THE CHROMATIC NUMBER OF THE KNESER GRAPH

IMRE BÁRÁNY

ABSTRACT. There exist several different proofs for Kneser's conjecture. Almost all of them use topology. In this talk I will explain some of the proofs, the ideas behind them, and further applications of algebraic topology in combinatorics and geometry.

RÉSUMÉ. Il existe plusieures différantes démonstrations pour l'hypothèse de Kneser. La plupart utilise la topologie algébraïque. Au cours de cette lecture je vais parler de ces démonstrations, des idées utilisées, et quelques autres applications de la topologie algébraïque au sujets combinatorielles et géometriques.

1. KNESER GRAPHS

An early and extremely spectacular application of algebraic topology in combinatorics is Lovász's proof [4] of Kneser's conjecture. The latter says [3] that if the k-tuples of an n element set (with $n \geq 2k-1$) are partitioned into $n-2k+1$ classes, then one of the classes contains two disjoint k-tuples. It is easy and useful to restate the problem in the language of graph, or hypergraph, theory.

So let H be a hypergraph with finite vertex set X, and define the Kneser graph $KG(H)$ whose vertices are the edges in H, with two vertices $E, F \in H$ connected if $E \cap F = \emptyset$. For the hypergraph consisting of all the k element subsets of an n element set, the corresponding the Kneser graph is denoted by $KG(n,k)$. In this setting Kneser's conjecture just asks whether the chromatic number of $KG(n,k)$ is $n-2k+2$.

It is easy to see that $KG(n,k)$ has a proper colouring with this many colours: Assume the ground set is $\{1,2,\dots,n\}$, and colour vertices, that is, k-tuples by their largest element if this largest element is at least $2k$, and colour by colour 1 if the largest element is at most $2k-1$. This is a proper colouring using exactly $n-2k+2$ colours. The difficulty lies in showing that the chromatic number is at least $n-2k+2$.

2. USING TOPOLOGY

Kneser's conjecture, which has been a theorem since 1978, has several proofs, all of them based on algebraic topology. Lovász's original proof [4] first constructs a simplicial complex, the so-called neighbourhood complex $N(G)$, for any graph G. Then it shows that if $N(G)$ is d-connected, then the chromatic number of G, $\chi(G)$, is at least $d+3$. The proof of this result is based on Borsuk's theorem. There is a further difficulty to show that the neighbourhood complex of $KG(n,k)$ is highly connected. The $\chi(G) \geq d+3$ theorem has further applications.

There are "elementary" proofs as well, due to Bárány [1], and Dol'nikov [2]. Both of them use Borsuk's theorem and a geometric realization of the vertex set of the underlying hypergraph H. Bárány's proof was extended by Schrijver [7] to find a vertex critical subgraph of $KG(n,k)$.

Dol'nikov's short and elegant argument starts with defining the *chromatic defect* of the hypergraph H, $\mathrm{cd}(H)$, as the minimum size of a set $Y \subset X$ such that $X \setminus Y$ can be two-coloured so that none of the edges of H disjoint from Y is monochromatic. The following general theorem is true [2]. For every hypergraph H, we have

$$\chi(KG(H)) \geq \mathrm{cd}(H).$$

Supported by a grant from the Hungarian National Foundation T 032542.

This implies Kneser's conjecture since, as it is very easy to see, the chromatic defect of hypergraph consisting of all k-subsets of an n-set is at least $n-2k+2$. The proof of Dol'nikov's theorem is based on a natural representation of the ground set of H in R^d where $d = \chi(KG(H))$, and an application of Borsuk's theorem, again. The method can be used to establish lower bounds on chromatic number of Kneser hypergraphs.

Sarkaria [6] proves a general theorem which needs too much preparation to be given here. It is a powerful result implying, among other things, Kneser's conjecture and the Van Kampen-Flores non-realizability theorem.

3. A combinatorial proof

Very recently, J. Matoušek [5] has given a purely combinatorial proof for Kneser's conjecture. His paper contains, in fact, two proofs. The first relies on Tucker's lemma, which is a combinatorial version of Borsuk's theorem, or, if you wish, the combinatorial core of Borsuk's theorem. In the second proof even this combinatorial core is avoided: a direct colouring and path-following algorithm is applied. This is an interesting approach yet I think that in general algebraic topology is going to play a direct and important role in combinatorics.

In the talk some of these proofs will be explained, and further recent applications of algebraic topology in geometry and combinatorics will be discussed.

References

[1] Bárány, I., "A short proof of Kneser's conjecture." *J. Comb. Theory A.* **25**(1978), 325–326.
[2] Dol'nikov, V. L., "Transversals of families of sets in R^d and a connection between the Helly and Borsuk Theorems." *Russian Acad. Sci. Sb. Math.* **79**(1994), 93–107.
[3] Kneser, M. Aufgabe 260. *Jahresbericht Deutschen Math. Ver.* **58**(1955), 27.
[4] Lovász, L., "Kneser's conjecture, chromatic number and homotopy." *J. Comb. Theory A.* **25**(1978), 319–324.
[5] Matoušek, J., "A combinatorial proof of Kneser's conjecture." *Combinatorica.* to appear (2001).
[6] Sarkaria, K., "A generalized Kneser conjecture." *J. Comb. Theory B.* **49**(1990), 236–240.
[7] Schrijver, A., "Vertex critical subgraphs of Kneser graphs." *Nieuw Arch. Wisk.* **26**(1978), 454–461.

E-mail address: `barany@renyi.hu`

Rényi Institute, PoB 127, 1364 Budapest