# BASES FOR THE FLAG $f$-VECTORS OF EULERIAN POSETS

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Abstract. We investigate various bases for the flag $f$-vectors of Eulerian posets. Many of the change-of-basis formulas are seen to be triangular. One change-of-basis formula implies the following: If the Charney-Davis Conjecture is true for order complexes, then certain sums of cd-coefficients are non-negative in all Gorenstein* posets. In particular, cd-coefficients with no adjacent $c$'s are non-negative. A convolution formula for cd-coefficients, together with the proof by M. Davis and B. Okun of the Charney-Davis Conjecture in dimension 3, imply that certain additional cd-coefficients are non-negative for all Gorenstein* posets. In particular we verify, up to rank 6, Stanley's conjecture that the coefficients in the cd-index of a Gorenstein* ranked poset are non-negative.

Résumé. Nous étudions certaines bases pour les $f$-vecteurs drapeaux des ensembles Euleriens partiellement ordonnés. Plusieurs des formules de changement de base sont triangulaires. L'une d'elles implique que si la Conjecture de Charney-Davis est vraie pour les complexes de chaînes, alors certaines sommes de coefficients $cd$ sont non-negatives dans tous les ensembles Gorenstein* partiellement ordonnés. En particulier, les coefficients $cd$ sans $c$ adjacents sont non-negatifs. Une formule de convolution pour les coefficients $cd$ ainsi qu'une preuve de la conjecture de Charney-Davis en 3 dimensions par M. Davis et B. Okun implique que certains autres coefficients $cd$ sont non-negatifs pour tous les ensembles Gorenstein* partiellement ordonnés. En particulier, nous vérifions jusqu'au rang 6 la conjecture de Stanley qui dit que tous les coefficients de l'index $cd$ d'un ensemble Gorenstein* partiellement ordonné sont non-negatifs.

## 1. Introduction

Much of the enumerative information about a graded poset is contained in its *flag $f$-vector*, which counts chains of elements according to the ranks they visit. Many naturally arising graded posets are Eulerian, that is, their Möbius function has the simple formula $\mu(x,y) = (-1)^{\rho(y)-\rho(x)}$, where $\rho$ is the rank function (see [24, Sections 3.8, 3.14]). In this paper we study various bases for the flag f-vectors of Eulerian posets. M. Bayer and L. Billera [3] proved a set of linear relations on the flag f-vector of an Eulerian poset, now commonly called the Bayer-Billera relations. They also proved that the Bayer-Billera relations and the relation $f_\emptyset = 1$ are the complete set of affine relations satisfied by the flag f-vectors of all Eulerian posets. They exhibited a basis for the linear span $\mathcal{E}_n$ of flag $f$-vectors of Eulerian posets of rank $n+1$, and thereby showed the dimension of $\mathcal{E}_n$ to be $F_n$, where $F_n$ is the Fibonacci number with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

Another important basis for the space of Eulerian flag $f$-vectors is the cd-index $\Phi$, introduced by M. Bayer and A. Klapper [5], following a suggestion by J. Fine. The cd-index is a polynomial in non-commuting variables $c$ and $d$. Some of the results in this paper are motivated by a conjecture of R. Stanley [25] that the coefficients of the cd-index are non-negative whenever $P$ is Gorenstein*, that is, whenever the order complex of $P - \{\hat{0}, \hat{1}\}$ triangulates a homology sphere. The coefficient of a cd-word $w$ in the cd-index of $P$ is denoted $\langle w|\Phi_P\rangle$.

An Eulerian poset $P$ has the *Charney-Davis property* if $\langle d^k|\Phi_{[x,y]}\rangle \geq 0$ for any interval $[x,y] \subseteq P$ of rank $2k+1$, for any $k$. The name of this property refers to the Charney-Davis conjecture about Gorenstein* flag simplicial complexes (see [13, 23] for details). In the special case of order complexes, the conjecture is that any Gorenstein* poset has the Charney-Davis property. (This connection between the Charney-Davis conjecture and the cd-index was observed by E. Babson [13]). The conjecture is trivial for Gorenstein* flag simplicial complexes of dimension $\leq 2$, and M. Davis and B. Okun [14] proved it for Gorenstein* flag complexes of dimension 3 and thus, trivially, for dimension 4. This implies in particular that Gorenstein* posets of rank $\leq 6$ (corresponding to order complexes of dimension $\leq 4$) have the Charney-Davis property.

The main results of this paper are the following: A common indexing set $\mathcal{F}_n$ is given for bases of $\mathcal{E}_n$. The Fibonacci partial order and three stronger orders are defined on $\mathcal{F}_n$. Each of these orders can be extended to a lexicographic order. Many of the basis-changes are shown to be upper- or lower-triangular in the Fibonacci order, or in one of its strengthenings. Most of the remaining basis-changes are not triangular.

We define the Charney-Davis index $\Gamma$, a non-commutative polynomial which serves as a basis for $\mathcal{E}_n$, and has non-negative coefficients when $P$ has the Charney-Davis property. The equation relating the Charney-Davis index to the cd-index (Equation (19)), leads directly to the following: ,

**Theorem 1.** *Let $P$ be an Eulerian poset with the Charney-Davis property. Then for any cd-word $w$:*

$$\sum_{v \geq w} 2^{l(v)} \langle v | \Phi_P \rangle \geq 0.$$

Here, "$\leq$" is the Fibonacci order. If $w$ has no two adjacent $c$'s, it is maximal in the Fibonacci order. Thus Theorem 1 has the following consequence:

**Corollary 2.** *Let $P$ be an Eulerian poset with the Charney-Davis property. If $w$ is a cd-word with no two adjacent $c$'s, then $\langle w|\Phi_P\rangle \geq 0$.*

Other closely related changes-of-basis (Equations (17) and (18)) imply a convolution formula (Proposition 13) for cd-coefficients which is used to prove the following theorem, which also appears in [2]:

**Theorem 3.** *Let $P$ be an Eulerian poset of rank $n+1$. The coefficients of $dc^{n-2}$, $c^{n-2}d$, $cdc^{n-3}$ and $c^{n-3}dc$ are all non-negative. Also, let $v$ be a cd-monomial starting and ending in $d$ and alternating $dcdc\cdots cd$ with at least one $c$, such that $c^k v c^m$ has degree $n$. Then $\langle c^k v c^m|\Phi_P\rangle \geq 0$.*

In [2], it is also shown that there are no other lower or upper bounds on cd-coefficients of Eulerian posets.

The convolution formula and the result of Davis and Okun give the following theorem, which supports Stanley's conjecture.

**Theorem 4.** *Let $P$ be a Gorenstein* poset of rank $n+1$. Let $w$ be a cd-monomial of degree $n$ with $w = c^k d^{e_1} c d^{e_2} c \cdots c d^{e_j} c^m$ for $j \geq 1$ and $e_i \in \{1,2\}$ for each $i$. (If $e_1 = 2$, require that $k \in \{0,1\}$; if $e_j = 2$, require that $m \in \{0,1\}$).*

*Then $\langle w|\Phi_P\rangle \geq 0$.*

The same proof gives the following strengthening of Corollary 2.

**Theorem 5.** *Let $P$ be a poset of rank $n+1$ with the Charney-Davis property. Let $w$ be a cd-monomial of degree $n$ with $w = c^k d^{e_1} c d^{e_2} c \cdots c d^{e_j} c^m$ for $j \geq 1$ and $e_i \geq 1$ for each $i$. (If $e_1 > 1$, require that $k \in \{0,1\}$; if $e_j > 1$, require that $m \in \{0,1\}$).*

*Then $\langle w|\Phi_P\rangle \geq 0$.*

For $n \leq 5$, any cd-monomial $w$ satisfies the hypotheses of Theorem 4. Therefore:

**Corollary 6.** *If $P$ is a Gorenstein* poset with rank$(P) \leq 6$, then $\Phi_P$ has non-negative coefficients.*

For Gorenstein* posets of rank 7, the only coefficients of $\Phi$ not shown in this paper to be non-negative are the coefficients of $ccdd$, $dccd$, $ddcc$ and $ddd$. The most that the present work can say about these coefficients is that $2\langle v|\Phi\rangle + \langle ddd|\Phi\rangle \geq 0$ for $v = ccdd$, $dccd$ or $ddcc$. (See the proof of Theorem 1).

The paper is structured as follows: In Section 2, we review five bases for the flag $f$-vectors of graded, not-necessarily-Eulerian posets, and give several change-of-basis formulas. In Section 3, a common indexing set $\mathcal{F}_n$ is given for bases of $\mathcal{E}_n$, the linear span of flag $f$-vectors of rank $n+1$

Eulerian posets. Several partial orders are defined on $\mathcal{F}_n$. Section 4 contains a list of bases for $\mathcal{E}_n$, and a summary of relationships between the various bases. In Sections 5 and 6, we give several change-of-basis formulas in $\mathcal{E}_n$. Section 7 contains the definition of the Charney-Davis index, its relations to other bases, and the proof of Theorem 1. Section 8 contains the remaining changes of basis. Convolution formulas for $\Phi$ and the proof of Theorem 3 appear in Section 9, and Theorems 4 and 5 are proven in Section 10.

## 2. Bases for the Flag $f$-Vectors of Graded Posets

Throughout this paper, $P$ is a graded poset of rank $n+1$ with rank function $\rho$. For a chain $c$ in $P-\{\hat{0},\hat{1}\}$ define $\rho(c)=\{\rho(x) : x \in c\}$. For any $S \subseteq [n]$, let $C(S)$ be the set of chains $c$ in $P-\{\hat{0},\hat{1}\}$ such that $\rho(c)=S$. The *flag $f$-vector* is $f_S := |C(S)|$. The flag $f$-vector is written $f_S(P)$ when it is important to specify the poset $P$ explicitly. It is useful to think of each entry $f_S(\cdot)$ of the flag $f$-vector as a linear functional on graded posets, as in [10]. The notation $f_S$ here corresponds to $f_S^{n+1}$ in [10] for $S \subseteq [n]$. When we speak of bases for flag $f$-vectors, we mean bases for the vector space of linear combinations of these functionals.

Later on, $P$ will be assumed to be Eulerian. But first, consider bases for the flag $f$-vectors of graded posets $P$ of rank $n+1$, not necessarily Eulerian. There is the flag $h$-vector $h_S$, related to the flag $f$-vector by the equations:

$$h_S := \sum_{T \subseteq S} (-1)^{|S-T|} f_T \tag{1}$$

$$f_S = \sum_{T \subseteq S} h_T. \tag{2}$$

The flag $k$-vector is defined in [8], where it is used to determine the integral span of the flag $f$-vectors of zonotopes. It is related to the flag $h$-vector by

$$k_S := \sum_{T \subseteq S} (-1)^{|S-T|} h_T \tag{3}$$

$$h_S = \sum_{T \subseteq S} k_T. \tag{4}$$

Similar formulas can be written to relate the flag $k$-vector to the flag $f$-vector.

In [9], the flag $l$-vector is used to describe the cone generated by the flag $f$-vectors of graded posets. In what follows and throughout the paper, $S^c := [n]-S$.

$$l_S := (-1)^{n-|S|} \sum_{T \supseteq S^c} (-1)^{|T|} f_T \tag{5}$$

$$f_S = \sum_{T \subseteq S^c} l_T. \tag{6}$$

The flag $L$-vector is introduced in [4]. It is a variant on the flag $l$-vector, used to describe some of the facets of the closed cone generated by the flag $f$-vectors of Eulerian posets.

$$L_S := (-1)^{n-|S|} \sum_{T \supseteq S^c} (-2)^{-|T|} f_T \tag{7}$$

$$f_S = 2^{|S|} \sum_{T \subseteq S^c} L_T. \tag{8}$$

By combining formulas and reversing the order of summation, it is easy to see that:

$$L_S = (-1)^{|S|} \sum_{T \supseteq S} 2^{-|T|} k_T \tag{9}$$

$$k_S = (-2)^{|S|} \sum_{T \supseteq S} L_T. \tag{10}$$

## 3. Orders on Sets Counted by the Fibonacci Numbers

The dimension of $\mathcal{E}_n$ is $F_n$, the Fibonacci number with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Thus it is necessary to index basis elements by a set counted by $F_n$. A convenient indexing set is $\mathcal{F}_n$, the set of words in the set $\{1, 2\}$ with the sum of the letters equal to $n$. These words are referred to as "1-2-words." The sum of the letters in a 1-2-word $w$ is called the *degree* $\deg(w)$ and the *length* $l(w)$ is the number of letters in $w$. Many of the change-of-basis formulas involve partial orders on $\mathcal{F}_n$.

The Fibonacci order "$\leq$," is a partial order on $\mathcal{F}_n$, generated by the following covering relations: $v \gtrdot w$ if $w$ is obtained from $v$ by replacing some 2 with 11. Intervals in the Fibonacci order are boolean algebras, so if $w \leq v$ then the Möbius function is $\mu(w, v) = (-1)^{l(v)-l(w)}$. To avoid confusion, the phrase "Fibonacci order," the interval notation $[v, w]$ and the symbol "$\leq$" always mean the "ordinary" Fibonacci order. Relations and intervals in all other orders defined below are always marked with subscripts.

Most of the bases discussed here are built on subsets of $[n]$. There are several natural ways to choose $F_n$ subsets of $[n]$. The *sparse* subsets of $[n]$ are the subsets which contain no pair of adjacent elements and which do not contain $n$. If $S^c$ is sparse, then $S$ is called *dense*. Each 1-2-word $w$ corresponds to a sparse set $S_w$ by lining $w$ up with the set $[n]$ such that each 2 covers two elements of $[n]$. Then for each 2 in $w$, the left element covered by the 2 is included in $S_w$ and the right element covered by the 2 is excluded from $S_w$. Elements covered by 1's are also excluded. This construction also gives rise to a dense set $D_w := (S_w)^c$. So, for example, if $w = 12221212$, then $S_w = \{2, 4, 6, 9, 12\} \subset [13]$ and $D_w = \{1, 3, 5, 7, 8, 10, 11, 13\}$. The Fibonacci order on $\mathcal{F}_n$ is ordinary containment of sparse sets. Also, $l(w) = |D_w| = n - |S_w|$.

Even and anti-even sets are also counted by $F_n$. A set $S \subseteq [n]$ is *even* if it is a disjoint union of intervals of even cardinality. A set is *anti-even* if its complement is even. Given a 1-2-word $w$, an even set $E_w$ is obtained by lining up $w$ with the set $[n]$ as before, and including all elements covered by 2's. The corresponding anti-even set is $A_w := (E_w)^c$. So if $w = 12221212$, then $E_w = \{2, 3, 4, 5, 6, 7, 9, 10, 12, 13\}$ and $A_w = \{1, 8, 11\}$. The Fibonacci order on $\mathcal{F}_n$ corresponds to *even containment* of even sets, as defined in [2]. We recall the definition here: If $U$ and $V$ are even sets we say $V$ contains $U$ evenly and write $U \subseteq_e V$ if $U \subseteq V$ and if $V - U$ is an even set. The length function is $l(w) = n - \frac{1}{2}|E_w| = \frac{1}{2}(|A_w| + n)$.

A natural extension of the Fibonacci order is the *strong Fibonacci order*, "$\leq_s$," corresponding to ordinary containment of even sets. The strong Fibonacci order is defined by the covering relations: $v \gtrdot_s w$ if $w$ is obtained from $v$ by replacing some $2^k$ with $12^{k-1}1$ for any $k \geq 1$. The Möbius function of the strong Fibonacci order, denoted by $\mu_s$, has the following useful, easily-proved property:

**Proposition 7.** *If $v \leq_s w = w_1 c w_2$, then $v = v_1 c v_2$ for $v_1 \leq_s w_1$ and $v_2 \leq_s w_2$, and $[v, w]_s \cong [v_1, w_1]_s \times [v_2, w_2]_s$. In particular,*

$$\mu_s(v, w) = \mu_s(v_1, w_1) \cdot \mu_s(v_2, w_2). \quad \square$$

Two extensions of the strong Fibonacci order are also required. The *right Fibonacci order* "$\leq_r$" is defined by covers: $v \gtrdot_r w$ if $w$ is obtained from $v$ by replacing some 12 with 21, or by replacing an initial 2 with 11. It is easy to see that the right Fibonacci order indeed extends the strong Fibonacci order. The *left Fibonacci order* "$\leq_l$" is defined similarly by covers, and is isomorphic to the right Fibonacci order on by reversing the order of words. Also, $\leq_r$ and $\leq_l$ are dual orders on constant-length subsets of $\mathcal{F}_n$.

The right Fibonacci order is extended by a lexicographic order, reading from the right of the word, with $2 > 1$. The same is true of the left Fibonacci order, reading from the left of the word. Also, both strong and ordinary Fibonacci order are extended by either lexicographic order. Thus all the triangular basis-changes given below are in particular triangular with respect to some lexicographic order.

## 4. Bases for $\mathcal{E}_n$

The sparse flag $f$-vector is indexed by $w \in \mathcal{F}_n$, with $f^{sp}_w := f_{S_w}$. The dense flag $f$-vector is $f^d_w := f_{D_w}$. The even flag $f$-vector is $f^e_w := f_{E_w}$ and the anti-even flag $f$-vector is $f^a_w := f_{A_w}$. Similarly for $h$, $k$, $l$ and $L$.

Some of the bases are not overtly related to subsets of $[n]$: in particular, the cd-index $\Phi$ and the ce-index $\Lambda$. The ce-index, defined by Stanley [25], is a polynomial in non-commuting variables $c$ and $e$ with $\deg(c) = \deg(e) = 1$, and $e$'s occurring only in even-length strings. It is simple and useful to index these bases by $\mathcal{F}_n$ as well. A cd-monomial is obtained from $w \in \mathcal{F}_n$ by replacing 1 with $c$ and 2 with $d$. To obtain a ce-monomial, replace 2 with $ee$ instead. We abuse notation slightly by writing $\langle w|\Phi\rangle$ for $w \in \mathcal{F}_n$ when we mean the coefficient of the cd-monomial corresponding to $w$. Similarly for $\langle w|\Lambda\rangle$.

A consequence of the Bayer-Billera relations is that for all Eulerian posets, $h_S = h_{S^c}$. Therefore the sparse flag $h$-vectors are equivalent to the dense flag $h$-vectors, and the even and anti-even flag $h$-vectors are equivalent. The following turn out to be bases for $\mathcal{E}_n$ (We will define $\Gamma$ later):

**Bases for $\mathcal{E}_n$:**

| | |
|---|---|
| $f^{sp}$ | The sparse flag $f$-vectors |
| $h^{sp}$ | The sparse flag $h$-vectors, or equivalently the dense flag $h$-vectors $h^{sp}$ |
| $k^{sp}$ | The sparse flag $k$-vectors |
| $l^d$ | The dense flag $l$-vectors |
| $f^a$ | The anti-even flag $f$-vectors, or equivalently the even flag $l$-vectors $l^e$ |
| $k^e$ | The even flag $k$-vectors |
| $\Phi$ | The cd-index |
| $\Lambda$ | The ce-index, or equivalently the even flag $L$-vectors $L^e$ |
| $\Gamma$ | The Charney-Davis index. |

Given a partial order "$\leq$", we say a change-of-basis formula is triangular "in $\leq$" if it is triangular in any linear extension of $\leq$. Most of the relations between the above bases are either not triangular, or are triangular in one of the four partial orders defined in the previous section. Figure 8 is a summary of which bases are related triangularly, in which partial orders.

**Figure 8. Summary of bases and relations.** *The symbols* ⋱, ◺ *and* ◹ *indicate diagonal, lower- and upper-triangular relationships respectively. Subscripts indicate strong, left and right Fibonacci orders. Absence of a subscript indicates ordinary Fibonacci order. The symbol* □ *indicates that the relationship is not triangular in any order. The entry* $◺_r◹_l$ *denotes that either* $◺_r$ *or* $◹_l$ *is appropriate.*

| | $f^{sp}$ | $l^d$ | $h^{sp}$ | $k^{sp}$ | $\Phi$ | $\Gamma$ | $\Lambda$ | $k^e$ | $f^a$ |
|---|---|---|---|---|---|---|---|---|---|
| $f^{sp}$ | id | | | | | | | | |
| $l^d$ | ⋱ | id | | | | | | | |
| $h^{sp}$ | ◺ | ◺ | id | | | | | | |
| $k^{sp}$ | ◺ | ◺ | ◺ | id | | | | | |
| $\Phi$ | ◺$_r$ | ◺$_r$ | ◺$_r$ | ◺$_r$◹$_l$ | id | | | | |
| $\Gamma$ | ? | ? | □ | ◹$_l$ | ◹ | id | | | |
| $\Lambda$ | ? | ? | □ | ◹$_l$ | ◹ | ⋱ | id | | |
| $k^e$ | □ | □ | □ | ◹$_l$ | ◹$_s$ | ◹$_s$ | ◹$_s$ | id | |
| $f^a$ | ? | ? | □ | □ | □ | ◺$_s$ | ◺$_s$ | □ | id |

## 5. Sparse and Dense Bases

In this section, we discuss the bases $f^{sp}$, $h^{sp}$, $k^{sp}$ and $l^d$. Bayer and Billera [3] showed that the sparse flag $f$-vectors are a basis for the linear span of flag $f$-vectors of Eulerian posets. Since the Fibonacci order is containment on sparse sets, Equations (1) through (6) imply relations between these sparse and dense bases. For example, for all $w \in \mathcal{F}_n$:

$$h^{sp}_w = \sum_{v \leq w} (-1)^{l(v)-l(w)} f^{sp}_v \tag{11}$$

$$k^{sp}_w = \sum_{v \leq w} (-1)^{l(v)-l(w)} h^{sp}_v \tag{12}$$

$$f^{sp}_w = (-1)^{l(w)} l^d_w. \tag{13}$$

Similar relations between any two of $f^{sp}$, $h^{sp}$, $k^{sp}$ and $l^d$ can be obtained by combining equations or by Möbius inversion. Since the Möbius function of Fibonacci order is $(-1)^{l(w)=l(v)}$, Möbius inversion is straightfoward.

Equation (13) is verified by the following fact.

**Proposition 9.** *Let $P$ be a graded poset of rank $n+1$ with $h_S = h_{S^c}$ for every $S \in [n]$ (e.g. if $P$ is Eulerian). Then for any $S \in [n]$, $l_S = (-1)^{|S|} f_{S^c}$.* □

## 6. Even and Anti-Even Bases

In this section we deal with the bases $f^a$, $k^e$ and $\Lambda$. First, notice that the even flag $l$-vector is the same as the anti-even flag $f$-vector. By Proposition 9, for all $w$:

$$l^e_w = (-1)^{2n-2l(w)} f^a_w = f^a_w.$$

One can prove that the $f^a$ span, using the Bayer-Billera relations [10, Corollary 3.7]. Alternately, one can appeal to a result about flag $L$-vectors. In [4], it is shown that in an Eulerian poset, $L_T = 0$ when $T$ is not even, and $L_T = \langle w|\Lambda\rangle$ when $T = E_w$. Restricting Equation (8) to anti-even flag $f$-vectors yields:

$$f^a_w = 2^{2l(w)-n} \sum_{v \leq_s w} \langle v|\Lambda\rangle. \tag{14}$$

Since the coefficients of the ce-index span, so do the $f^a_w$. Similarly, Equation (10) can be used to show that the even flag $k$-vector spans, by relating $k^e$ to $\Lambda$.

There is a simple convolution formula for $f^a$ [10, Proposition 1.2]. If the degree of $w_1$ is $k-1$, then

$$f^a_{w_1 c w_2} = \sum_{\substack{x \in P \\ \rho(x)=k}} f^a_{w_1}([\hat{0},x]) \cdot f^a_{w_2}([x,\hat{1}]). \tag{15}$$

The following is straightforward, by combining Equation (15) with Equation (14), and appealing to Proposition 7.

**Proposition 10.** *If* $w = w_1 c w_2$, *with* $\deg(w_1) = k-1$ *then*

$$2\langle w|\Lambda_P\rangle = \sum_{\substack{x \in P \\ \rho(x)=k}} \langle w_1|\Lambda_{[\hat{0},x]}\rangle \cdot \langle w_2|\Lambda_{[x,\hat{1}]}\rangle.$$

$\square$

For each $c \in C(S)$, write $c = (\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_{m+1} = \hat{1})$. Proposition 10 and induction imply the following:

**Proposition 11.**

$$2^{2l(w)-n}\langle w|\Lambda_P\rangle = \sum_{C(A_w)} \prod_{i=0}^{m} \langle e^{\rho(x_{i+1})-\rho(x_i)-1}|\Lambda_{[x_i,x_{i+1}]}\rangle.$$

$\square$

## 7. The cd-Index and the Charney-Davis Index

The ce-index $\Lambda$ is related to the cd-index $\Phi$ by the formula

$$\Lambda(c,e) = \Phi(c,(c^2-e^2)/2). \tag{16}$$

As observed in [2, Equation 1] and as a consequence of Equation (16), above:

$$\langle w|\Phi\rangle = (-2)^{n-l(w)} \sum_{v \geq w} \langle v|\Lambda\rangle \tag{17}$$

$$\langle w|\Lambda\rangle = (-2)^{-n}(-1)^{l(w)} \sum_{v \geq w} 2^{l(v)}\langle v|\Phi\rangle. \tag{18}$$

For an Eulerian poset $Q$ of rank $2k+1$, let $\kappa(Q) := \langle d^k|\Phi_Q\rangle$. Proposition 11 and Equation (18) imply:

**Proposition 12.**

$$\sum_{v \geq w} 2^{l(v)}\langle v|\Phi_P\rangle = 2^{n-l(w)} \sum_{C(A_w)} \prod_{i=0}^{k} \kappa([x_i,x_{i+1}]). \quad \square$$

The form of Proposition 12 motivates the definition of the *Charney-Davis index* $\Gamma$, a polynomial in non-commuting variables $\gamma$ and $\delta$, degrees 1 and 2 respectively. Elements of $\mathcal{F}_n$ correspond to $\gamma\delta$-monomials $w$ of degree $n$, and $\Gamma$ is defined coefficient-wise:

$$\langle w|\Gamma\rangle := \sum_{C(A_w)} \prod_{i=0}^{k} \kappa([x_i,x_{i+1}]).$$

Thus Proposition 12 says:

$$\langle w|\Gamma\rangle = 2^{l(w)-n} \sum_{v \geq w} 2^{l(v)}\langle v|\Phi\rangle. \tag{19}$$

Combining formulas and Möbius inversion yield the triangular relations.

Equation (19) and the definition of $\Gamma$ are the proof of Theorem 1. If $P$ has the Charney-Davis property, then the coefficients of $\Gamma$ are non-negative, and therefore by Equation (19),

$$\sum_{v\geq w} 2^{l(v)}\langle v|\Phi\rangle = 2^{n-l(w)}\langle w|\Gamma\rangle \geq 0.$$

## 8. Other Changes of Basis

In this section, we account for the remaining triangular basis-change formulas. Each remaining triangular relation is a consequence of the change-of-basis formula relating $\Phi$ to $k^{sp}$. In [8], $k^{sp}$ is defined as a sum of cd-coefficients, and is shown to be related to $f^sp$ as indicated above. In [7], the cd-coefficients are given as a signed sum of sparse $k$-vectors. Formulas relating the cd-index to the full flag $k$-vector can be found in [15].

To relate $\Phi$ to $k^{sp}$ the following two operations on 1-2-words are useful. The *left 2-shift* of $w$ is ${}^2w$, obtained by removing the rightmost 2 from $w$ and placing it at the beginning of the word. The *right 1-shift* of $w$ is $w^1$, obtained by removing the leftmost 1 from $w$ and placing it at the end of the word. The inversion number $i(w)$ is the number of instances of a 2 appearing before a 1 in $w$. For example, if $w = 12221212$, then ${}^2w = 21222121$, $w^1 = 22212121$ and $i(w) = 7$. The following are respectively [8, Definition 6.5] and [7, Proposition 7.1], translated into the language of the right Fibonacci order.

$$k^{sp}_w = \sum_{w^1\leq_r v\leq_r w} \langle v|\Phi\rangle \tag{20}$$

$$\langle w|\Phi\rangle = \sum_{{}^2w\leq_r v\leq_r w} (-1)^{i(v)-i(w)}k^{sp}_v. \tag{21}$$

These equations can be used to explain the rest of the entries in the table in Figure 8. Recall that on constant-length subsets of $\mathcal{F}_n$, the left and right Fibonacci orders are dual. The intervals $[w^1, w]_r$ and $[{}^2w, w]_r$ are constant-length subsets. Thus $\Phi$ is related to $k^{sp}$ lower-triangularly in the right order, or upper-triangularly in the left order. Because the right Fibonacci order extends the ordinary and strong Fibonacci orders, Equations (20) and (21) combine with previous equations, relating $\Phi$ to $h^{sp}$, $f^{sp}$ and $l^d$ lower-triangularly in the right Fibonacci order. Similarly, Equations (20) and (21) combine with previous equations, relating $k^{sp}$ to $\Lambda$, $\Gamma$ and $k^e$ upper-triangularly in the left Fibonacci order.

## 9. Convolution Formulas for cd-Coefficients

The change-of-basis formulas can be applied to give convolution formulas for some of the bases. This was already done in Proposition 10, where Equation (15), a trivial convolution formula for $f^a$, became a less trivial convolution for $\Lambda$. In the same way, the convolution formula for $\Lambda$ yields a convolution formula for $\Phi$, which has important consequences for non-negativity of cd-coefficients. The proof uses Proposition 10, Equations (17) and (18), and the following observation: If $w = w_1dcdw_2$, then $v \geq w$ if and only if $v = v_1cv_2$ for $v_1 \geq w_1d$ and $v_2 \geq dw_2$.

**Proposition 13.** *If $w = w_1dcdw_2$ with $\deg(w_1d) = k-1$ then*

$$2\langle w|\Phi_P\rangle = \sum_{\substack{x\in P\\ \rho(x)=k}} \langle w_1d|\Phi_{[\hat{0},x]}\rangle \cdot \langle dw_2|\Phi_{[x,\hat{1}]}\rangle.$$

*If $w = cdw_2$, then*

$$2\langle w|\Phi_P\rangle = \sum_{\substack{x\in P\\ \rho(x)=1}} \langle dw_2|\Phi_{[x,\hat{1}]}\rangle.$$

Proposition 13 leads to the proof of Theorem 3 from the Introduction.

**Theorem 3.** *Let $P$ be an Eulerian poset of rank $n+1$. The coefficients of $dc^{n-2}$, $c^{n-2}d$, $cdc^{n-3}$ and $c^{n-3}dc$ are all non-negative. Also, let $v$ be a cd-monomial starting and ending in $d$ and alternating $dcdc\cdots cd$ with at least one $c$, such that $c^k v c^m$ has degree $n$. Then $\langle c^k v c^m | \Phi_P\rangle \geq 0$.*

*Proof.* For any Eulerian poset $P$ of rank $n+1$:

$$f_{\{i\}} = \begin{cases} 2 + \langle dc^{n-2}|\Phi_P\rangle & \text{if } i = 1 \\ 2 + \langle c^{i-1}dc^{n-i-1}|\Phi_P\rangle + \langle c^{i-2}dc^{n-i}|\Phi_P\rangle & \text{if } 2 \leq i \leq n \\ 2 + \langle c^{n-2}d|\Phi_P\rangle & \text{if } i = n \end{cases} \tag{22}$$

Because $f_{\{1\}} \geq 2$ and $f_{\{n\}} \geq 2$, $\langle dc^{n-2}|\Phi_P\rangle$ and $\langle c^{n-2}d|\Phi_P\rangle$ are non-negative. The remaining coefficients are non-negative by Proposition 13, and induction. □

## 10. Non-Negative cd-Index Coefficients of Gorenstein* Posets

In this section, we prove Theorem 4. The proof of Theorem 5 is nearly identical. In what follows, topological statements about a poset $P$ apply to the geometric realization of the order complex of $P - \{\hat{0}, \hat{1}\}$ [24, Section 3.8]. The $m$-skeleton of a graded poset is the subposet $Sk_m(P) = \{p \in P : \text{rank}(p) \leq m+1\} \cup \{\hat{1}\}$. If $P$ is the face poset of a CW-complex [11], this corresponds to the usual notion of skeleton.

**Theorem 4.** *Let $P$ be a Gorenstein* poset of rank $n+1$. Let $w$ be a cd-monomial of degree $n$ with $w = c^k d^{e_1} c d^{e_2} c \cdots c d^{e_j} c^m$ for $j \geq 1$ and $e_i \in \{1,2\}$ for each $i$. (If $e_1 = 2$, require that $k \in \{0,1\}$; if $e_j = 2$, require that $m \in \{0,1\}$).*

*Then $\langle w|\Phi_P\rangle \geq 0$.*

*Proof.* Davis and Okun [14] proved that $\langle d^2|\Phi\rangle$ is non-negative for a Gorenstein* poset of rank 5. We use their result, and induction on $j$. The case where $j = 1$ and $e_1 = 1$ is Proposition 14, below. If $j = 1$ and $e_1 = 2$, then the Davis-Okun result and Proposition 13 give non-negativity. For $j > 1$, the result follows by Proposition 13 and induction. □

**Proposition 14.** *If $P$ is an Eulerian poset of rank $n+1$ then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = (-1)^m \tilde{\chi}(Sk_m(P)) - 1.$$

*If $P$ is Gorenstein* then*

$$\langle c^m dc^{n-m-2}|\Phi_P\rangle = \mathit{rank}[H_m(Sk_m(P))] - 1 \geq 0.$$

*Proof.*

$$(-1)^m \tilde{\chi}(Sk_m(P)) = f_{\{m+1\}} - f_{\{m\}} + \cdots + (-1)^m f_{\{1\}} + (-1)^{m+1}.$$

By Equation (22), this is $\langle c^m dc^{n-m-2}|\Phi_P\rangle + 1$. Also,

$$(-1)^m \tilde{\chi}(Sk_m(P)) = \sum_i (-1)^{m-i} \text{rank}[H_i(Sk_m(P))].$$

If $P$ is a homology sphere then by the Rank-Selection Theorem for Cohen-Macaulay [12] posets, $H_i(Sk_m(P)) = 0$ for $i < m$. To show that $H_m(Sk_m(P)) \neq 0$, consider any element $p \in P$ of rank $m+2$. The interval $[\hat{0}, p]$ is a Gorenstein* poset, i.e. the order complex of $(\hat{0}, p)$ is a homology $m$-sphere. The orientation class of that homology sphere is a non-zero element of $H_m(Sk_m(P))$. □

The first assertion of Proposition 14 was also observed by I. Novik [22, Section 2].

## 11. Acknowledgments

The author wishes to thank his advisor V. Reiner for helpful conversations, and in particular for suggesting that Proposition 13 might be true and would prove Corollary 6. The author also wishes to thank L. Billera for helpful conversations, and R. Ehrenborg and M. Readdy for suggesting coproduct proofs of Equations (19) and Proposition 13.

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