to a certain extent: see [11] for paths, [5] for constellations, and [10, 14] for two different recursive descriptions of bijections between two-stack sortable permutations and non-separable planar maps, which are known to be counted by (4) (see [7]). There is no doubt either that the functional equation approach has allowed us to *prove* these results, and that it is applicable to a wider class of problems for which no simple formula exists.

3. Three variables, and more: examples and open questions

3.1. "Elementary" linear equations. Let us first kill the hope that any reasonable three-variable extension of equation (2) will systematically yield algebraic generating functions. The following linear equation [4]

$$F(t,u,v) = 1 + t(v^2/u + u^2/v)F(t,u,v) - t^2uvF(t,u,v) - \frac{tu^2}{v}F(t,u,0) - \frac{tu^2}{u}F(t,0,v)$$

defines a unique power series $F(t,u,v)$ in t, with polynomial coefficients in u and v. It can be shown to satisfy

$$F(t,0,1) = \sum_{n\geq 0}(-1)^n t^{3(2^n-1)}.$$

This one-variable series $f(t)$ is neither algebraic, nor D-finite, nor even differentially algebraic: in other words, it does not satisfy any non-trivial polynomial equation in $t, f(t), f'(t), f''(t), \ldots, f^{(k)}(t)$, for any value of k.

However, similar equations do have a D-finite solution. For instance, the following equation was recently studied in connection with the dichromatic polynomial of certain planar maps [1]:

$$F(t,u,v) = 1 + tv\frac{uF(t,u,v) - F(t,1,v)}{u-1} + tu\frac{vF(t,u,v) - F(t,u,1)}{v-1},$$

and it was proved (or rather: checked) that its solution satisfies

$$F(t,1,1) = 2\sum_{n\geq 0}\frac{t^n}{(n+2)^2(n+1)}\sum_{i=1}^{n+1}\binom{n+2}{i-1}\binom{n+2}{i}\binom{n+2}{i+1},$$

which is a D-finite series (see [18, Chap. 6] for the definition and elementary properties of D-finite series). These examples suggest the following natural question:

> Take a linear relation between $F(t,u,v)$, $F(t,u,0)$, $F(t,0,v)$ and 1. Assume that it uniquely defines its three unknowns as power series in t, with polynomial coefficients in u and v. When, and why, is the solution algebraic? D-finite? differentially algebraic?

The kernel method [4] shows that this question boils down to the nature of series $f(t,u)$ satisfying equations of the form

$$f(t,u) = A(t,u) + B(t,u)f(t,\Phi(t,u)),$$

where A, B and Φ are algebraic functions of t and u.

3.2. Specific examples. Another, less systematic approach is to focus on instances of three-variate functional equations that are known (or conjectured) to have algebraic solutions. An example is provided by the enumeration of 2143-avoiding (or *vexillary*) involutions. The technique of generating trees gives the functional equation

$$\left(1 + \frac{t^2u^2v}{1-u} + \frac{t^2v}{1-v}\right)F(t,u,v) = \frac{t^2u^2v^2}{(1-tv)(1-tuv)} + t\left(1 + \frac{tv}{1-v}\right)F(t,uv,1) + \frac{t^2u^2v}{1-u}F(t,1,v).$$

The solution turns out to be algebraic, and to satisfy [15]:

$$F(t,1,1) = \frac{1-t-\sqrt{(1+t)(1-3t)}}{2t^2} = \sum_{n\geq 0}M_n t^n,$$

where M_n is the nth Motzkin number.

Another example is the enumeration of planar lattice walks avoiding a half-line (Figure 4). A naive step-by-step construction yields the functional equation:

$$F(t,u,v) = 1 + t\,(u+1/u)\,(v+1/v)\,F(t,u,v) - \frac{2t}{u}F_1(t) - 2tuF_2(t,u),$$

where

$$F_1(t) = [uv]F(t,u,v) := \sum_{n\ge 0} t^n [uvt^n]F(t,u,v),$$

and

$$F_2(t,u) = [u^{<}v]F(t,u,v) := \sum_{n\ge 0}\sum_{i<0} t^n u^i [u^i v t^n]F(t,u,v).$$

This strange equation is actually not difficult to solve, and one obtains [3, 6]:

$$F(t,u,v) = \frac{\left(1-8t^2(1+1/u^2)+\sqrt{1-16t^2}\right)^{1/2}}{\sqrt{2}(1-t(u+1/u)(v+1/v))}.$$

Moreover, this approach can be generalized in a straightforward manner to walks taking their steps in any subset of $\mathbb{Z}^2$, so that it does not cost much more to solve, for instance, the models of Figure 5.

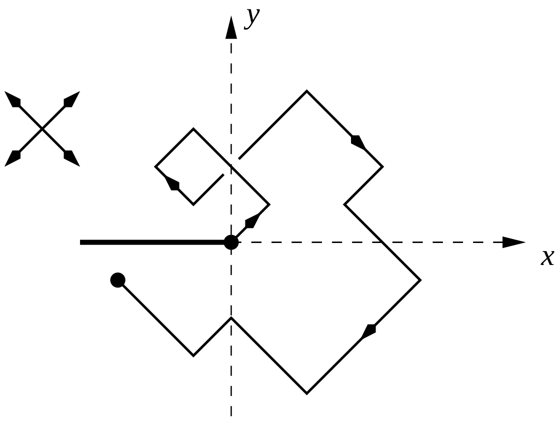

FIGURE 4. A path on the diagonal square lattice avoiding a half-line.

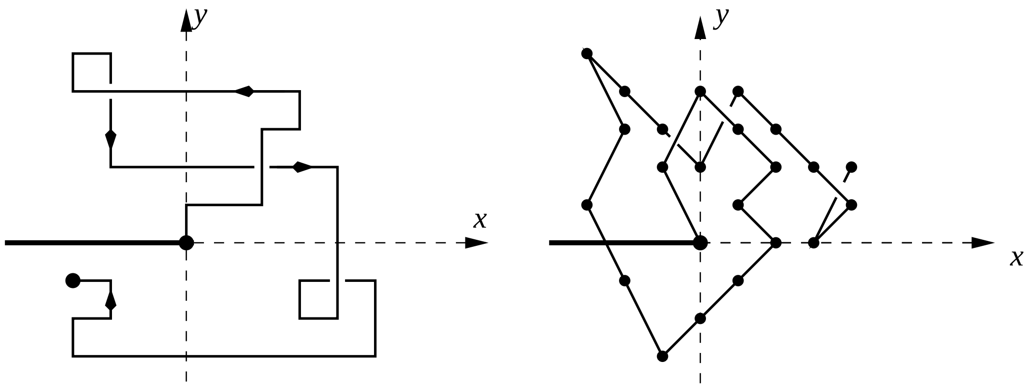

FIGURE 5. Two variations on the slit plane model.

3.3. **q-Analogues.** Another three-variable problem would be to find a q-analogue of the method of Section 2. This question is motivated, for instance, by the enumeration of two-stack sortable permutations by (a variation of) their inversion number. Instead of Eq. (3), one has to solve [2]:

$$F(t,u) = \frac{1}{1-u} + t\,[1+uF(tq,u)]\ \frac{F(t,u)-F(t,0)}{u},$$

and we would need a q-analogue of the quadratic method to solve this equation (one can see that the case $q=1$ is equivalent, up to a simple change in the unknown function, to (3)). More precisely, it is an open question to determine whether $F(t,0)$ is q-algebraic (meaning that a polynomial equation relates $t, q, F(t,0), F(tq,0), \ldots, F(tq^k,0)$ for some value of k).

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