A COMBINATORIAL CONSTRUCTION FOR SIMPLY–LACED LIE ALGEBRAS

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Abstract. This paper shows how to uniformly associate Lie algebras to the simply-laced Dynkin diagrams excluding E_8 by constructing explicit combinatorial models of minuscule representations using only graph-theoretic ideas. This involves defining raising and lowering operators in a space of ideals of certain distributive lattices associated to sequences of vertices of the Dynkin diagram.

1. Introduction

Our goal is to show how to (almost) uniformly construct the simply-laced Lie algebras using only graph theoretic ideas from the Dynkin diagrams. We will thus construct the Lie algebras corresponding to A_n, D_n, E_6 and E_7 using a method which is independent of type. The only case not covered is that of E_8, for which more sophisticated techniques must be used.

Apart from the generators and relations approach of Serre, (which constructs a generating set but not a basis), the only general construction of exceptional Lie algebras known to this author is that of Tits [14].

Technically no knowledge of Lie theory is assumed. Root systems are introduced in a simple fashion by examining mutation/reflection operators on graphs, in the spirit of affine Lie algebras ([3], [4], [6]). This approach is dual to the numbers game as studied recently by Proctor [10], and is systematically developed in [16].

We associate labelled distributive lattices called heaps (this terminology follows Viennot [15]) to particular graphs and construct representations of Lie algebras by raising and lowering operators on spaces of ideals of heaps.

The posets occurring are related to Bruhat orders in Coxeter groups ([7]), minuscule representations ([2], [8], [13]), the geometry of Schubert cells ([7], [12]), conformal field theory ([5]), and combinatorics ([9], [11]) . This paper provides another approach to their study using only graph theoretical considerations.

One of the key points is the definition of a parity function (taking on values ± 1) on certain convex subsets of distributive lattices associated to sequences of vertices of a graph.

From our construction we are able to identify Chevalley bases of the corresponding Lie algebras, clarify the associated structure constants, construct new models for spin representations and present very explicit realizations of the exceptional Lie algebras E_6 and E_7. The theory here generalises, to non simply-laced Dynkin diagrams, to Kac-Moody Lie algebras and to more general representations, but some of this involves considerable additional development, still in progress.

2. Neighbourly heaps for a graph

Let X be a simple graph. By an X-*sequence* we mean a sequence $s = (x_1, \ldots, x_n)$ of vertices of X. If we transform s to s' by switching x_i and x_{i+1} for some i then there are three possibilities:

(1) x_i and x_{i+1} are neighbours in X—(an X-*switch*)
(2) x_i and x_{i+1} are distinct and not neighbours—(a free *switch*)
(3) $x_i = x_{i+1}$—(a redundant *switch*).

Any X-sequence s' obtainable from s by free switches is defined to be *equivalent* to s; we write $s \simeq s'$ and let $[s]$ denote the equivalence class of s, which we call an X-*string*. We refer to the

x_i in $s = (x_1, \ldots, x_n)$ as the *occurrences* in s; as occurrences they are considered distinct even if as vertices of X there may be repetitions. We partially order the occurrences x_i in s by declaring $x_i < x_j$ if $i < j$ and x_i, x_j are neighbours in X. The resulting poset P_s is unchanged by free switches and so depends only on the X-string $[s]$. We refer to $P_s = P_{[s]}$ as the X-*heap* of $[s]$.

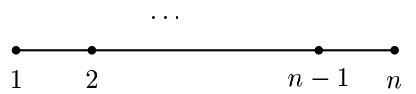
Proposition 2.1. *The X-string $[s]$ consists exactly of the total orderings of $P_{[s]}$ consistent with the partial order.*

If s and s' are X-sequences with s' obtainable from s by applying p X-switches and q free switches then let $\epsilon(s, s') = (-1)^p$.

Proposition 2.2. $\varepsilon(s, s') = (-1)^p$ *is well–defined, and depends only on $[s]$ and $[s']$.*

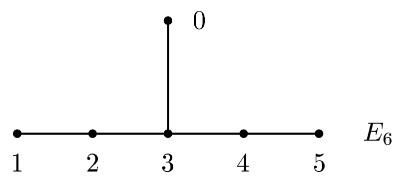
Thus $\varepsilon(s, s') = \varepsilon([s], [s'])$. This quantity will be called the *relative parity* of the X-strings $[s]$ and $[s']$, or of the corresponding heaps $P_{[s]}$ and $P_{[s']}$.

Example 1. Suppose $X = A_n$ labelled as shown.

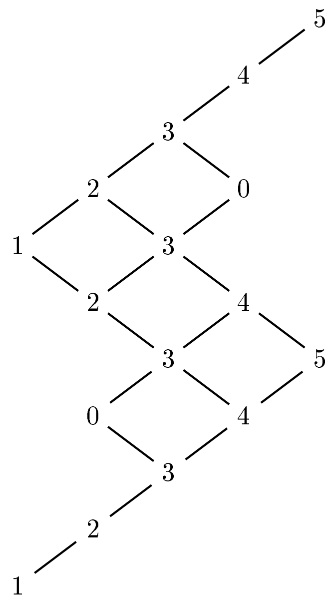


If we consider only X-sequences which are permutations of $\{1, \ldots, n\}$, the associated heaps are 'stock market graphs' where each successive node is either up or down from the previous. We get naturally a map from S_n to the set of sequences $\{(\varepsilon_1, \ldots, \varepsilon_{n-1}) \mid \varepsilon_i = \pm 1\} = T$. It is natural to ask for the distribution of this map: how many permutations map to a given $t \in T$? When t is the zigzag sequence alternating plus and minus one, this is known as André's Problem, and the answer is given by Euler numbers, or Entringer numbers. The general case is related to the number of skew tableau of a 'staircase' shape.

Example 2. Suppose $X = E_6$ labelled as shown



The X-sequence $s = (1, 2, 3, 0, 4, 5, 3, 2, 4, 3, 1, 0, 2, 3, 4, 5)$ has heap



$F(E_6,1)$

An X-sequence $s=(x_1,\ldots,x_n)$ will be called *neighbourly* if between any two consecutive occurrences of a vertex x there are at least two occurrences of some neighbour or neighbours of x. This property is preserved by free switches, so we also speak of neighbourly X-strings and X-heaps.

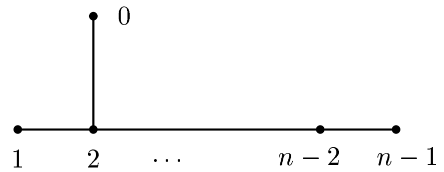
A neighbourly X-sequence s will be called *maximal* if F cannot be extended by the addition of a vertex x in any position to a larger neighbourly X-sequence s', and similarly for X-strings and heaps. The neighbourly E_6-heap of Example 2 is maximal.

A neighbourly X-string or X-heap will be called *two-neighbourly* if there are exactly two occurrences of some neighbour or neighbours of x between any two consecutive occurrences of any vertex x. The heap $F(E_6,1)$ of Example 2 is two-neighbourly.

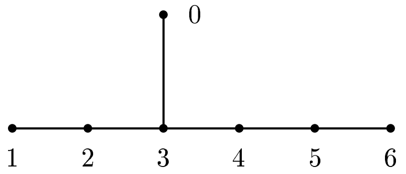
Recall that a *lattice* is a poset such that for $a,b\in L$ the least upper bound $a\vee b$ and greatest lower bound $a\wedge b$ exist uniquely. When these operations satisfy the usual distributive laws, the lattice is called *distributive*. If P is any poset, an *ideal* of P is a subset I such that $x\in I,\ y\leq x$ implies $y\in I$. Let $J(P)$ denote the poset of all ideals of P ordered by inclusion. Then $J(P)$ is always a distributive lattice, and any distributive lattice is of the form $J(P)$ for some poset P.

Proposition 2.3. *If F is a maximal neighbourly X-heap for some graph X, then F is a lattice.*

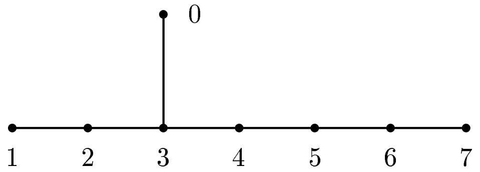
Recall the family of graphs D_n, and E_7 and E_8 labelled as shown



D_n



E_7



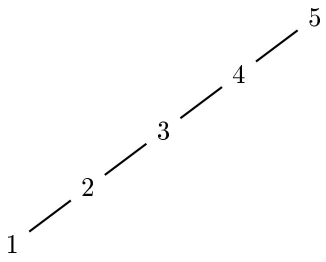
E_8

Theorem 2.1. *Let X be a simple graph for which there exists a maximal neighbourly X-heap F which is two-neighbourly. Then X is one of the graphs $A_n, n \geq 1,\ D_n,\ n \geq 4,\ E_6$ or E_7. There are exactly n such X-heaps for A_n, three for D_n, two for E_6 and one for E_7. Each of the these lattices is distributive.*

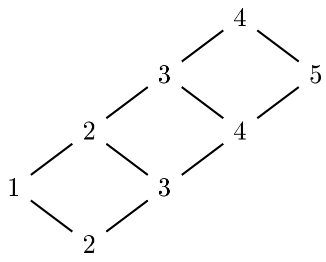
We now describe these X-heaps, which we call *minuscule*. The curious terminology is motivated by Lie theory and will be justified later.

a) The case A_n. We label the minuscule A_n-heaps $F(A_n, k)$ $k = 1, \ldots, n$. Hopefully the following example will make the general case clear.

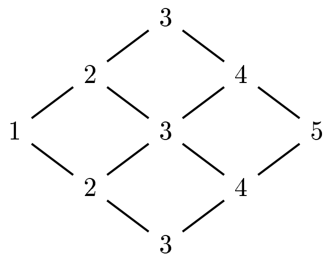
For $n = 5$



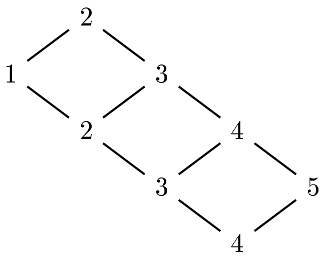
$F(A_5, 1)$



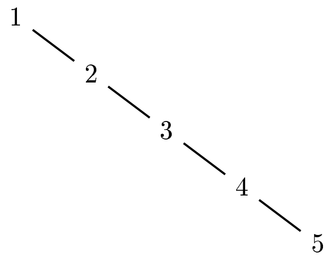
$F(A_5, 2)$



$F(A_5, 3)$

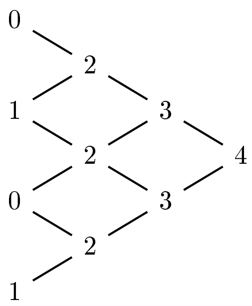


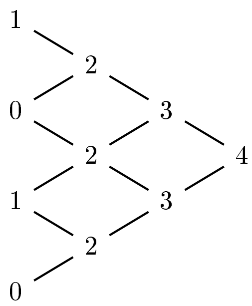
$F(A_5, 4)$

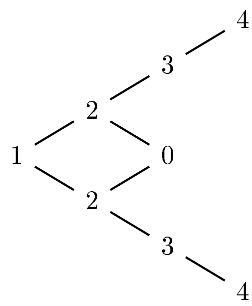


$F(A_5, 5)$

b) The case D_n. The minuscule D_n-heaps are labelled $F(D_n, 0)$, $F(D_n, 1)$ and $F(D_n, n-1)$. The following example for $n = 5$ should make the general case clear.





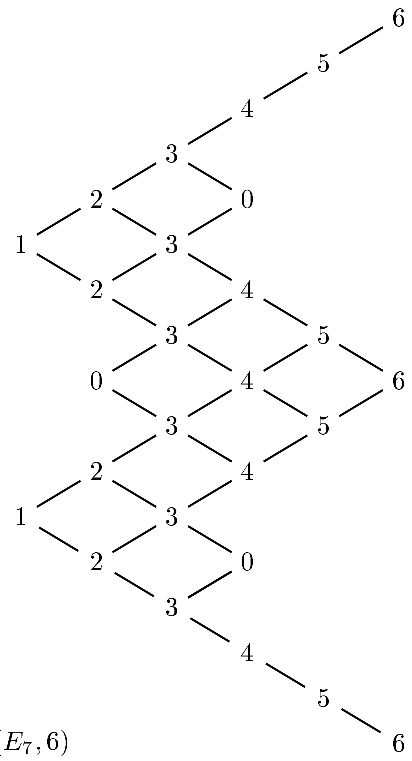


$F(D_5, 1)$ $F(D_5, 0)$ $F(D_5, 4)$

The heaps $F(D_n, 0)$ and $F(D_n, 1)$ have the same triangular shape with $n(n-1)/2$ elements, while $F(D_n, n-1)$ consists of a square symmetrically placed between two chains, and has $2(n-1)$ elements.

c) The case E_6. There are two minuscule E_6-heaps labelled $F(E_6, 1)$ and $F(E_6, 5)$. The heap $F(E_6, 1)$ appeared in Example 2. The heap $F(E_6, 5)$ has the same shape and is the inverse of $F(E_6, 1)$.

d) The case E_7. There is only one minuscule E_7 heap labelled $F(E_7, 6)$.



$F(E_7, 6)$

This lovely lattice, which we might call the *swallow*, is symmetric, spindle shaped, Sperner, Gaussian and enjoys other interesting combinatorial properties (see [7], [9], [11]).

Note that in each case the graph X is an ideal of the minuscule X-heap and that the minimal vertex appears in the label of that X-heap.

3. Roots of a simple graph

Let X be a simple graph. We will define a distinguished class of integer valued functions on the vertices of X which we call the *roots* of X. Let $P(X)$ denote the set of all integer valued functions on X, with $P^+(X)$ and $P^-(X)$ the non-negative and non-positive functions in $P(X)$ respectively. For a vertex x, let δ_x denote the function which is 1 at x and 0 elsewhere. We call an element of $P(X)$ a *population* and refer to δ_x as a *singleton* population.

For each vertex x, define $s_x : P(X) \to P(X)$ by

$$(ps_x)(y) = \begin{cases} p(y) & \text{if } \; y \neq x \\ \\ \sum_{z \in N(x)} p(z) - p(x) & \text{if } y = x \end{cases}$$

where $N(x)$ denotes the set of neighbours of x. Call s_x the *mutation-reflection* at x.

There is a useful physical model for visualising such reflections. We may imagine X as representing a pattern of cities and roads on Mars, which contains Martians and anti-Martians. If a Martian and an anti-Martian appear together in a city, they mutually annihilate each other, so that each city contains only Martians or anti-Martians or is empty. If a given city *mutates*, its inhabitants turn to anti-inhabitants and simultaneously each neighbouring city sends a cloned copy of its population into the mutating city.

A *root* population of X is any population obtainable from a singleton population by a sequence of reflections s_{x_i}. We let $R(X)$ denote the set of all root populations, and $R^+(X) = R(X) \cap P^+(X)$, $R^-(X) = R(X) \cap P^-(X)$, the *positive* and *negative* root populations respectively. We refer informally to root populations as *roots*.

Lemma 3.1. (1) $s_x^2 = id$ *for all* x
(2) $s_x s_y = s_y s_x$ *for all* x, y *which are not neighbours*
(3) $s_x s_y s_x = s_y s_x s_y$ *if* x, y *are neighbours.*

Proposition 3.1. *The group* W *generated by all* s_x *is a Coxeter group with the relations in the previous Lemma as the only relations.*

Proposition 3.2. $R(X)$ *is finite* $\Leftrightarrow$ W *is finite* $\Leftrightarrow$ X *is one of the graphs* A_n $n \geq 1$, D_n $n \geq 4$, E_6, E_7 *or* E_8.

Proposition 3.3. *For any simple graph* X, $R(X) = R^+(X) \cup R^-(X)$.

This last rather remarkable result is a consequence of the theory of Coxeter groups; the author knows of no direct combinatorial proof (sadly).

We say X is an *ADE graph* iff it is in the list in Proposition 3.2. For such graphs, the set of roots is a root system of classical type. To connect our discussion with the usual approach, we define an inner product $(\ ,\ )$ on $P(X)$ for general X by

$$(\delta_x, \delta_y) = \begin{cases} 2 & \text{if } x = y \\ -1 & \text{if } x \text{ and } y \text{ are neighbours} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.2. *For* $p,\ q \in P(X)$ *and* x *a vertex of* X

$$(p, q) = (ps_x, qs_x).$$

Proposition 3.4. $(\ ,\ )$ *is positive definite* $\Leftrightarrow$ X *is of ADE type.*

Thus for X an ADE graph, $R(X)$ is a finite root system in the usual sense since each s_x is indeed the reflection in the hyperplane determined by δ_x and preserves $R(X)$. It seems interesting to inquire as to the properties of the root systems $R(X)$ for general graphs. For example, to what extent does the following generalise?

Proposition 3.5. *If X is an ADE graph, then*

$$R(X) = \{p \in P(X) \mid (p,p) = 2\}.$$

4. Constructions of Lie algebras from minuscule heaps

Let X be an ADE graph with a minuscule X-heap F. A subset L of F is *convex* if $\forall x, y \in L$, any z such that $x < z < y$ is also in L. We will refer to convex subsets as *layers*. For any layer L of F, define the *content* of L to be the population $c(L)(x) = \#$ times x appears in L.

For $\alpha \in R^+(X)$, define a layer L to be an *α-layer* iff $c(L) = \alpha$, and let $\mathcal{L}_\alpha(F)$ denote the set of α-layers of F. For any subset S of F let $I(S) = \{x \in F \mid \exists y \in S,\ x \le y\}$ be the ideal generated by S. Partially order layers by declaring $L_1 \le L_2$ if $L_1 \subseteq I(L_2)$.

Proposition 4.1. *For any $\alpha \in R^+(X)$, $\mathcal{L}_\alpha(F)$ is non-empty and contains a unique minimal α-layer L_α with respect to the above partial order.*

If L is any α-layer then we define $\varepsilon(L) = \varepsilon(L, L_\alpha)$, and call it the *parity* of L.

Now let $V_F = \text{span}\{v_I \mid I \text{ is an ideal of } F\}$. For any layer $L \subseteq F$ define operators X_L and Y_L on V_F by

$$X_L(v_I) = \begin{cases} v_{I\cup L} & \text{if } I \cup L \text{ is an ideal and } I \cap L = \phi \\ 0 & \text{otherwise} \end{cases}$$

$$Y_L(v_I) = \begin{cases} v_{I\setminus L} & \text{if } I \supseteq L \text{ and } I\setminus L \text{ is an ideal} \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha \in R^+$, define operators X_α, Y_α, and H_α on V_F by

$$X_\alpha = \sum_{L \in \mathcal{L}_\alpha(F)} \varepsilon(L) X_L$$

$$Y_\alpha = \sum_{L \in \mathcal{L}_\alpha(F)} \varepsilon(L)\ Y_L$$

$$H_\alpha(v_I) = \begin{cases} v_I & \text{if } \exists\ \alpha\text{-layer } L \subseteq I \text{ such that } I\setminus L \text{ is an ideal} \\ -v_I & \text{if } \exists\ \alpha\text{-layer } L, \text{ such that } I \cup L \text{ is an ideal and } I \cap L = \phi \\ 0 & \text{otherwise.} \end{cases}$$

For a vertex x of X, let us write $H_{\delta_x} = H_x$.

Proposition 4.2. *For any $\alpha \in R^+$,*

$$H_\alpha = \sum_x \alpha(x)\ H_x.$$

Thus the operators $H_\alpha, \alpha \in R^+$ are not linearly independent. The main result is the following.

Theorem 4.1. *Let X be a simple graph with minuscule X-heap F. Then the set of operators $\{X_\alpha \mid \alpha \in R^+\} \cup \{Y_\alpha \mid \alpha \in R^+\} \cup \{H_x \mid x \text{ a vertex of } X\}$ on V_F is linearly independent and its span forms a Lie algebra. This Lie algebra is simple, depends only on X, and is the usual Lie algebra with Dynkin diagram X.*

The proof relies on some remarkable properties of both α-layers in minuscule X-heaps and the parity functions $\varepsilon(L)$.

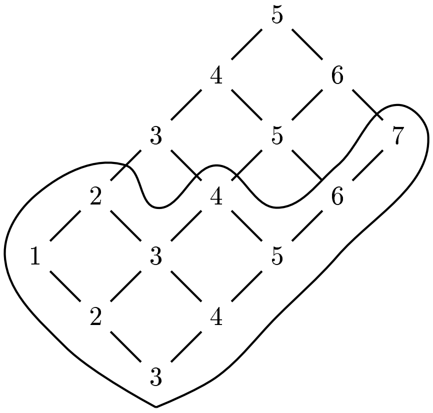
Theorem 4.1 gives an explicit combinatorial construction of a Lie algebra $\mathfrak{g}$ of operators on V_F. Furthermore the basis given in the Theorem is a Chevalley basis for $\mathfrak{g}$. All structure constants are integers and can be explicitly read off from the minuscule heap using the formulae for $X_\alpha, Y_\alpha, H_\alpha$ above.

The particular representations so constructed coincide with the so-called *minuscule representations* for simply-laced Lie algebras, defined by the condition that all weight spaces are conjugate under the Weyl group. The reason that we cannot construct E_8 this way is that E_8 has no minuscule representations—the smallest representation is the adjoint representation which has the zero weight space (with multiplicity 8) as well as the root spaces.

5. Examples

We will now give some brief descriptions of the representations constructed by this method. This includes all the fundamental representations of $sl(n)$, the two spin representations and the standard representation of the even orthogonal Lie algebra $so(2n)$, and the 27 and 56 dimensional representations of E_6 and E_7 respectively.

a) The case A_n. For $1 \leq k \leq n$ the minuscule A_n-heap $F(A_n, k)$ is the poset commonly known as $k \times (n - k + 1)$.

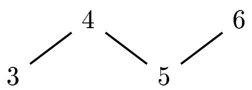


Shown is an ideal I in $F(A_7, 3)$. This ideal is specified by 3 numbers $\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5$ lying along the northeast lines as shown. A general ideal I in $F(A_7, 3)$ is determined by one of the 56 triples $(\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 5.$$

In general there are $\binom{n+1}{k}$ solutions of $0 \leq \lambda_1 \leq \lambda_2 \ldots \leq \lambda_k \leq n - k + 1$ and so this is the dimension of the corresponding representation of $sl(n)$.

An α-layer is just a string of adjacent elements of the given substring defining α (what we called a 'stock market poset' earlier). The minimal α-layer L is a subposet of the minimal copy of A_n in F, and the parity of an arbitrary α-layer L is $(-1)^j$ where j is the number of bonds by which L differs from L_0.



In the above diagram where $\alpha = \delta_3 + \ d_4 + \delta_5 + \delta_6$, the α-layer differs from L_α by just one bond so has parity -1. Thus $X_\alpha(v_I) = -v_{I\cup L}$ in this particular example, and $H_\alpha(v_I) = -v_I$.

Note also that

$$\begin{array}{rclcrcl} H_1(v_I) & = & 0 & \quad & H_5(v_I) & = & -v_I \\ H_2(v_I) & = & v_I & & H_6(v_I) & = & 0 \\ H_3(v_I) & = & -v_I & & H_7(v_I) & = & v_I \\ H_4(v_I) & = & v_I & & & & \end{array}$$

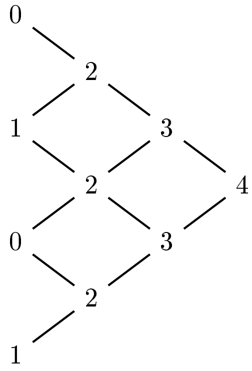
and indeed $H_\alpha = H_3 + H_4 + H_5 + H_6$.

Using Theorem 2.1 to compute structure constants we get for example that

$$\begin{array}{llll} [X_1, X_2] & = -X_{12} & [X_{12}, X_3] & = -X_{123} \\ [X_{1234}, X_5] & = X_{12345} & [X_3, X_4] & = X_{34} \qquad \text{etc.} \end{array}$$

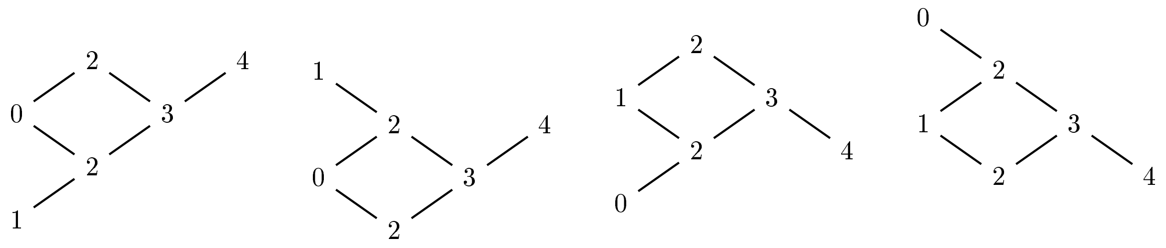
Note that our Chevalley basis $\{X_\alpha, Y_\alpha, H_x\}$ and the corresponding structure equations of $\mathfrak{g}$ depend on k.

b) The case D_n. For D_n labelled as previously, we refer to the minuscule D_n-heaps $F(D_n, 1)$ and $F(D_n, 0)$ as the *spin-heaps*. Applying the construction gives us the two spin representations of the orthogonal groups. The results are completely general but we illustrate them with the case $n = 5$.



$F(D_5, 1)$

The lattice of ideals of $F(D_5, 1)$ is isomorphic to the E_6-heaps $F(E_6, 1)$ or $F(E_6, 5)$, and contains 16 elements. In general the spin-heaps $F(D_n, 1)$ and $F(D_n, n-1)$ have 2^{n-1} ideals, which is thus the dimension of the corresponding (spin) representations. The Clifford algebra usually used to define these representations is here encapsulated by the parity functions. Let's illustrate the spin representation by exhibiting the raising operator X_α for $\alpha = \delta_1 + 2\delta_2 + \delta_3 + \delta_4 + \delta_0$. There are four α-layers whose shapes are the following.



(The relative parity of these four layers is $+, +, -, -$ respectively). Thus X_α acts in non-zero fashion only on the four ideals directly below these layers, and sends each to $\pm$ the union with the corresponding layer. Apart from the spin-heaps for D_5 there is also the heap $F(D_5, 4)$ which corresponds to the so-called 'standard representation' of dimension 10 of $so(10)$. More generally the D_n-heap $F(D_n, n-1)$ has a lattice of ideals isomorphic to $F(D_{n+1}, n)$ with $2n$ elements, the dimension of the corresponding standard representation of $so(n)$.

c) The case E_6. Each of the minuscule E_6-heaps $F(E_6, 1)$ and $F(E_6, 5)$ have 27 ideals. The corresponding 27 dimensional realizations of E_6 are the smallest possible, and are related to the 27 lines on a cubic. The lattice of ideals of each of the above heaps is isomorphic as a distributive lattice to $F(E_7, 6)$. Each of the 36 raising and lowering operators may be concretely visualised as transformations of this lattice in that each node is sent to a multiple of another node or to zero. Since in practice most of these operators are quite simple, it is not impossible with some patience to represent the entire Lie algebra on a large copy of $F(E_7, 6)$ with signed arrows for the raising operators between appropriate vertices.

d) The case E_7. There is only one minuscule E_7-heap, $F(E_7, 6)$, and it is not hard to count that there are 56 ideals of this lattice, so the corresponding representation of E_7 is 56 dimensional (also the smallest possible). Each of the 63 raising and lowering operators may again be concretely visualised as transformations of this lattice of ideals, which is related to E_8.

These constructions are very explicit and amenable to investigation. They all have the rather remarkable property that the Lie algebra has a basis for which the corresponding operators all act on a basis of the representation space by transformations that in matrix form have at most one non-zero element in each column, that non-zero element being either ± 1.

In practice this means that the corresponding operators may be visualised acting on a lattice of ideals by arrows between nodes with labels ± 1. It is worth pointing out that the lattice of ideals in each case is itself a distributive lattice, which with the exception of $F(E_7, 1)$ is one of the minuscule lattices. This is part of a remarkable 'cascading' phenomenon which links all the root systems together in a pleasant inductive pattern (see [16]). These remarks are related to observations of Steinberg and Proctor (see [9]).

We leave it to the reader to experiment with these representations to find many further remarkable features.

References

[1] Behrend R., Pearce P. and Zuber, J-B., Integrable boundaries, conformal boundary conditions and $A-D-E$ fusion rules, J. Phys. A, 31, no.50, L763-L770, (1998).

[2] Iosifescu M. and Scutaru H., Kronecker products, minuscule representations, and polynomial identities J. Math. Phys. 31, no.2, 264-277 (1990).

[3] Kac V.G., Infinite root systems, representations of graphs and invariant theory, Invent. Math. 56, 57-92 (1980).

[4] Kac V.G., Infinite root systems, representations of graphs and invariant theory II, Journal of Algebra 77, 141-162 (1982).

[5] Marsh R., Canonical bases for the minuscule modules of the quantized enveloping algebras of types B and D, Comm. Algebra 25, no.5, 1387-1411 (1997).
[6] Moody R.V., Root systems of Hyperbolic Type, Advances in Mathematics 33, 144-160 (1979).
[7] Hiller H., Geometry of Coxeter groups, Research Notes in Mathematics, 54 Pitman(Advanced Publishing Program), Boston, 1982.
[8] Hiroyuki O., Root lattices and weight lattices of exceptional simple Lie algebras of type E Comment. Math. Univ. St. Paul 46, no.1, 23-31 (1997).
[9] Proctor R.A., Bruhat Lattices, Plane Partition Generating Functions, and Minuscule Representations, Europ. J. Combinatorics 5, 331-350 (1984).
[10] Proctor R.A., Minuscule elements of Weyl groups, the numbers game, and d-complete posets, J. Algebra 213, no.1, 272-303 (1999).
[11] Proctor R.A., Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets, J. Algebraic Combin. 9, no.1, 61-94. (1999)
[12] Seshadri C.S., Geometry of G/P. I. Theory of standard monomials for minuscule representations, in C.P. Ramanujam—a tribute, 207–239, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin, 1978.
[13] Stembridge J.R., On minuscule representations, plane partitions and involutions in complex Lie groups, Duke Math. J., no.2, 469-490 (1994).
[14] Tits J., Les groupes de Lie exceptionnels et leur interpretation géométrique, Bull. Soc. Math. Belg., 8, 48–81 (1956).
[15] Viennot, G. X. Heaps of pieces I: Basic definitions and combinatorial lemmas, in 'Combinatoire Énumérative' (G. Labelle and P. Leroux, eds.), pp. 321–350, Lect. Notes in Math. Vol 1234, Springer–Verlag, Berlin, 1985.
[16] Wildberger N.J., The mutation game and Coxeter graphs, preprint (2001).

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