Extended Abstract

HOMOLOGY OF MATCHING AND CHESSBOARD COMPLEXES

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Abstract. We study the topology of matching and chessboard complexes. Our main results are as follows.

(1) We prove conjectures of A. Björner, L. Lovász, S. T. Vrećica, and R. T. Živaljević on the connectivity of these complexes.
(2) We show that for almost all n, the first nontrivial homology group of the matching complex on n vertices has exponent three.
(3) We prove similar but weaker results on the exponent of the first nontrivial homology group of the m-by-n chessboard complex for all pairs $m < n$ for which m is sufficiently large and $n - m$ is sufficiently small.
(4) We give a basis for the top homology group of the m-by-n chessboard complex.
(5) We prove that a certain skeleton of the matching complex is shellable. This result answers a question of Björner, Lovász, Vrećica, and Živaljević and is analogous to a result of G. Ziegler on chessboard complexes.

A *matching* on vertex set V is a graph in which each vertex is contained in at most one edge. If $G = (V, E)$ is a graph then the collection of all subgraphs (V, E') of G which are matchings determines a simplicial complex $M(G)$, as follows. The vertices of $M(G)$ are (in correspondence with) the edges of G, and the k-dimensional faces of $M(G)$ are the subgraphs of G which are matchings with $k+1$ edges. If G is the complete graph on vertex set $[n] := \{1, 2, \ldots, n\}$ for some positive integer n then we write M_n for $M(G)$. Similarly, if G is the complete bipartite graph with parts $[m]$ and $[n]' := \{1', 2', \ldots, n'\}$ for positive integers m, n then we write $M_{m,n}$ for $M(G)$.

The complexes M_n are called *matching complexes*. Topological properties of these complexes were first examined by S. Bouc in [Bo], in connection with the Quillen complexes at the prime 2 for the symmetric groups S_n. One of Bouc's main results shows that the homology of M_n with complex coefficients behaves quite nicely. Note that the natural action of S_n determines an action of S_n on M_n, which in turn determines a representation of S_n on each reduced homology group of M_n. As is well known, the irreducible complex representations of S_n are indexed by partitions of n. For a partition λ, let $\mathcal{S}^\lambda$ be the irreducible representation corresponding to λ, let λ' be the partition conjugate to λ and let $d(\lambda)$ be the sidelength of the largest square which fits inside the Young diagram of λ (see [MacD],[Sa] or [St, Chapter 7] for definitions). Bouc's result is as follows.

Theorem 1 ([Bo, Proposition 5]). *For all $i \geq 1$ and all $n \geq 2$, there is an isomorphism*

$$\widetilde{H}_{i-1}(M_n, \mathbb{C}) \cong \bigoplus_{\substack{\lambda : \lambda \vdash n, \\ \lambda = \lambda', \\ d(\lambda) = n - 2i}} \mathcal{S}^\lambda$$

of S_n-modules.

Theorem 1 has been rediscovered more than once. In fact, a result equivalent to this theorem (see [KRW]) was proved in an earlier paper ([JoWe]) of T. Jósefiak and J. Weyman, who used their result to provide a representation-theoretic interpretation for the identity

$$\prod_{i<j}(1 - x_i x_j) \prod_i (1 - x_i) = \sum_{\lambda = \lambda'} (-1)^{\frac{|\lambda| + d(\lambda)}{2}} s_\lambda,$$

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where s_λ is the Schur function associated with λ. (This identity was originally due to D. E. Littlewood). In [Si], S. Sigg proves the same result as Jósefiak and Weyman in order to determine the homology of some nilpotent Lie algebras, and in [ReRo], V. Reiner and J. Roberts prove a generalization of Theorem 1 during their investigation of free resolutions of certain modules over certain quotients of polynomial rings. In [DW], X. Dong and M.L. Wachs show that discrete Hodge theory can be used to obtain an elegant proof of Theorem 1.

Matching complexes are also studied by A. Björner, L. Lovász, S. Vrećica, and R. Živaljević in [BLVZ]. Both [BLVZ] and [Bo] contain the following result on the connectivity of matching complexes. Recall that a topological space T is called k-connected if for $0 \le i \le k$ every continuous map from the i-sphere to T can be extended to a continuous map from the $(i+1)$-ball to T. The next theorem is a special case of [BLVZ, Theorem 4.1] and also follows immediately from [Bo, Lemme 2, Proposition 2 and Proposition 6].

Theorem 2. *For a positive integer n, set*

$$\nu(n) := \lfloor \frac{n+1}{3} \rfloor.$$

Then M_n is $(\nu(n)-2)$-connected.

In [BLVZ] it is conjectured that this result is the best one possible.

Conjecture 3 ([BLVZ])**.** *If $n > 2$ then M_n is not $(\nu(n)-1)$-connected.*

It is known (see for example [Bj, 9.16]) that for $k \ge 1$ a complex Δ is k-connected if and only if Δ is simply connected and $\widetilde{H}_i(\Delta, \mathbb{Z}) = 0$ for all $i \le k$. So, to prove Conjecture 3 it suffices to show that

$$\widetilde{H}_{\nu(n)-1}(M_n, \mathbb{Z}) \neq 0 \tag{1}$$

for all $n > 2$. Note that by Theorem 1, if $\widetilde{H}_{\nu(n)-1}(M_n, \mathbb{C}) \neq 0$ then there is some partition λ of n such that

$$n - 2\nu(n) = d(\lambda) \le \sqrt{n}. \tag{2}$$

It is easy to see that $n - 2\nu(n) > \sqrt{n}$ whenever $n \ge 12$. In fact, Condition (2) is satisfied if and only if $n \in \{3,4,5,6,8,9,11\}$. So, if Conjecture 3 is true then $\widetilde{H}_{\nu(n)-1}(M_n, \mathbb{Z})$ is a nontrivial finite group for all remaining $n > 2$. Most of the work necessary to prove Conjecture 3 is done in [Bo], where the following results appear. From now on we write $\widetilde{H}_k(M_n)$ for $\widetilde{H}_k(M_n, \mathbb{Z})$.

Proposition 4 ([Bo, Proposition 7])**.** *If $k \ge 1$ then $\widetilde{H}_k(M_{3k+4}) \cong \mathbb{Z}_3$.*

Proposition 5 ([Bo, Proposition 8])**.** *If $k \ge 3$ then $\widetilde{H}_k(M_{3k+3})$ is a nontrivial 3-group of exponent at most 9.*

To prove Propositions 4 and 5, Bouc first shows that there is a long exact sequence

$$\cdots \xrightarrow{\delta} \bigoplus_{a,h} \tilde{H}_{t-1}(M_{n-3}) \xrightarrow{\phi} \tilde{H}_t(M_n) \xrightarrow{\psi} \bigoplus_{i,j} \tilde{H}_{t-2}(M_{n-4}) \xrightarrow{\delta}$$
$$\bigoplus_{a,h} \tilde{H}_{t-2}(M_{n-3}) \xrightarrow{\phi} H_{t-1}(M_n) \xrightarrow{\psi} \cdots \qquad ,$$

where a ranges over the set $\{1,2\}$ and h,i,j range over the set $\{3,\ldots,n\}$ with $i \neq j$. For $n \equiv 0,1 \bmod 3$ the long exact sequence ends with

$$\bigoplus_{a,h} \tilde{H}_{\nu(n-3)-1}(M_{n-3}) \xrightarrow{\phi} H_{\nu(n)-1}(M_n) \to 0. \tag{3}$$

Bouc uses the surjection ϕ and induction to obtain Propositions 4 and 5. His calculation $\widetilde{H}_2(M_7) \cong \mathbb{Z}_3$ ([Bo, Proposition 3]) provides the base step of the induction.

For $n \equiv 2 \bmod 3$, Bouc's long exact sequence ends with

$$\bigoplus_{a,h} \tilde{H}_{\nu(n-3)-1}(M_{n-3}) \xrightarrow{\phi} \tilde{H}_{\nu(n)-1}(M_n) \xrightarrow{\psi} \bigoplus_{i,j} \tilde{H}_{\nu(n-4)-1}(M_{n-4}) \to 0 \tag{4}$$

It is straightforward to use this to prove the following result.

Proposition 6. *$\widetilde{H}_k(M_{3k+2}) \neq 0$ for all $k \in \mathbb{N}$.*

Conjecture 3 follows. With more work, we use Bouc's long exact sequence to obtain the following result.

Theorem 7.
1. *If $n \in \{3,4,5,6,8\}$ then $\widetilde{H}_{\nu(n)-1}(M_n)$ is nontrivial and free.*
2. *If $n \in \{9,11\}$ then $\widetilde{H}_{\nu(n)-1}(M_n)$ is the direct sum of a nontrivial free group and a nontrivial group of exponent 3.*
3. *If $n \geq 15$ or $n \in \{7,10,12,13\}$ then $\widetilde{H}_{\nu(n)-1}(M_n)$ has exponent 3.*

Note that the case $n = 14$ is not covered in Theorem 7. Of course we know that $\widetilde{H}_4(M_{14})$ is a nontrivial finite group by Theorem 1 and Proposition 6, but we do not know that this group has exponent 3. For $n \leq 13$, the entire homology $\widetilde{H}_*(M_n)$ has been determined, using a computer program first developed by F. Heckenbach and later updated by J.-G. Dumas, Heckenbach, D. Saunders and V. Welker. The results for $n \leq 12$ appear in [BBLSW]. The main points of Theorem 7 are the improvement of Bouc's exponent result from Proposition 5 and the fact that $\widetilde{H}_k(M_{3k+2})$ has exponent 3 for $k \geq 5$. The improvement of Bouc's Proposition 5 is a consequence of the computer calculation

$$\widetilde{H}_3(M_{12}) \cong \mathbb{Z}_3^{56},$$

which provides a base step for an induction proof whose induction step is provided by the surjection ϕ in (3).

The key idea in the proof of the exponent result for $n \equiv 2 \bmod 3$ is to show that the group $\bigoplus_{i,j} \widetilde{H}_{\nu(n-4)-1}(M_{n-4})$ is generated by elements whose preimages under ψ of (4) have the form $\alpha * \beta$, where α is a cycle in $\widetilde{H}_{\nu(5)-1}(M_A)$, for some subset $A \subseteq [n]$ of cardinality 5, β is a cycle in $\widetilde{H}_{\nu(n-5)-1}(M_{[n]\setminus A})$ and $*$ is the operation associated with concatenating oriented simplices. This enables us to prove by induction that $\widetilde{H}_{\nu(n)-1}(M_n)$ is generated by cycles of the form $\alpha * \beta$, where α and β are as above. Since $3(\alpha * \beta) = \alpha * 3\beta$, if $\widetilde{H}_{\nu(n-5)-1}(M_{n-5})$ has exponent 3 then so does $\widetilde{H}_{\nu(n)-1}(M_n)$. Hence $\widetilde{H}_{\nu(n)-1}(M_n)$ has exponent 3 for $n \geq 17$. Since $\widetilde{H}_2(M_9)$ has a nontrivial free part, we cannot say anything about the exponent of $\widetilde{H}_4(M_{14})$.

Not much is known about the order of the finite groups $\widetilde{H}_{\nu(n)-1}(M_n)$ when $n \geq 14$. We are able however to derive the following bounds. For all $n \geq 2$, let r_n be the rank (i.e., minimum number of generators) of $\widetilde{H}_{\nu(n)-1}(M_n)$.

Theorem 8. *Let $n \geq 9$.*
1. *If $n \equiv 0 \bmod 3$ then $n - 1 \leq r_n \leq 2(n-2)r_{n-3}$.*
2. *If $n \equiv 2 \bmod 3$ then $(n-1)(n-3) - 1 \leq r_n \leq (n-2)(n-3) + 2(n-2)r_{n-3}$.*

The lower bound on r_n in the case that $n \equiv 0 \bmod 3$, was obtained by Bouc [Bo]. We extend his technique to obtain the lower bound for the other case. The upper bounds are derived from the exact sequences (3) and (4).

We turn now to the *chessboard complexes* $M_{m,n}$. The name "chessboard complex" comes from the fact that $M_{m,n}$ is isomorphic to the complex of all collections S of squares on an $m \times n$ chessboard such that if a rook is placed on each square in S, none of these rooks threatens any other. These complexes were first studied by P. Garst in [Ga], as coset complexes for collections of point stabilizers in symmetric groups. They also appear in the work of S. Vrećica and R. Živaljević on combinatorial geometry ([VrZi]) and the commutative algebra work ([ReRo]) of Reiner and Roberts mentioned above. The connectivity properties of $M_{m,n}$ are investigated in [BLVZ], where the following theorem appears.

Theorem 9 ([BLVZ, Theorem 1.1]). *For $m, n \in \mathbb{N}$ with $m \leq n$, define*

$$\nu(m,n) := \min\left\{m, \lfloor\frac{m+n+1}{3}\rfloor\right\}.$$

Then the chessboard complex $M_{m,n}$ is $(\nu(m,n)-2)$-connected.

Again, it was conjectured in [BLVZ] that this result is the best one possible.

Conjecture 10 ([BLVZ, Conjecture 1.5]). *For all $m \leq n$, the complex $M_{m,n}$ is not $(\nu(m,n)-1)$-connected.*

The complex homology of $M_{m,n}$ was determined by J. Friedman and P. Hanlon in [FrHa]. Note that the group $S_m \times S_n$ acts on $M_{m,n}$, making the complex homology groups of $M_{m,n}$ modules for $S_m \times S_n$. Friedman and Hanlon give a description of the decomposition of each $\widetilde{H}_i(M_{m,n}, \mathbb{C})$ into irreducible $S_m \times S_n$-modules. This description is similar to that given for the matching complex in Theorem 1, although somewhat more complicated. A consequence of this description is the following result.

Theorem 11 ([FrHa, Theorem 7]). *We have*

$$\widetilde{H}_{\nu(m,n)-1}(M_{m,n}, \mathbb{C}) \neq 0$$

if and only if $n \geq 2m-4$ or $(m,n) \in \{(6,6),(7,7),(8,9)\}$.

It follows that if Conjecture 10 is true, the group $\widetilde{H}_{\nu(m,n)-1}(M_{m,n}, \mathbb{Z})$ will be nontrivial but finite for all pairs (m,n) which do not satisfy one of the conditions given in Theorem 11. We have proved Conjecture 10 and obtained some information on torsion similar to that given in Theorems 7 and 8. As above, we suppress the symbol $\mathbb{Z}$ from our notation for integral homology.

Theorem 12. *Let $m, n \in \mathbb{N}$ with $m \leq n$.*

1. *For all such m, n we have $\widetilde{H}_{\nu(m,n)-1}(M_{m,n}) \neq 0$.*
2. *If $m+n \equiv 1 \bmod 3$ and $5 \leq n \leq 2m-5$ then*
$$\widetilde{H}_{\nu(m,n)-1}(M_{m,n}) \cong \mathbb{Z}_3.$$
3. *If $m+n \equiv 0 \bmod 3$ and $9 \leq n \leq 2m-9$ then $\widetilde{H}_{\nu(m,n)-1}(M_{m,n})$ is a 3-group of exponent at most 9. Also, the rank $r_{m,n}$ of $\widetilde{H}_{\nu(m,n)-1}(M_{m,n})$ satisfies*
$$n \leq r_{m,n} \leq (m-1)r_{m-2,n-1} + (n-1)r_{n-1,m-2}.$$
4. *If $m+n \equiv 2 \bmod 3$ and $13 \leq n \leq 2m-13$ then $\widetilde{H}_{\nu(m,n)-1}(M_{m,n})$ is a 3-group of exponent at most 9. Also, the rank $r_{m,n}$ of $\widetilde{H}_{\nu(m,n)-1}(M_{m,n})$ satisfies*
$$n(n-1)-1 \leq r_{m,n} \leq (m-1)(n-1) + (m-1)r_{m-2,n-1} + (n-1)r_{n-1,m-2}.$$

We conjecture that the exponent of $\tilde{H}_{v(m,n)-1}(M_{m,n})$ is in fact 3 when the pair (m,n) satisfies the conditions of Theorem 12. We prove that in order to establish this conjecture one needs only verify it for $m = n = 9$. However the software of Dumas, Heckenbach, Saunders and Welker is not yet able to produce results for these values.

It is natural to ask whether $\widetilde{H}_{\nu(m,n)-1}(M_{m,n})$ is a 3-group whenever (m,n) does not satisfy the conditions of Theorem 11. However, computations performed by Dumas using the software of Dumas, Heckenbach, Saunders and Welker indicate that $\widetilde{H}_4(M_{7,8})$ contains 2-torsion.

The proof of Theorem 12 uses an adaptation of Bouc's long exact sequence. Namely, there is a long exact sequence

$$\cdots \xrightarrow{\delta} \bigoplus_{a} \widetilde{H}_{t-1}(M_{m-2,n-1}) \oplus \bigoplus_{a'} \widetilde{H}_{t-1}(M_{m-1,n-2}) \xrightarrow{\phi} \widetilde{H}_t(M_{m,n}) \xrightarrow{\psi}$$

$$\bigoplus_{a,a'} \widetilde{H}_{t-2}(M_{m-2,n-2}) \xrightarrow{\delta} \bigoplus_{a} \widetilde{H}_{t-2}(M_{m-2,n-1}) \oplus \bigoplus_{a'} \widetilde{H}_{t-2}(M_{m-1,n-2}) \xrightarrow{\phi} \cdots,$$

where a runs through $\{2,\dots,m\}$ and a' runs through $\{2',\dots,n'\}$. However, it is not as easy to exploit this sequence in proving Theorem 12 as it was to use the original sequence of Bouc when proving Theorem 7. In fact, it is necessary to understand the top homology $\widetilde{H}_{m-1}(M_{m,n})$ for certain pairs (m,n) in order to proceed with the induction process used in our proof. Indeed, we use the long exact sequence to show that $\widetilde{H}_{\nu(m,n)-1}(M_{m,n})$ is generated by elements of the form $\alpha * \beta$, where α is an element in the *top* homology of some chessboard complex $M_{I,J}$, $I \subseteq [m], J \subseteq [n]'$, β is an element in the homology of $M_{[m]\setminus I,[n]'\setminus J}$ and $*$ is the operation associated with concatenating oriented simplices. We then need to decompose α into "smaller" cycles in order to apply induction. This is accomplished by constructing a basis for the top homology of the chessboard complex.

Our construction of a basis for the top homology and cohomology of $M_{m,n}$ for all pairs (m,n) is based on the Robinson-Schensted correspondence. It follows from Friedman and Hanlon's decomposition of the homology of $M_{m,n}$ into irreducible $S_m \times S_n$-modules that the rank of the top homology of the chessboard complex $M_{m,n}$ is the number of pairs of standard Young tableaux (S,T) such that S has m cells, T has n cells and the Young diagram underlying S is obtained from that underlying T by removing the first row. Let $\mathcal{P}_{m,n}$ be the set of such pairs of standard tableau. We construct for each $(S,T) \in \mathcal{P}_{m,n}$, an element $\rho(S,T) \in \widetilde{H}_{m-1}(M_{m,n})$ and an element $\gamma(S,T) \in \widetilde{H}^{m-1}(M_{m,n})$, and show that these elements form bases for homology and cohomology, respectively.

Let $(S,T) \in \mathcal{P}_{m,n}$. First add a cell with entry ∞ to the bottom of each of the first $n-m$ columns of S (possibly including some empty columns) to obtain a semistandard tableau S^* of the same shape as T. (Here ∞ represents a number larger than m.) The inverse of the Robinson-Schensted bijection applied to (S^*,T) produces a permutation σ of the multiset $\{1,2,\dots,m,\infty^{n-m}\}$. The multiset permutation σ corresponds naturally to the oriented simplex (ie. matching) of $M_{m,n}$ given by

$$((\sigma(i_1),i_1),(\sigma(i_2),i_2),\dots,(\sigma(i_m),i_m)),$$

where $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_m)$ is the subword of σ obtained by removing the ∞'s. This oriented simplex is clearly a cocycle since it is in the top dimension. Let $\gamma(S,T)$ be the coset of the coboundary group $B^{m-1}(M_{m,n})$ that contains this oriented simplex.

The construction of the cycles is a bit more involved. Recall that in the inverse Robinson-Schensted procedure, an entry "pops" from a cell in the top row of the left tableau when an entry is "crossed out" of the right tableau. For each top cell, we must keep track of the entries of S^* that are popped and the corresponding entries of T that are crossed out. For each $i = 1,2,\dots,n-m$ let A_i^* be the multiset of entries that are popped from the ith cell of the top row of S^* and let B_i be the corresponding set of entries that are crossed out of T. One can easily see that A_i^* is actually a set and $\infty \in A_i^*$ for all i. Now let $A_i = A_i^* \setminus \{\infty\}$. So $|A_i| = |B_i| - 1$. It is easily observed that $M_{m,n}$ is an orientable pseudomanifold whenever $m = n-1$ which implies that its top homology is cyclic. For $i = 1,\dots,n-m$, let α_i be a generator of the cyclic group $\widetilde{H}_{|A_i|-1}(M_{A_i,B_i})$. Now define

$$\rho(S,T) = \alpha_1 * \cdots * \alpha_{n-m},$$

which is unique up to sign.

Theorem 13. *Let $m \le n$. Then*

(1) *$\{\rho(S,T) \mid (S,T) \in \mathcal{P}_{m,n}\}$ is a basis for $\widetilde{H}_{m-1}(M_{m,n})$.*
(2) *$\{\gamma(S,T) \mid (S,T) \in \mathcal{P}_{m,n}\}$ is a basis for the free part of $\widetilde{H}^{m-1}(M_{m,n})$.*

Theorem 13 is proved by finding an ordering of the pairs of standard tableaux

$$(S_1, T_1), \ldots, (S_t, T_t)$$

in $\mathcal{P}_{m,n}$ such that the matrix

$$(\langle \rho(S_i, T_i), \gamma(S_j, T_j) \rangle)_{i,j=1,\ldots,t}$$

is triangular with 1's on the diagonal. Here $\langle , \rangle$ denotes a pairing of homology and cohomology. The invertibility of the matrix establishes independence of the cycles (and cocyles). The result then follows from the Friedman-Hanlon determination of the rank of rational homology.

The final result we mention here is an answer to another question from [BLVZ]. Given the connectivity properties of M_n and $M_{m,n}$ described in Theorems 2 and 9, the authors of that paper ask whether the $(\nu(n)-1)$-skeleton of M_n is shellable and whether the $(\nu(m,n)-1)$-skeleton of $M_{m,n}$ is shellable. The second question was answered affirmatively by G. Ziegler in [Zi]. We provide an affirmative answer to the first question.

Theorem 14. *For every $n \geq 2$, the $(\nu(n)-1)$-skeleton of M_n is shellable.*

We prove Theorem 14 by exhibiting an ordering of the facets of the given skeleton which is a shelling. We will now describe this ordering. For any graph G, let $V(G)$ denote the vertex set of G and let $E(G)$ denote the edge set of G. An ordered partition of a graph G is a sequence $(G_1, \ldots, G_r)$ of subgraphs such that $\{V(G_1), \ldots, V(G_r)\}$ is a partition of $V(G)$ and $\{E(G_1), \ldots, E(G_r)\}$ is a partition of $E(G)$. We define for each $G \in M_n$ an ordered partition $P(G) = (G_1, \ldots, G_r)$ of G, as follows. Set $G_0 = (\emptyset, \emptyset)$ and set $i = 1$. If G_j has been defined for $0 \leq j < i$, set

$$X_i := [n] \setminus \bigcup_{j<i} V(G_j).$$

If $X_i = \emptyset$ then stop. If $|X_i| = 1$ then set $G_i = (X_i, \emptyset)$ and stop. Otherwise, let x_i, y_i be the two smallest elements of X_i and let G_i be the smallest subgraph of G that contains vertices x_i, y_i and all incident edges. Increase i by one and continue.

For example, if $n = 10$ and $E(G) = \{16, 24, 37\}$ then

$$P(G) = (G_1, G_2, G_3, G_4),$$

where

$$G_1 = (\{1,2,4,6\}, \{16,24\}), \quad G_2 = (\{3,5,7\}, \{37\}), \quad G_3 = (\{8,9\}, \emptyset), \quad G_4 = (\{10\}, \emptyset).$$

Next we define a partial order on the set of all graphs on subsets of $[n]$ as follows: $G_1 \prec G_2$ if either $|V(G_1)| < |V(G_2)|$ or G_1 is a single edge uv and G_2 consists of isolated vertices u, v. This partial order induces a partial order on the set of all ordered partitions of graphs on $[n]$, namely lexicographical order. Lexicographical order induces a partial order on the facets of the $(\nu(n)-1)$-skeleton of M_n given by $G \prec_L H$ if $P(G)$ precedes $P(H)$ in the lexicographical order on ordered partitions of graphs. We prove that any linear extension of the partial order $\prec_L$ is a shelling.

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