EXAMPLES AND COUNTEREXAMPLES FOR PERLES' CONJECTURE

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Abstract. The combinatorial structure of a d–dimensional simple convex polytope – as given, for example, by the set of the $(d-1)$–regular subgraphs of facets – can be reconstructed from its abstract graph [3] [10]. However, no polynomial/efficient algorithm is known for this task, although a polynomially checkable certificate for the correct reconstruction exists [9].

A much stronger certificate would be given by the following characterization of the facet subgraphs, conjectured by M. Perles: "*The facet subgraphs of a simple d–polytope are exactly all the $(d-1)$–regular, connected, induced, non–separating subgraphs*" [12].

We state Stolletz' examples for the validity of Perles' conjecture: In particular, it holds for the duals of cyclic polytopes, and for the duals of stacked polytopes [13].

On the other hand, we observe that for any 4–dimensional counterexample, the boundary of the (simplicial) dual polytope P^* contains a 2–complex without a free edge, and without 2–dimensional homology. Examples of such complexes are known; we use a simple modification of "Bing's house" (two walls removed) to construct explicit 4–dimensional counterexamples to Perles' conjecture.

1. Introduction

If P is a d–dimensional simple polytope, then its graph $G = G(P)$ is a d–regular, d–connected graph. If F is any facet of P, then $G_F = G(F)$ is a $(d-1)$–regular, induced, connected, non–separating subgraph of P. Whether these properties already characterize the subgraphs of facets of a simple polytope was asked by Perles [12] a long time ago:

Does every $(d-1)$–regular, induced, connected, non–separating subgraph of the graph of a simple d–polytope correspond to a facet of P?

This question is important in the context of reconstruction of polytopes. (See [11] for a general source on reconstruction of polytopes, and in particular for Perles' unpublished work in this field.) Simple polytopes can theoretically be reconstructed from their graphs – this was originally conjectured by Perles, and first proved by Blind & Mani [3], and particularly elegantly by Kalai [10]. However, although Kalai's proof can be implemented to reconstruct polytopes of reasonable size [1], the reconstruction is far from easy or efficient (in the theoretical or practical sense).

Using ideas from Kalai's paper [10], Kaibel & Körner [9] derived a certificate for the reconstruction of a simple polytope that can indeed be checked in polynomial time. They do, however, not (yet) have a polynomial time algorithm to find/construct such a certificate. If Perles' question had a positive answer, then the reconstruction would be much easier, as was noted by Kalai [10], and by Achatz & Kleinschmidt [1]. In particular, Perles' conjecture proposes an efficient criterion for recognizing the facet subgraphs of simple polytopes, and thus an easy check whether a given list of subgraphs is indeed the complete list of facet subgraphs.

Here we first establish that Perles' conjecture is true for non–trivial classes of polytopes, including all the duals of stacked polytopes, and the duals of cyclic polytopes (Section 3).

However, our main result is that, perhaps surprisingly, Perles' conjecture is *not true* in general. To prove this, we first identify certain topological obstructions for Perles' conjecture (Section 4). We then take (known) 2–dimensional complexes that realize this obstruction – the simplest one to handle being known as "Bing's house with two rooms" (with two walls removed). We describe fairly general methods to produce counterexamples from such 2–complexes. A small, explicit counterexample derived this way will be presented as an electronic geometry model (`www.eg-models.de`).

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2. Versions of Perles' Conjecture

We will distinguish two slightly different versions of Perles' conjecture, where the first and stronger version is the one stated in [10], see also [15, p. 100, Problem 3.13*].

Definition 1. A simple d–polytope P *satisfies Perles' conjecture* if every induced, connected, $(d-1)$–regular, non–separating subgraph of $G(P)$ is the graph of a facet of P.
A simple d–polytope P *weakly satisfies Perles' conjecture* if every induced, $(d-1)$–connected, $(d-1)$–regular, non–separating subgraph of $G(P)$ is the graph of a facet of P.

If P is a simple polytope, then its polar dual P^Δ is simplicial. The vertices and edges of P correspond to the facets and ridges of P^Δ respectively. The different notions in Perles' conjecture for P have immediate translations to the combinatorics of P^Δ. This yields the "puzzle" reformulation of [3].

- An induced subgraph of the graph of P corresponds to a collection of facets and ridges of P^Δ such that, whenever two adjacent facets belong to the collection, then their common ridge also is in the collection. We will present this collection of facets and ridges in terms of the pure simplicial $(d-1)$–complex it generates. The dual graph of this "pseudomanifold" is the one that Perles' conjecture in the original form refers to.
- If the induced subgraph is $(d-1)$–regular, then every $(d-1)$–simplex in the corresponding subcomplex has exactly one free $(d-2)$–simplex.
- If the induced subgraph is connected, then the corresponding subcomplex is dually connected, i.e., every two $(d-1)$–simplices can be joined by a chain of successively adjacent facets and ridges. (This is what is often called "strongly connected" for pseudomanifolds.)
- If the induced subgraph does not separate its complement (graph theoretically), then the corresponding subcomplex does not separate its complement (topologically).
- If the induced subgraph is the graph of a facet, then the corresponding subcomplex is a vertex star: the collection of all facets that contain a given vertex.

Definition 2. A simplicial d–polytope P^Δ *satisfies Perles' conjecture* if every pure, $(d-1)$–dimensional, dually connected, non–separating subcomplex of its boundary such that every maximal simplex has exactly one free face is a vertex star.

A simplicial d–polytope P^Δ *weakly satisfies Perles' conjecture* if every pure, $(d-1)$–dimensional, dually $(d-1)$–connected, non–separating subcomplex of its boundary such that every maximal simplex has exactly one free face is a vertex star.

3. Positive results

Stolletz' Diplomarbeit [13] contains a series of results about classes of simplicial polytopes that satisfy Perles' conjecture, among them:

Theorem 1 (cf. Stolletz [13, Kap. 4] for $d = 4$)**.** *The duals of cyclic polytopes $C_d(n)^\Delta$ satisfy Perles' conjecture.*

Theorem 2 (Stolletz [13, Satz 3.1.3])**.** *All dual to stacked polytopes (that is, the multiple truncations of simplices) satisfy Perles' conjecture. More generally, the operation of "cutting off a simple vertex" preserves the property to satisfy Perles' conjecture.*

4. The obstruction

We restrict our discussion to the case $d = 4$, where we will construct counter–examples to Perles' conjecture[1]. From these, iterated bipyramids yield counter–examples in all dimensions $d > 4$.

The discussion in this section motivates our construction; it leads us to well–characterized obstructions to the validity of Perles' conjecture (for any specific polytope). For simplicity, we formulate this for $d = 4$; the generalization to $d \geq 4$ is immediate.

[1] The pictures try to illustrate the situation in $d = 3$, though.

Proposition 3. *Let P^Δ be a simplicial 4–polytope whose boundary complex $\Delta := \Delta(\partial P^\Delta)$ contains a dually connected pure 3–dimensional subcomplex Γ, all whose tetrahedra have exactly one free triangle in Γ.*

Then Γ (and in particular Δ) contains a pure 2–dimensional, dually connected subcomplex $\mathrm{core}(\Gamma)$ *without free edges. Moreover, Γ separates the boundary of P^Δ if and only if* $\mathrm{core}(\Gamma)$ *does, and* $\mathrm{core}(\Gamma)$ *is empty if and only if Γ is a vertex star.*

In the case $d = 3$, $\mathrm{core}(\Gamma)$ would be 1–dimensional, with neither a cycle (non–separating) nor a free face: a leafless tree. This (re)proves Perles' conjecture for 3–polytopes.

Proof. We specify a pure 2–dimensional subcomplex $\mathrm{core}(\Gamma)$ of Γ which will comply with our conditions (the definition is due to Carsten Schultz):

Every tetrahedron $\sigma \in \Gamma$ has a unique vertex $v(\sigma)$ opposite to its free face. A triangle τ belongs to $\mathrm{core}(\Gamma)$ if both its neighboring tetrahedra σ_1, σ_2 belong to Γ, and if $v(\sigma_1) \neq v(\sigma_2)$.

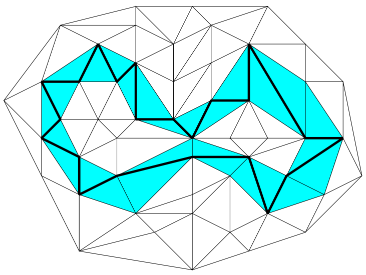

Figure 1: If the grey 2–complex in the figure is Γ, then the black 1–dimensional subcomplex is its core.

Claim **1.** $\mathrm{core}(\Gamma)$ has no free faces.

Suppose that ϱ is a free edge of the triangle $\tau \in \mathrm{core}(\Gamma)$. The tetrahedra of Δ which contain ϱ correspond to the vertices of the 2–face ϱ^Δ of P. They are cyclically ordered: $\sigma_1, \dots, \sigma_n$, as in figure 2.

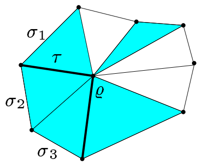

Figure 2

We can assume that $\tau = \sigma_1 \cap \sigma_2$ and $\sigma_3 \in \Gamma$. For, if σ_3 and σ_n were both not in Γ, then this would imply $v(\sigma_1) = v(\sigma_2)$, and hence $\tau \notin \mathrm{core}(\Gamma)$.

Because $\sigma_2 \cap \sigma_3 \notin \mathrm{core}(\Gamma)$, we have $v(\sigma_3) = v(\sigma_2)$. This vertex lies in $\sigma_2 \cap \sigma_3$, and it cannot be the vertex of σ_2 that is opposite to τ; thus we get that $v(\sigma_3) = v(\sigma_2) \in \varrho$. In particular, this means that $\sigma_3 \cap \sigma_4$ is not the free face of σ_3, and thus $\sigma_4 \in \Gamma$. Iterating these arguments, we see that $\sigma_1, \dots, \sigma_n \in \Gamma$ and $v(\sigma_n) = \ldots = v(\sigma_2) \neq v(\sigma_1)$. So, $\sigma_n \cap \sigma_1$ is another triangle in $\mathrm{core}(\Gamma)$.

Claim **2.** $\mathrm{core}(\Gamma)$ is empty if and only if Γ is a vertex star.

If $\mathrm{core}(\Gamma)$ is empty, as Γ is dually connected, all the vertices $v(\sigma)$ are equal, say, v_0. Hence Γ is part of $\mathrm{star}_\Delta(v_0)$. But then, Γ must be the whole star, because there is only one free face per tetrahedron.

Claim **3.** If $\mathrm{core}(\Gamma)$ separates then so does Γ.

Alexander duality implies that a subcomplex separates if and only if it has non–trivial 2–dimensional homology. The 3–dimensional complex Γ collapses down to a 2–dimensional subcomplex Γ', which contains $\mathrm{core}(\Gamma)$. Since Γ' has dimension 2, it is clear that the (relative) homology group $H_3(\Gamma', \mathrm{core}(\Gamma))$ vanishes. Hence, the map ι in the exact sequence

$$\begin{array}{ccccccc} H_3(\Gamma, \mathrm{core}(\Gamma)) & \longrightarrow & H_2(\mathrm{core}(\Gamma)) & \xrightarrow{\iota} & H_2(\Gamma) & \longrightarrow & \cdots \\ \| & & & & & & \\ H_3(\Gamma', \mathrm{core}(\Gamma)) = 0 & & & & & & \end{array}$$

is injective, and the claim is proved. □

The same proof can be carried out in a PL–manifold in place of the boundary of a simplicial polytope.

Let us further analyze the above situation. The aim is to reveal the combinatorial properties of complexes that appear as a Schultz–complex (in addition to the topological ones that we have already seen). In Section 5 we will show how to guarantee these properties using stellar subdivisions.

Consider now the structure of Γ locally, restricted to the (open) star of a vertex v_0. This vertex star is divided by $\mathrm{core}(\Gamma)$ into several pieces and Γ is a union of some of these pieces:

Lemma 1. *Let $\sigma, \sigma' \in \mathrm{star}_\Delta(v_0)$ be maximal simplices in the same piece. I.e., there is a dual path in $\mathrm{star}_\Delta(v_0)$ between σ and σ' which does not meet $\mathrm{core}(\Gamma)$. If $\sigma \in \Gamma$ and $v(\sigma) = v_0$ then $\sigma' \in \Gamma$ and $v(\sigma') = v_0$.*

Corollary 4. *For every vertex v_0 of $\mathrm{core}(\Gamma)$ the complex which is generated by $\{\sigma \in \Gamma : v(\sigma) = v_0\}$ is (the closure of) a union of components of $\mathrm{star}_\Delta(v_0) \backslash \mathrm{core}(\Gamma)$.*

Furthermore, two such components intersect at most in codimension one, and if so, then this intersection is included in $\mathrm{core}(\Gamma)$.

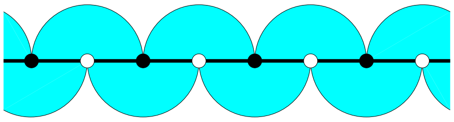

Figure 3

5. A COUNTEREXAMPLE

Proposition 3 asks for a 2–dimensional simplicial complex Γ, without a free edge (every edge is contained in at least two triangles), which does not separate (that is, $H_2(\Gamma, \mathbb{Z}) = \{0\}$), and which is embeddable in $\mathbb{R}^3$ (in particular, $H_1(\Gamma, \mathbb{Z})$ has no torsion). Such 2–dimensional complexes do exist. The most prominent examples are probably Borsuk's "dunce hat" [4, 14], and Bing's "house with two rooms" [2], which are even contractible.

To keep the constructions for the following simple, we shall put an extra condition on the complexes we look at: we want them to be "essentially manifolds," that is, we want every point to have a closed neighborhood that is homeomorphic either to two or to three triangles that are joined together at a common edge. (Thus, we admit singular curves, along which a manifold branches into three parts, but we do not admit singular points that may be more complicated than the points on a singular curve.)

Our point of departure is a modification $\mathcal{B}$ of Bing's house. Two extra walls are missing that would usually be added to make the interior of each room simply connected. (Cf. figure 4)

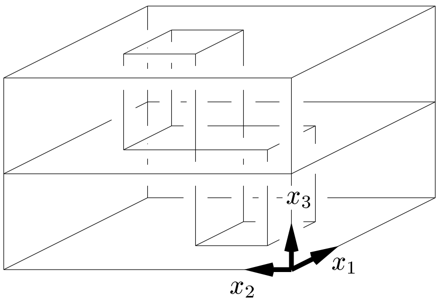

Figure 4: The complex $\mathcal{B}$ has two rooms: The downstairs room is connected to the outside via the chimney through the upstairs room, the upstairs room is connected to the outside via the chimney through the downstairs room.

$\mathcal{B}$ can be embedded into a pile of $2 \times 3 \times 4$ cubes – a cubical complex. Triangulate the pile by the hyperplane arrangement formed by hyperplanes of the form $x_i - x_j = k$ to get a simplicial complex (cf. figure 5). Add a cone over the boundary to get the boundary of a simplicial 4–polytope Q^Δ (the subdivision by a hyperplane arrangement is "coherent"/"regular"; cf. [15, Lect. 5]).

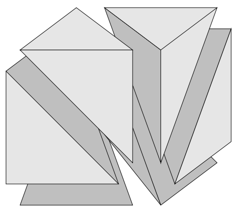

Figure 5

This polytope contains a triangulation of $\mathcal{B}$ as a subcomplex, which we also denote by $\mathcal{B}$. The vertices of this subcomplex can be assigned colors 0, 1, or 2, by assigning to each vertex the sum of its coordinates modulo 3. The only edges that join vertices of the same color are the diagonals of the cubes which are not in $\mathcal{B}$. Now the two chimneys look as in figure 6.

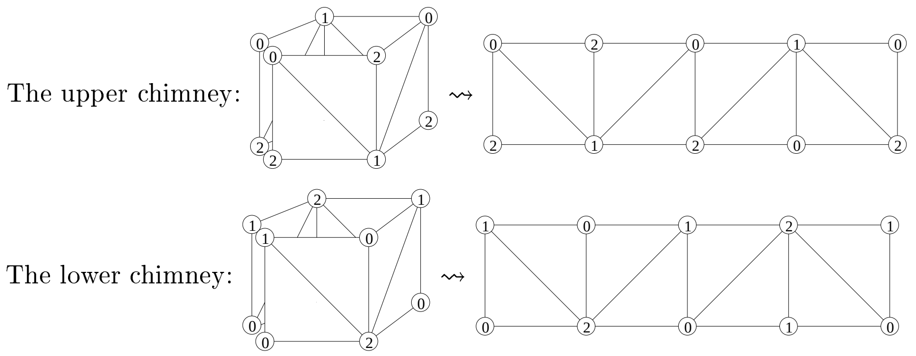

Figure 6

We will perform a sequence of stellar subdivisions on faces of Q^Δ in order to arrive at the final polytope P^Δ.

- Perform stellar subdivisions on vertical edges in the chimneys; in the upper chimney the two edges with labels $\genfrac{}{}{0pt}{}{0}{2}$, and in the lower chimney the two edges with labels $\genfrac{}{}{0pt}{}{1}{0}$ (cf. figure 7).

The upper chimney The lower chimney

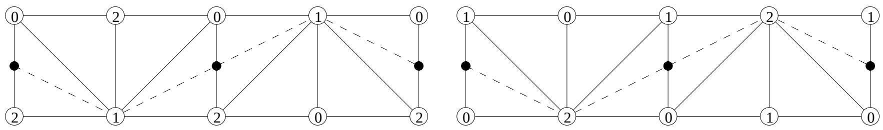

Figure 7

The resulting new edges in the chimneys (four each) are interpreted as marking the boundary between 'inside' and 'outside' in the chimneys. No color is given to these four new vertices.

• Perform stellar subdivisions on simplices outside $\mathcal{B}$ all whose faces belong to $\mathcal{B}$ (e.g., the diagonals of the cubes). Then $\mathcal{B}$ becomes an induced subcomplex.

• Whenever there are vertices $v_1, v_2 \in \mathcal{B}$ of the same color, and a vertex $w \notin \mathcal{B}$ such that both $\{w, v_1\}$ and $\{w, v_2\}$ are edges (e.g. v_1, v_2 on the boundary of the pile and w the cone vertex), perform a stellar subdivision on one of these edges. (A more efficient approach is to make these subdivisions only if the vertex w lies "outside" respectively "in the upper room" respectively "in the lower room," if the vertices v_1, v_2 have color 0 respectively color 1 respectively color 2. A more brutal approach is to subdivide all the edges not in $\mathcal{B}$.)

The result of all these operations is P^{Δ}. It remains to name the vertex star pieces that we want to attach to $\mathcal{B}$ in order to get a Perles–contradicting subcomplex. For that purpose we use the vertex labels, and proceed according to the scheme

Label		Partial vertex star
0	$\longleftrightarrow$	outside
1	$\longleftrightarrow$	upstairs room
2	$\longleftrightarrow$	downstairs room

If a vertex with label 2 does not touch the downstairs room, then no partial vertex star is assigned, etc. The union Γ of these partial vertex stars is a non–separating subcomplex, all whose tetrahedra have exactly one free triangle opposite to 'their' vertex of $\mathcal{B}$.

Finally, Γ is 'at least as connected as' $\mathcal{B}$:

Proposition 5. *Let G be the graph whose nodes are the two–faces of $\mathcal{B}$, and whose edges correspond to pairs of two–faces that share a one–face. If G is 3–connected, then Γ is dually 3–connected.*

Proof. Delete two three–faces from Γ. The first observation is that any of the remaining three–faces can be joined by a dual path to a three–face which has a two–face in core(Γ). The strategy to join two of these three–faces is to guarantee the existence of a path in G between the two corresponding nodes, and then to lift this path to a dual path in Γ.

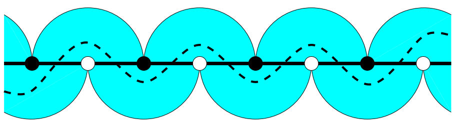

Figure 8

$\square$

6. Remarks

It seems reasonable to ask whether every induced, 3–connected, 3–regular, non–separating and planar (!) subgraph of the graph of a simple 4–polytope is the graph of a facet. For $d > 4$, the natural condition that could replace the "planar" would be "isomorphic to the graph of a $(d-1)$–polytope." But this is not a good condition.

Does the 120–cell satisfy Perles' conjecture?

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REFERENCES

[1] H. ACHATZ & P. KLEINSCHMIDT: *Reconstructing a simple polytope from its graph,* in: "Polytopes – Combinatorics and Computation" (G. Kalai, G.M. Ziegler, eds.), *DMV Seminar* **29**, Birkhäuser, Basel 2000, pp. 155-165..

[2] R H BING: *Some aspects of the topology of 3–manifolds related to the Poincaré conjecture,* in: *Lectures on Modern Mathematics, Vol. II* (T.L. Saaty, ed.), Wiley 1964, 93-128.

[3] R. BLIND & P. MANI–LEVITSKA: *On puzzles and polytope isomorphisms, Aequationes Math.* **34** (1987), 287-297.

[4] K. BORSUK: *Über das Phänomen der Unzerlegbarkeit in der Polyedertopologie, Commentarii Math. Helv.* **8** (1935), 142-148.

[5] E. GAWRILOW & M. JOSWIG: `polymake`: *A framework for analyzing convex polytopes,* in: "Polytopes – Combinatorics and Computation" (G. Kalai, G.M. Ziegler, eds.), *DMV Seminar* **29**, Birkhäuser, Basel 2000, pp. 43-73.

[6] E. GAWRILOW & M. JOSWIG: *Polymake: A framework for analyzing convex polytopes,* 1997-2000, `http://www.math.tu-berlin.de/diskregeom/polymake/doc/`.

[7] G. BOHUS, W. JOCKUSCH, C. W. LEE & N. PRABHU: *On a triangulation of the 3–ball and the solid torus, Discrete Math.* **187** (1998), 259-264.

[8] M. JOSWIG: *Reconstructing a non–simple polytope from its graph,* in: "Polytopes – Combinatorics and Computation" (G. Kalai, G.M. Ziegler, eds.), *DMV Seminar* **29**, Birkhäuser, Basel 2000, pp. 167-176.

[9] V. KAIBEL & F. KÖRNER: *A polynomial certificate for the reconstruction of simple polytopes,* TU Berlin 2000, in preparation.

[10] G. KALAI: *A simple way to tell a simple polytope from its graph, J. Combinatorial Theory, Ser. A* **49** (1988), 381-383.

[11] G. KALAI: *Some aspects of the combinatorial theory of convex polytopes,* in: "Polytopes: Abstract, Convex and Computational" (T. Bisztriczky, P. McMullen, and A. Weiss, eds.), Proc. NATO Advanced Study Institute, Toronto 1993, Kluwer Academic Publishers 1994, pp. 205-230.

[12] M. PERLES: *Results and problems on reconstruction of polytopes,* Jerusalem 1970, unpublished [10]; problems posed in Oberwolfach 1984 and 1986 [3].

[13] R. STOLLETZ: *Facetten–Graphen einfacher Polytope,* Diplomarbeit, TU Berlin, 77 pages.

[14] E. C. ZEEMAN: *On the dunce hat, Topology* **2** (1963), 341-358.

[15] G. M. ZIEGLER: *Lectures on Polytopes, Graduate Texts in Mathematics* **152**, Springer–Verlag New York Berlin Heidelberg 1995; Revised edition, 1998.

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