(-1)–ENUMERATION OF PLANE PARTITIONS WITH COMPLEMENTATION SYMMETRY

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Abstract. We compute the weighted enumeration of plane partitions contained in a given box with complementation symmetry where adding one half of an orbit of cubes and removing the other half of the orbit changes the weight by -1 as proposed by Kuperberg in [8, pp.25/26]. We use nonintersecting lattice path families to accomplish this for transpose–complementary, cyclically symmetric transpose–complementary and totally symmetric self–complementary plane partitions. For symmetric transpose–complementary and self–complementary plane partitions we get partial results.
Résumé.

1. Introduction

A plane partition P can be defined as a finite set of points (i,j,k) with $i,j,k>0$ and if $(i,j,k)\in P$ and $1\le i'\le i$, $1\le j'\le j$, $1\le k'\le k$ then $(i',j',k')\in P$. We interpret these points as midpoints of cubes and represent a plane partition by stacks of cubes (see Figure 1). If we have $i\le a$, $j\le b$ and $k\le c$ for all cubes of the plane partition, we say that the plane partition is contained in a box with sidelengths a,b,c.

Plane partitions were introduced by MacMahon. One of his main results is the following [11, Sec. 429, $q\to 1$, proof in Sec. 494]:

The number of all plane partitions contained in a box with sidelengths a,b,c equals

$$B(a,b,c)=\prod_{i=1}^{a}\frac{(c+i)_b}{(i)_b}, \tag{1}$$

where $(a)_n:=a(a+1)(a+2)\dots(a+n-1)$ is the usual shifted factorial.
A plane partition can have several kinds of symmetries which we now list.
A plane partition P is called

- *symmetric* if whenever $(i,j,k)\in P$ then also $(j,i,k)\in P$,
- *cyclically symmetric* if whenever $(i,j,k)\in P$ then also $(j,k,i)\in P$,
- *totally symmetric* if it is both symmetric and cyclically symmetric.

A plane partition P contained in the box $a\times b\times c$ is called

- *self–complementary* if whenever $(i,j,k)\in P$ then $(a+1-i,b+1-j,c+1-k)\notin P$.

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1991 *Mathematics Subject Classification.* Primary 05A15; Secondary 05A19 05B45 33C20 52C20.
Key words and phrases. lozenge tilings, rhombus tilings, plane partitions, determinants, pfaffians, nonintersecting lattice paths.

- *transpose–complementary* if whenever $(i,j,k) \in P$ then $(a+1-j, a+1-i, c+1-k) \notin P$.

The various combinations of these symmetries lead to ten symmetry classes (cf. [18]). It is known that for each of the ten classes the number of plane partitions contained in that class is given by a "nice" closed formula (see [18, 9, 3, 22]). Additionally, the four symmetry classes without complementation admit a natural q–enumeration. In the case of no symmetry this is also a result of MacMahon. The weight is just $q^{\#\text{cubes}}$. If one counts symmetric plane partitions or cyclically symmetric plane partitions with respect to this weight then one obtains nice closed formulas as well (see [1, 2, 13]). The second weight which has been considered for the symmetry classes without complementation is $q^{\#\text{orbits}}$. (Here, we mean orbits with respect to the symmetries of the applicable class.) Aside of the case of no symmetries (clearly, in this case the weights $q^{\#\text{cubes}}$ and $q^{\#\text{orbits}}$ are identical), there exist nice closed formulas for the enumeration of symmetric plane partitions (see [1]) and, conjecturally, for totally symmetric plane partitions (see [18]).

Amazingly, upon setting $q=-1$ in these enumerations we get the *plain enumeration* of plane partitions with complementation symmetry (see [20, 21]). If we consider the plane partitions in a symmetry class with complementation symmetry, then there seems to be no natural q–enumeration (except in the case of self–complementary plane partitions [18]). In particular, counting these plane partitions with respect to $q^{\#\text{cubes}}$ gives nothing new because this statistic is constant for all the plane partitions in the symmetry class. (Obviously, counting with respect to $q^{\#\text{orbits}}$ makes no sense as well.) However, a natural (-1)–enumeration for plane partitions with complementation symmetry has been recently proposed by Kuperberg (cf. [8, pp.25/26]).

This (-1)–enumeration is defined as follows: A plane partition with complementation symmetry contains exactly one half of each orbit. Let a move consist of removing one half of an orbit and adding the other half. Two plane partitions are connected either by an odd or by an even number of moves, so it is possible to define a relative sign. The sign becomes absolute if we assign a certain plane partition the weight 1.

For example, in the case of transpose–complementary plane partition this can be realized by counting the number $n(P)$ of cubes contained in the upper half of the plane partition P and doing the enumeration $\sum_P (-1)^{n(P)}$.

In [8], Kuperberg conjectures that this (-1)–enumeration has a nice expression for all the six symmetry classes with complementation symmetry. He gives the result for the case of transpose–complementary plane partitions derived by the method of "forcing planarity". This result is stated below in Theorem 1 and proved in a different way. Kuperberg has also found a proof for the case of cyclically symmetric self–complementary plane partitions. The same method could be used to prove Theorem 6 below except for the sign.

The main purpose of the present paper is to prove Kuperberg's conjecture in almost all other cases. We determine the expressions for the (-1)–enumeration of cyclically symmetric transpose–complementary plane partitions and totally symmetric self–complementary plane partitions. We get partial results depending on the parity of the sidelengths for symmetric transpose–complementary plane partitions and self–complementary plane partitions.

All these results are stated in Theorems 1–7 below. In fact, as it turns out, Kuperberg's conjecture is only partially "true," because in the case of symmetric transpose–complementary plane partitions there is one case in which there is apparently no compact expression for the (-1)–enumeration.

We now present our results.

Theorem 1 (Kuperberg). *The enumeration of transpose–complementary plane partitions in a box with sides $a \times a \times 2b$ with weight $(-1)^{n(P)}$ equals 0 for b odd and a even and*

$$\prod_{j=1}^{\lceil a/2 \rceil - 1} \frac{(\lfloor b/2 \rfloor + j)(a-j)_b}{(j)_{b+1}}$$

else, where $(a)_n$ denotes the shifted factorial $a(a+1)\dots(a+n-1)$.

Theorem 2. *The enumeration of symmetric transpose–complementary plane partitions in a box with sides $2\alpha \times 2\alpha \times 2b$ with weight $(-1)^{n(P)}$ equals*

$$\begin{array}{rl} \dfrac{\prod_{k=1}^{\frac{\alpha}{2}} (b+2k)_{\alpha-1}}{\prod_{k=1}^{\frac{\alpha}{2}} (2k)_{\alpha-1}} & \text{for } \alpha \text{ even and } b \text{ even,} \\ \dfrac{\prod_{k=1}^{\frac{\alpha-1}{2}} (b+2k)_{\alpha}}{\prod_{k=1}^{\frac{\alpha-1}{2}} (2k)_{\alpha}} & \text{for } \alpha \text{ odd and } b \text{ even,} \\ 0 & \text{for } b \text{ odd.} \end{array}$$

Theorem 3. *The enumeration of symmetric transpose–complementary plane partitions in a box with sides $(2\alpha+1) \times (2\alpha+1) \times 2b$ with weight $(-1)^{n(P)}$ has the form*

$$\prod_{k=1}^{\alpha/2} (b/2+k)_{\alpha/2+1} (b+2\alpha+2) \cdot p(b) \quad \text{for } \alpha \text{ even and } b \text{ even,}$$

and similar expressions for the other parities of α and b. Here, p is a polynomial of degree $(\alpha/2)^2$.

Theorem 4. *The enumeration of cyclically symmetric transpose–complementary plane partitions in a box with sides $2\alpha \times 2\alpha \times 2\alpha$ with weight $(-1)^{n(P)}$ equals*

$$\begin{array}{rl} \left(\displaystyle\prod_{k=1}^{(\alpha-1)/2} \frac{(6k-2)!}{(2k+\alpha-1)!} \right)^2 & \text{for } \alpha \text{ odd,} \\ 0 & \text{else.} \end{array}$$

Theorem 5. *The enumeration of totally symmetric self–complementary plane partitions in a box with sides $2\alpha \times 2\alpha \times 2\alpha$ with weight $(-1)^{n(P)}$ equals*

$$\begin{array}{rl} \displaystyle\prod_{k=1}^{(\alpha-1)/2} \frac{(6k-2)!}{(2k+\alpha-1)!} & \text{for } \alpha \text{ odd,} \\ 0 & \text{else.} \end{array}$$

This is also the number of vertically symmetric alternating sign matrices of size $(\alpha+2)$.

Theorem 6. *For even a, b, c, the enumeration of self–complementary plane partitions in a box with sides $a \times b \times c$ with weight $(-1)^{n(P)}$ equals*

$$B\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right),$$

where $B(a,b,c)$ is defined in Equation (1).

Theorem 7 (Kuperberg)**.** *The (-1)–enumeration of cyclically symmetric self–complementary plane partitions in a box with sides $2\alpha \times 2\alpha \times 2\alpha$ with weight $(-1)^{n(P)}$ is the square root of the ordinary enumeration, that is*

$$\pm \prod_{k=0}^{\alpha-1} \frac{(3k+1)!}{(\alpha+k)!}.$$

This is also the number of alternating sign matrices (see [25]), the number of totally symmetric self–complementary plane partitions [22] and the number of descending plane partitions. Results for small values of α suggest that the sign is $+1$ for all α.

Thus, the only cases that are still open are the case of symmetric transpose–complementary plane partitions in a box with two odd sides (in which case no nice formula seems to exist) and the case of self–complementary plane partitions in a box with at least one odd side. Here, the enumeration seems to have a nice closed form. One subcase is stated in the following conjecture.

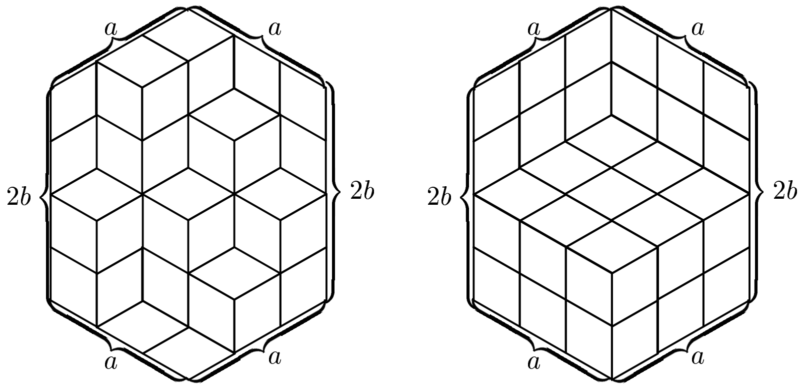

FIGURE 1. Two transpose–complementary plane partition. The one on the right has weight 1.

Conjecture. *For $a \equiv 0 \pmod 4$ and $b \equiv c \equiv 3 \pmod 4$, the enumeration of self–complementary plane partitions in a box with sides $a \times b \times c$ with weight $(-1)^{n(P)}$ equals up to sign*

$$B\left(\tfrac{a}{4}, \tfrac{b+1}{4}, \tfrac{c+1}{4}\right)^2 B\left(\tfrac{a}{4}, \tfrac{b-3}{4}, \tfrac{c+1}{4}\right) B\left(\tfrac{a}{4}, \tfrac{b+1}{4}, \tfrac{c-3}{4}\right)$$

where $B(a, b, c)$ is defined in Equation (1).

The other subcases of a even and b and c odd suggest similar expressions. The remaining case are the self–complementary plane partitions with a odd and b and c even. Also there, the (-1)–enumeration seems to have a nice closed form but we did not bother to work out precise conjectures.

We prove Theorems 1–6 by adjusting a well-known bijection between plane partitions and families of nonintersecting lattice paths. In the cases of transpose–complementary plane partitions, cyclically symmetric transpose–complementary plane partitions and totally symmetric self–complementary plane partitions, these path families can be enumerated by a determinant given by the Gessel–Viennot method.

In the case of symmetric transpose–complementary, self–complementary and cyclically symmetric self–complementary plane partitions the path families can be enumerated by a sum of minors that can be expressed as a Pfaffian by a theorem of Okada (see Lemma 11).

The resulting determinants and Pfaffians are then shown to be equal to the expressions given in the theorems.

Kuperberg's proof of Theorem 7 uses a correspondence between plane partitions and perfect matchings and the Hafnian–Pfaffian method (see [8]) to express the enumeration as a Pfaffian and compares this to the Pfaffian of a known enumeration.

Theorem 1 is proved in Section 2. The proofs of Theorem 2 and 3 are sketched in Section 3. The proof of Theorem 4 is sketched in Section 5. The proof of Theorem 5 is sketched in Section 6. The proof of Theorem 6 is sketched in Section 4.

A more detailed version of this paper can be found at `arXiv:math.CO/0011175`.

2. Transpose–complementary plane partitions

The aim of this section is to compute the enumeration of transpose–complementary plane partitions contained in a given box with weight $(-1)^{n(P)}$, where $n(P)$ is the number of cubes of P in the upper half of the box (cf. the paragraphs preceding Theorem 1). By the definition of transpose–complementary, the box must have sidelengths of the form $a \times a \times 2b$.

Step 1: From plane partitions to rhombus tilings

The first step is using a well–known bijection between plane partitions and rhombus tilings to convert the problem to a tiling enumeration problem. (In this paper by rhombus we always mean a rhombus consisting of two equilateral triangles of sidelength 1.) The rhombus tiling of a hexagon with sides

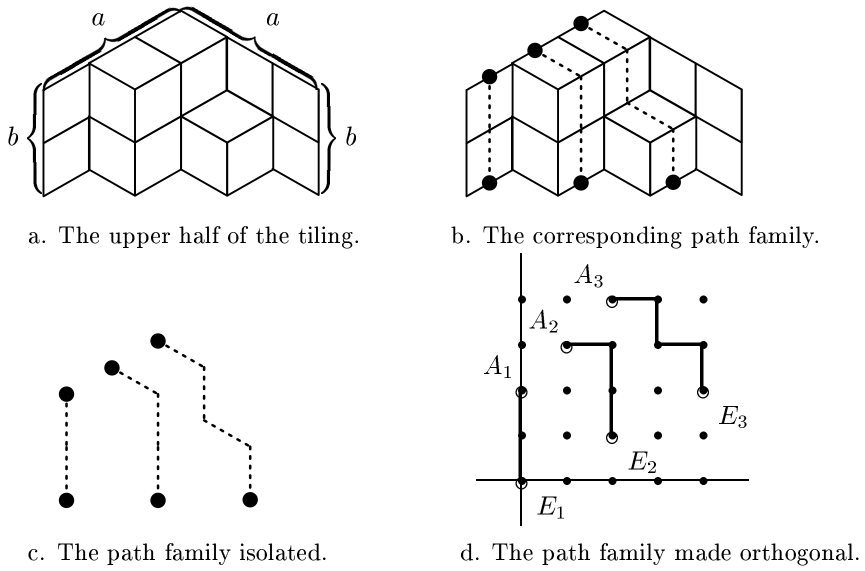

FIGURE 2.

$a, a, 2b, a, a, 2b$ corresponding to a given plane partition is simply a projection of the 3–dimensional stack of cubes to the plane (see Figure 1). It is easy to see that transpose–complementary plane partitions correspond to rhombus tilings with a horizontal symmetry axis.

Step 2: From rhombus tilings to families of nonintersecting lattice paths.
We use a variant of a well-known translation of rhombus tilings to families of nonintersecting lattice paths. We start with a rhombus tiling with a horizontal symmetry axis (see Figure 1). Clearly, the symmetry axis must be covered by horizontal rhombi. Since the tiling is symmetric we need only consider the rhombi lying above the symmetry axis (see Figure 2a). We mark the midpoints of the edges along the upper left side of length a and of the parallel edges on the zig-zag line (see Figure 2b). We connect these points by paths which follow the rhombi of the tiling as illustrated in Figure 2b. Clearly, the resulting paths are nonintersecting, i.e., no two paths have a common vertex. If we slightly distort the underlying lattice, we get orthogonal paths with south and east steps (see Figure 2d).

We can introduce a coordinate system in a way such that the coordinates of the starting points A_i and end points E_j are

$$A_i = (i-1, b+i-1), \qquad i = 1, \dots, a, \tag{2}$$

$$E_j = (2j-2, j-1), \qquad j = 1, \dots, a. \tag{3}$$

We have to find a description of the weight $(-1)^{n(P)}$ in terms of the path families. A horizontal rhombus in the tiling can be viewed as the top square of a stack of cubes in the plane partition interpretation. Therefore, it is assigned the weight $(-1)^n$, where n is the number of cubes (i, j, k) with $k > b$ below the square. In terms of paths this is the same as assigning to a path p the weight $(-1)^{\mathrm{area1}(p)}$ where $\mathrm{area1}(p)$ is the area between the path p and the horizontal line containing the lowest point of the path. We want to use instead the weight $(-1)^{\mathrm{area2}(p)}$ where $\mathrm{area2}(p)$ is the area between the path p and the x–axis to be able to apply Lemma 8 below. When we do this, we make an overall error of $(-1)^{1^2+2^2+\cdots+(a-1)^2} = (-1)^{a(a-1)/2}$. Hence, we have to count families of nonintersecting lattice paths with starting points A_i (see (2)), end points E_j (see (3)) and weight $(-1)^{\mathrm{area2}(p)}$ and multiply the result by $(-1)^{a(a-1)/2}$.

Step 3: From lattice paths to a determinant
By the main theorem on nonintersecting lattice paths (see [10, Lemma 1] or [4, Theorem 1]) the weighted count of such families of paths can be expressed as a determinant. We employ the notation $\mathcal{P}(A_i \to E_j)$ for the weighted count of paths from A_i to E_j and $\mathcal{P}(\mathbf{A} \to \mathbf{E}, \text{nonint.})$ for the weighted count of families of nonintersecting lattice paths with the ith path running from A_i to E_i, $i = 1, 2, \dots, n$. Note that the weight of a path is the product of the weights of its steps. Now we can state the theorem for paths with south and east steps in the integer lattice.

Lemma 8. *Let $A_1, A_2, \dots, A_n, E_1, E_2, \dots, E_n$ be integer points meeting the following condition: Any path from A_i to E_l has a common vertex with any path from A_j to E_k for any i, j, k, l with $i < j$ and $k < l$.*

Then we have

$$\mathcal{P}(\mathbf{A} \to \mathbf{E}, \text{nonint.}) = \det_{1 \le i,j \le n} (\mathcal{P}(A_i \to E_j)).$$

This is still applicable if some of the points are isolated, i.e., unconnected to any other point.

The lemma is clearly applicable to the enumeration problem formulated in Step 2 since area2 is the sum of the areas between each horizontal step and the x–axis. Now we have to determine the matrix entries $\mathcal{P}(A_i \to E_j)$ with A_i and E_j as in (2) and (3) and the weight $(-1)^{\text{area2}(p)}$. It is well-known that the enumeration of paths from (a, b) to (c, d) with weight $q^{\text{area1}(p)}$ is the q–binomial coefficient $\left[\begin{smallmatrix} c-a+b-d \\ c-a \end{smallmatrix}\right]_q$. The q–binomial coefficient (see [17, p. 26] for further information) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{j=n-k+1}^{n} (1 - q^j)}{\prod_{j=1}^{k} (1 - q^j)}.$$

Although it is not obvious from this definition, the q–binomial coefficient is a polynomial in q. So it makes sense to put $q = -1$. Since we want to use $(-1)^{\text{area2}(p)}$ there is an additional factor of $(-1)^{(j-1)(2j-1-i)}$. In summary, the matrix entries are

$$M_{ij} = (-1)^{(j-1)(i-1)} \begin{bmatrix} b+j-1 \\ 2j-i-1 \end{bmatrix}_{-1}.$$

Writing $M = (M_{ij}), 1 \le i, j \le a$, the remaining task is to compute $(-1)^{a(a-1)/2} \det M$.

Step 4: Evaluation of a useful determinant
For the evaluation of $\det M$, we make use of the following determinant lemma by Krattenthaler [6, Lemma 2.2]:

Lemma 9.

$$\det_{1 \le i,j \le n} ((X_j + A_n) \dots (X_j + A_{i+1})(X_j + B_i) \dots (X_j + B_2))$$
$$= \prod_{1 \le i < j \le n} (X_i - X_j) \prod_{2 \le i \le j \le n} (B_i - A_j).$$

□

This lemma implies the following determinant evaluation, which is crucial in this section.

Lemma 10.

$$\det_{1 \le i,j \le \alpha} \left(\binom{\beta + j}{2j - i - \gamma} \right) = \prod_{j=1}^{\alpha} \frac{(\beta + j)!(j-1)!(2\beta + \gamma + j + 1)_{j-1}}{(2j - 1 - \gamma)!(\beta + \gamma + j - 1)!}.$$

Proof. We start by taking out factors of the determinant:

$$\det_{1\le i,j\le\alpha}\left(\binom{\beta+j}{2j-i-\gamma}\right)=\prod_{j=1}^{\alpha}\frac{(\beta+j)!}{(2j-1-\gamma)!(\beta+\gamma+\alpha-j)!}(-2)^{\binom{\alpha}{2}}$$
$$\times\det_{1\le i,j\le\alpha}\left((-j+\tfrac{\gamma+1}{2})\dots(-j+\tfrac{\gamma+i-1}{2})(-j+\alpha+\beta+\gamma)\dots(-j+\beta+\gamma+i+1)\right).$$

Now we are in the position to apply Lemma 9 with $n=\alpha$, $X_k=-k$, $A_k=\beta+\gamma+k$, $B_k=\frac{\gamma+k-1}{2}$. After a little simplification we get the claimed result. □

Step 5: Evaluation of det M

We want to evaluate $(-1)^{a(a-1)/2}\det M$, where

$$M_{ij}=(-1)^{(i-1)(j-1)}\begin{bmatrix}b+j-1\\2j-i-1\end{bmatrix}_{-1},\quad 1\le i,j\le a.$$

It is easy to verify that

$$\begin{bmatrix}n\\k\end{bmatrix}_{-1}=\begin{cases}0 & n \text{ even}, k \text{ odd},\\ \binom{\lfloor n/2\rfloor}{\lfloor k/2\rfloor} & \text{else}.\end{cases}\tag{4}$$

So for b even, we have $M_{ij}=0$ for i even, j odd. If we rearrange the rows and columns of M so that the even–numbered ones come before the odd–numbered ones then we get a block matrix of the form $\begin{pmatrix}A&0\\ *&B\end{pmatrix}$, where A is a $\lfloor\frac{a}{2}\rfloor\times\lfloor\frac{a}{2}\rfloor$–matrix with $A_{ij}=-\binom{\frac{b}{2}+j-1}{2j-i-1}$ and B is a $\lceil\frac{a}{2}\rceil\times\lceil\frac{a}{2}\rceil$–matrix with $B_{ij}=\binom{\frac{b}{2}+j-1}{2j-i-1}$. Clearly, $\det M$ is now the product of the determinants of A and B. Therefore, we have

$$(-1)^{\frac{a(a-1)}{2}}\det M=\det_{1\le i,j\le\lfloor\frac{a}{2}\rfloor}\left(\binom{\frac{b}{2}+j-1}{2j-i-1}\right)\det_{1\le i,j\le\lceil\frac{a}{2}\rceil}\left(\binom{\frac{b}{2}+j-1}{2j-i-1}\right).$$

For the first determinant we use Lemma 10 with $\beta=b/2-1$, $\gamma=1$ and $\alpha=\lfloor a/2\rfloor$. For the second determinant we use Lemma 10 with $\beta=b/2-1$, $\gamma=1$ and $\alpha=\lceil a/2\rceil$. It is a routine computation to check that the product of these two determinants can be written as

$$\prod_{j=1}^{\lceil a/2\rceil-1}\frac{(b/2+j)(a-j)_b}{(j)_{b+1}}$$

which agrees with the claimed expression in Theorem 1.

The case b odd is done analogously.

3. Symmetric transpose–complementary plane partitions

In this section we sketch the proof of Theorem 2. The factors in Theorem 3 are found in a similar fashion. It is easy to see that the exponent of -1 is the number of cubes in the plane partition in the upper right quarter of the containing box with sidelengths $a\times a\times 2b$.

Analogously to Step 1 of Section 2 we view the symmetric transpose–complementary plane partitions as rhombus tilings with a horizontal and a vertical symmetry axis (see Figure 3). Therefore, the rhombus tiling is determined by its upper right quarter. This part of the tiling corresponds to certain families of nonintersecting lattice paths (see Figure 3).

Here, the starting points may vary since the horizontal rhombi on the vertical axis can be in different places. We can use Lemma 8 for the enumeration of lattice path families for every fixed choice of starting points. Therefore, instead of the single determinant $\det_{1\le i,j\le n}(\mathcal{P}(A_i\to E_j))$ we get a sum of determinants running over all possible sets of starting points.

This sum can be written as a single Pfaffian with the help of the following lemma, the minor–summation formula by Okada [14, Theorem 3] (cf. also [19, Theorem 3.1])

Our way of stating it is taken from [15, Corollary 3.2].

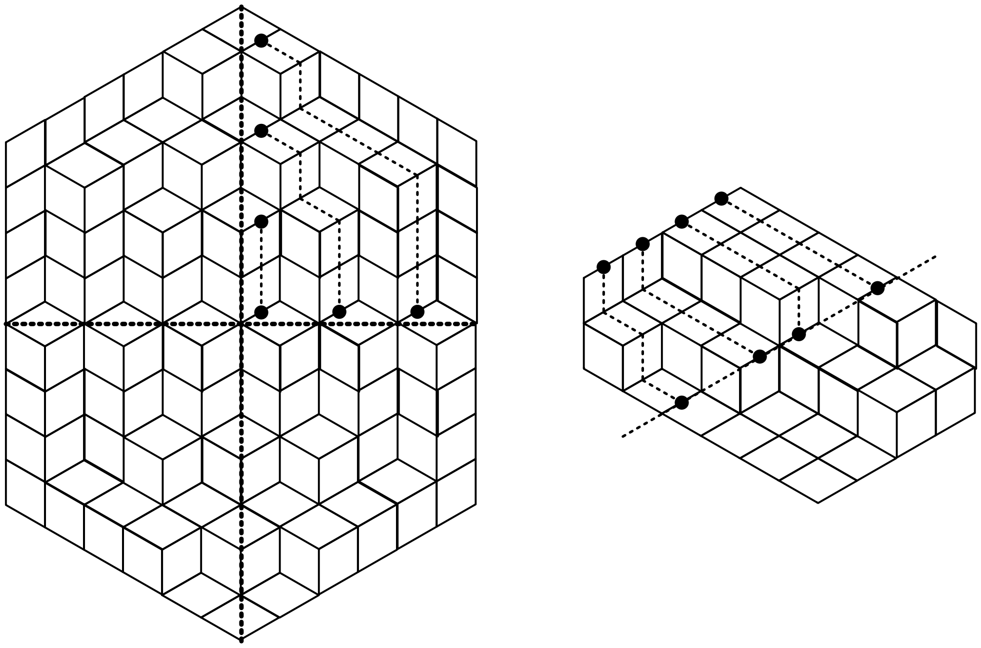

FIGURE 3. A symmetric transpose–complementary plane partition and a self-complementary plane partition.

Lemma 11. *Suppose that $n \le p$ and n is even. Let $T = (t_{ik})$ be a $p \times n$ matrix and $A = (a_{kl})$ be a $p \times p$ skew-symmetric matrix. Then we have*

$$\sum_{1 \le k_1 < \cdots < k_n \le p} \mathrm{Pf}\left(A^{k_1,\dots,k_n}_{k_1,\dots,k_n}\right) \det(T_{k_1,\dots,k_n}) = \mathrm{Pf}({}^tTAT),$$

where ${}^tT$ denotes the transpose of the matrix T, $T_{k_1,\dots,k_n}$ is the matrix composed of the rows of T with indices $k_1,\dots,k_n$, $A^{k_1,\dots,k_n}_{k_1,\dots,k_n}$ is the matrix composed of the rows and columns of A with indices $k_1,\dots,k_n$ and $\mathrm{Pf}(M)$ denotes the Pfaffian of a skew–symmetric matrix M.

The specialization $a_{ij} = \mathrm{sgn}(j-i)$ gives $\mathrm{Pf}\, A = 1$ and turns the left side into a sum of determinants exactly like ours.

The resulting Pfaffian can be reduced to a determinant of a matrix whose entries are polynomials in b. We show that certain linear factors $b + k$ divide the determinant by showing that they divide certain linear combinations of rows of the matrix. (This is the "identification of factors" method described in Section 2.4 of [7].)

In the case of Theorem 2 the determinant factors completely and the method gives all the factors. In the case of Theorem 3 a polynomial remains that seems to be irreducible in general.

In the course of the proof we use Zeilberger's algorithm [23, 24] and the Pfaff–Saalschütz summation formula ([16], (2.3.1.3); Appendix (III.2)).

4. SELF–COMPLEMENTARY PLANE PARTITIONS

In this section we sketch the proof of Theorem 6. Self–complementary plane partitions correspond to rhombus tilings with 180° rotational symmetry, see Figure 3 for an example.

The tiling is clearly determined by one half of the hexagon. Similarly to the previous cases, we find a bijection to families of nonintersecting lattice paths (see Figure 3). The end points are variable, on the symmetry axis and must be chosen symmetrically. The weight of a lattice path can again be expressed in terms of (-1)–binomial coefficients.

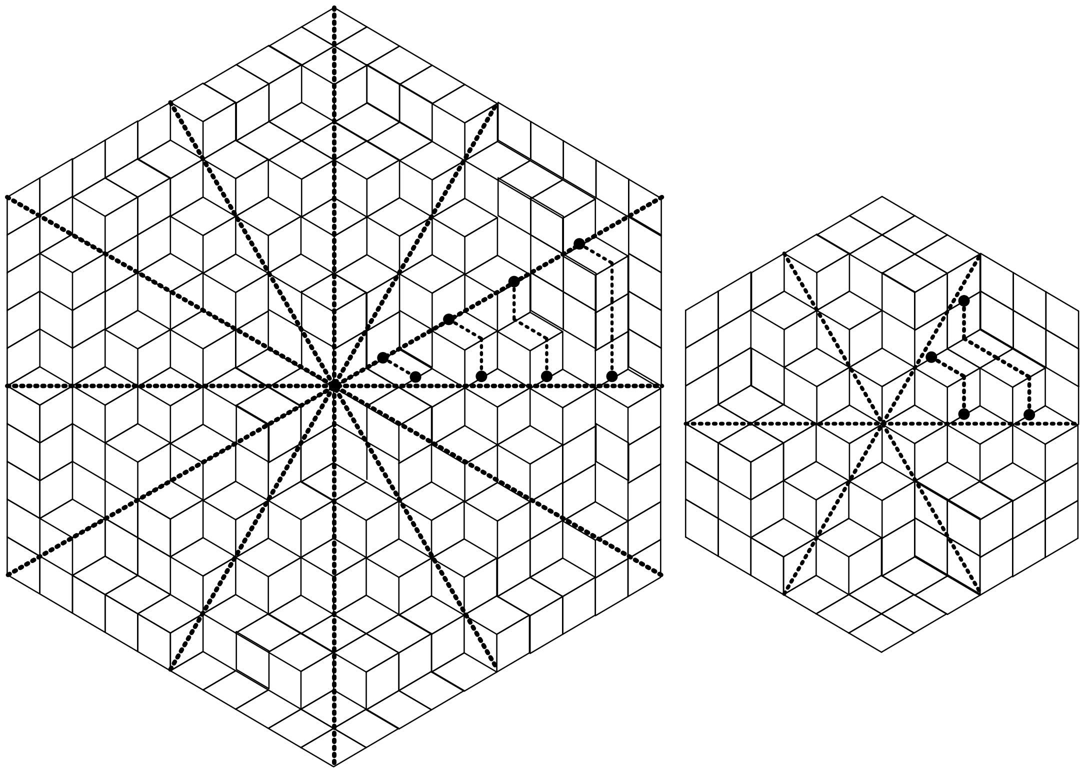

FIGURE 4. A totally symmetric self–complementary plane partition and a cyclically symmetric transpose–complementary plane partition.

Define T to be a $a \times (a+b)$–matrix with entry $T_{ij} = \mathcal{P}(A_i \to E_j)$. We can use Lemma 8 for any fixed set of starting points and sum over all possible sets. This gives

$$\sum_{1 \le k_1 < \cdots < k_{a/2} \le (a+b)/2} \det\left(T_{k_1}, \ldots, T_{k_{a/2}}, T_{a+b+1-k_{a/2}}, \ldots, T_{a+b+1-k_1}\right).$$

We can express this sum as a single Pfaffian using a special case of Lemma 11. The entries in the resulting determinant can be simplified by using Vandermonde's summation formula. Application of Lemma 9 gives the desired result. It is the last step of simplification of the matrix entries that does not work analogously if a side of the hexagon is odd, although the result seems to have a nice product formula as well.

5. Cyclically symmetric transpose–complementary plane partitions.

The rhombus tilings corresponding to these plane partitions have three symmetry axes as shown in Figure 4 (right hand side). It is enough to count the rhombus tilings in one of the resulting six regions (see Figure 4). As before, we can set up a correspondence to families of nonintersecting lattice paths and the appropriate weight can be described in terms of (-1)–binomial coefficients.

We can proceed analogously to Step 3 of Section 2 because the starting and end points are fixed in this case. The resulting determinant can be reduced to a special case of the following identity from [12]:

$$\det_{0 \le i,j \le n-1}\left(\binom{\mu+i+j}{2i-j}\right) = (-1)^{\chi(n\equiv 3(4))} 2^{\binom{n-1}{2}} \prod_{i=1}^{n-1} \frac{(\mu+i+1)_{\lfloor \frac{i+1}{2} \rfloor}(-\mu-3n+i+\frac{3}{2})_{\lfloor \frac{i}{2} \rfloor}}{(i)_i}. \tag{5}$$

It is a routine computation to verify that this leads to the expression in Theorem 4.

6. Totally symmetric self–complementary plane partitions

In this section we sketch the proof of Theorem 5. Analogously to Section 2, the plane partitions can be viewed as rhombus tilings with certain symmetries of a hexagon with sides $2\alpha \times 2\alpha \times 2\alpha$. The corresponding symmetry axes divide the hexagon in 12 parts (see Figure 4, left hand side). It is enough to consider the tiling of one of them. As before, it can be converted to a family of nonintersecting lattice paths (see Figure 4).

Again, the weight of a path can be described in terms of (-1)–binomial coefficients and we can use Lemma 11 again. This leads to a Pfaffian of a matrix containing many zeroes. Rearrangement of rows and columns and some straightforward calculations reduce the task to the evaluation of the determinant in Equation 5.

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