COMBINATORIAL K-THEORY

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Abstract. We show that the linear span of all stable Grothendieck polynomials is a bialgebra which is a K-theoretic generalization of the ring of symmetric functions. We apply this bialgebra to obtain a Littlewood-Richardson rule for the K-theory of Grassmann varieties, as well as a formula for the structure sheaf of a quiver variety.

1. Introduction

The ring of symmetric functions has for a long time been a fundamental tool in the study of representation theory, cohomology, degeneracy loci, and numerous other areas. The goal of this talk is to introduce a more general bialgebra which plays a similar role for K-theory of algebraic varieties.

Let $X = \mathrm{Gr}(d, \mathbb{C}^n)$ be the Grassmann variety of d dimensional subspaces of $\mathbb{C}^n$. For each partition $\lambda = (\lambda_1, \dots, \lambda_d)$ with at most d parts and $\lambda_1 \leq n-d$, there is a Schubert variety in X defined by

$$\Omega_\lambda = \{V \in X \mid \dim(V \cap \mathbb{C}^{n-d+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq d\}\,.$$

Here $\mathbb{C}^k \subset \mathbb{C}^n$ denotes the subset of vectors whose last $n-k$ components are zero. The codimension of Ω_λ is equal to the weight $|\lambda| = \sum \lambda_i$ of λ. If we identify partitions with their Young diagrams, then a Schubert variety Ω_μ is contained in Ω_λ if and only μ contains λ.

The Schubert structure sheaves $\mathcal{O}_{\Omega_\lambda}$ are known to be a basis for the Grothendieck ring $K^\circ X$, so we can define structure constants $c^\nu_{\lambda\mu}$ for the Grothendieck ring by

$$[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}] = \sum_\nu c^\nu_{\lambda\mu} [\mathcal{O}_{\Omega_\nu}]\,.$$

These structure constants depend only on the partitions λ, μ, and ν, not on the Grassmannian in which the Schubert varieties for these partitions are realized. Furthermore $c^\nu_{\lambda\mu}$ is non-zero only if $|\nu| \geq |\lambda| + |\mu|$. The constants $c^\nu_{\lambda\mu}$ for which $|\nu| = |\lambda| + |\mu|$ are the usual Littlewood-Richardson coefficients, i.e. the structure constants for the cohomology ring $H^*(X)$ with respect to the basis of Schubert classes. Another known case is a Pieri formula of Lenart which expresses the coefficient $c^\nu_{\lambda,(k)}$ for multiplying with the structure sheaf of a special Schubert variety $\Omega_{(k)}$ as a binomial coefficient [12].

Our main theorem for K-theory of Grassmannians [3] is an explicit combinatorial formula saying that $c^\nu_{\lambda\mu}$ is equal to $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of *set-valued tableaux* which satisfy certain properties. Set-valued tableaux are similar to semistandard Young tableaux, but allow a non-empty set of integers in each box rather than a single integer. When $|\lambda| + |\mu| = |\nu|$ our formula specializes to the classical Littlewood-Richardson rule.

Our formula implies that if $c^\nu_{\lambda\mu}$ is not zero, then ν is contained in the union of all partitions ρ of weight $|\rho| = |\lambda| + |\mu|$ such that the Littlewood-Richardson coefficient $c^\rho_{\lambda\mu}$ is non-zero. As a consequence, for fixed λ and μ, the constants $c^\nu_{\lambda\mu}$ are non-zero for only finitely many partitions ν, even though one might conceivably get arbitrarily many such constants by realizing the product $[\mathcal{O}_{\Omega_\lambda}] \cdot [\mathcal{O}_{\Omega_\mu}]$ in larger and larger Grassmannians.

This allows us to define a commutative ring $\Gamma = \bigoplus \mathbb{Z} \cdot G_\lambda$ with a formal basis $\{G_\lambda\}$ indexed by partitions and multiplication defined by $G_\lambda \cdot G_\mu = \sum c^\nu_{\lambda\mu} G_\nu$. The Grothendieck ring $K^\circ X$ is then

The author was partially supported by NSF Grant DMS-0070479.

the quotient of this ring by the ideal spanned by the basis elements G_λ for partitions that do not fit in a d by $n-d$ rectangle.

The ring Γ is the main object of study in this talk. In contrast to the Grothendieck ring of a fixed Grassmannian, Γ turns out to have a natural structure of bialgebra. The coproduct on Γ is related to the pullback of Grothendieck rings along an embedding $\operatorname{Gr}(d_1, \mathbb{C}^{n_1}) \times \operatorname{Gr}(d_2, \mathbb{C}^{n_2}) \subset \operatorname{Gr}(d_1+d_2, \mathbb{C}^{n_1+n_2})$.

Notice that if the Grothendieck ring of a Grassmann variety X is replaced with the cohomology ring $H^*(X)$ in the above definition of Γ, then the resulting ring is the ring of symmetric functions Λ. The bialgebra Γ can therefore be seen as a K-theoretic parallel to the ring of symmetric functions. Furthermore, if we define a filtration of Γ by ideals $\Gamma_p = \bigoplus_{|\lambda| \geq p} \mathbb{Z} \cdot G_\lambda$, then the associated graded bialgebra is naturally the ring of symmetric functions.

In the next section we will realize Γ as the linear span of all stable Grothendieck polynomials, which has a basis indexed by Grassmannian permutations. This again is analogous to the ring of symmetric functions, which is the linear span of all stable Schubert polynomials.

In section 3 we will use Γ to give a formula for the structure sheaf of quiver variety, which generalizes our earlier formula with Fulton for the cohomology class of a quiver variety. The difference between the cohomology formula and the K-theory formula literally consists of replacing the ring of symmetric functions with the bialgebra Γ.

Finally, in section 4 we will discuss some combinatorial properties of Γ. For example we will show that the linear endomorphism on Γ which sends a basis element G_λ to the element $G_{\bar{\lambda}}$, for the partition $\bar{\lambda}$ obtained by removing the top row of the Young diagram of λ, is a ring homomorphism. We will also discuss the structure of Γ as an abstract ring.

For constructing and studying the bialgebra Γ it has been invaluable to speak to S. Fomin, from whom we have learned very much about stable Grothendieck polynomials during several fruitful discussions. In particular we thank Fomin for providing the proof of a basic Lemma of [3]. We also thank W. Fulton for numerous helpful discussions and suggestions, and F. Sottile for useful input on stable Grothendieck polynomials.

2. Stable Grothendieck polynomials

Grothendieck polynomials were introduced by Lascoux and Schützenberger as representatives for the structure sheaves of the Schubert varieties in a flag variety [11, 9]. Fomin and Kirillov later defined stable Grothendieck polynomials as certain limits of Lascoux and Schützenberger's polynomials [7, 6]. In this talk we will only be concerned with stable Grothendieck polynomials for Grassmannian permutations, which are indexed by partitions.

If a and b are two non-empty subsets of the positive integers $\mathbb{N}$, we will write $a < b$ if $\max(a) < \min(b)$, and $a \leq b$ if $\max(a) \leq \min(b)$. We define a *set-valued tableau* to be a labeling of the boxes in a Young diagram or skew diagram with finite non-empty subsets of $\mathbb{N}$, such that the rows are weakly increasing from left to right and the columns strictly increasing from top to bottom. The shape $\operatorname{sh}(T)$ of a set-valued tableau T is the partition or skew diagram it is a labeling of. For example

$$\begin{array}{cccc} & & \boxed{1} & \boxed{2\,3} \\ & \boxed{1\,2} & \boxed{2\,3\,4} & \\ \boxed{2} & \boxed{3\,5} & \boxed{7} & \end{array}$$

is a set-valued tableau of shape $(4,3,3)/(2,1)$ containing the sets $\{2\}$, $\{3,5\}$, $\{7\}$, $\{1,2\}$, $\{2,3,4\}$, $\{1\}$, and $\{2,3\}$ when the boxes are read left to right and then bottom to top. Notice that a semistandard Young tableau is a set-valued tableau in which each box contains a single integer.

Given a set-valued tableau T, let x^T be the monomial in which the exponent of x_i is the number of boxes in T which contain the integer i. If T is the tableau displayed above we get $x^T = x_1^2 x_2^4 x_3^3 x_4 x_5 x_7$. We let $|T|$ denote the total degree of this monomial, i.e. the sum of the cardinalities of the sets in the boxes of T.

Definition 2.1. For any partition λ we set

$$G_\lambda = G_\lambda(x_1, x_2, \dots) = \sum_{\mathrm{sh}(T)=\lambda} (-1)^{|T|-|\lambda|}\, x^T \;\in\; \mathbb{Z}\llbracket x_1, x_2, \dots \rrbracket$$

where the sum is over all set-valued tableaux T of shape λ.

This power series G_λ is equal to the stable Grothendieck polynomial for the Grassmannian permutation given by λ. This in particular implies that G_λ is symmetric in the variables x_i. The definition we have given is a consequence of a general formula for Grothendieck polynomials of Fomin and Kirillov [7].

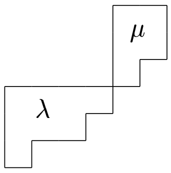
Notice that if λ is not the empty partition, then G_λ does not have bounded degree, and is therefore not an element of the ring of symmetric functions. Notice also that the term of lowest degree in G_λ is the Schur function s_λ. In particular the power series G_λ are linearly independent. We define Γ to be the linear span of these power series:

$$\Gamma = \bigoplus \mathbb{Z} \cdot G_\lambda \subset \mathbb{Z}\llbracket x_1, x_2, \dots \rrbracket\,.$$

In [3] it is shown that this group Γ is the linear span of all stable Grothendieck polynomials, i.e. every stable Grothendieck polynomial is a linear combination of the basis elements G_λ.

Define the *word* of a set-valued tableau T to be the sequence of integers contained in its boxes, when these are read left to right, then bottom to top, and the integers in each box are put in increasing order. The word of the tableau displayed above is $(2, 3, 5, 7, 1, 2, 2, 3, 4, 1, 2, 3)$. A sequence of positive integers $w = (w_1, \dots, w_\ell)$ has *content* $(c_1, c_2, \dots, c_r)$ if w consists of c_1 1's, c_2 2's, and so on up to c_r r's. If the content of each subsequence $(w_i, \dots, w_\ell)$ is a partition, then w is called a *reverse lattice word.*

If λ and μ are partitions, we let $\lambda * \mu$ denote the skew diagram obtained by attaching the Young diagrams for λ and μ corner to corner as shown:

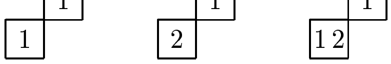
$\lambda * \mu \;=$ 

Theorem 2.2. *The group Γ is closed under multiplication. In fact we have*

$$G_\lambda \cdot G_\mu = \sum_\nu c^\nu_{\lambda\mu}\, G_\nu$$

where $c^\nu_{\lambda\mu}$ is $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of set-valued tableaux T of shape $\lambda * \mu$ such that the word of T is a reverse lattice word with content ν.

For example, for the shape $(1) * (1)$ we can find the following set-valued tableaux whose words are reverse lattice words:

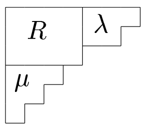


It follows that $G_1 \cdot G_1 = G_2 + G_{(1,1)} - G_{(2,1)}$.

Given three partitions λ, μ, and ν we define

$$d^\nu_{\lambda\mu} = c^\rho_{\nu R}$$

where R is any rectangular partition containing λ and μ, and ρ is the partition obtained by attaching the Young diagrams for λ and μ to the sides of R as shown:

$\rho \;=$ 

It follows from the formula for $c^{\nu}_{\lambda\mu}$ of Theorem 2.2 that this definition is independent of the choice of R.

Theorem 2.3. *For each partition ν and integer $m \geq 0$ we have*

$$G_\nu(x_1,\dots,x_m,y_1,\dots,y_m) = \sum_{\lambda,\mu} d^{\nu}_{\lambda\mu}\, G_\lambda(x_1,\dots,x_m)\cdot G_\mu(y_1,\dots,y_m)\,.$$

Here $G_\lambda(x_1,\dots,x_m)$ denotes the polynomial obtained by setting x_i equal to zero for $i > m$.

Now define a linear map $\Delta : \Gamma \to \Gamma \otimes \Gamma$ by $\Delta G_\nu = \sum_{\lambda,\mu} d^{\nu}_{\lambda,\mu}\, G_\lambda \otimes G_\mu$. It follows immediately from Theorem 2.3 that Δ is compatible with the product $\Gamma \otimes \Gamma \to \Gamma$.

Corollary 2.4. *The group $\Gamma = \bigoplus \mathbb{Z}\cdot G_\lambda$ is a commutative and cocommutative bialgebra with unit and counit.*

The coproduct is cocommutative because the power series G_ν is symmetric. The counit of Γ is the linear map $\Gamma \to \mathbb{Z}$ which maps $G_{(0)} = 1$ to one and all other basis elements G_λ to zero.

We finish this section by explaining the relation of Γ with K-theory of Grassmannians. If $E = L_1 \oplus \cdots \oplus L_m$ is a vector bundle on a variety X which is a direct sum of line bundles, we set

$$G_\lambda(E) = G_\lambda(1 - L_1^\vee,\dots,1 - L_m^\vee) \;\in\; K^\circ X\,.$$

Since G_λ is symmetric, this can be written as a polynomial in the exterior powers of the dual bundle $E^\vee$. The definition of $G_\lambda(E)$ therefore makes sense even if E is not a sum of line bundles.

Now let $X = \operatorname{Gr}(d,\mathbb{C}^n)$ be a Grassmann variety and $S \subset \mathbb{C}^n \times X$ the tautological subbundle of rank d on X. If λ is a partition which is contained in a d by $n-d$ rectangle, then a theorem of Fulton and Lascoux [8] implies that the structure sheaf of the Schubert variety $\Omega_\lambda \subset X$ is given by

$$[\mathcal{O}_{\Omega_\lambda}] = G_\lambda(S^\vee) \in K^\circ X\,.$$

The Littlewood-Richardson rule for the K-theory of X stated in the introduction therefore becomes a special case of Theorem 2.2. Furthermore, the kernel of the ring homomorphism $\Gamma \to K^\circ X$ which maps G_λ to $G_\lambda(E^\vee)$ is the ideal $I \subset \Gamma$ spanned by all basis elements G_λ for partitions λ which are not contained in a d by $n-d$ rectangle. We conclude that $K^\circ \operatorname{Gr}(d,\mathbb{C}^n) = \Gamma/I$.

3. Applications to quiver varieties

Let X be a non-singular variety and $E_0 \to E_1 \to E_2 \to \cdots \to E_n$ a sequence of vector bundles and bundle maps over X. A set of *rank conditions* for this sequence is a collection $r = \{r_{ij}\}$ of non-negative integers for $0 \leq i < j \leq n$. Given this data we define the *quiver variety* $\Omega_r = \Omega_r(E_\bullet)$ as the degeneracy locus

$$\Omega_r = \{x \in X \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leq r_{ij}\ \forall i < j\}\,.$$

We will demand that the rank conditions can *occur*, i.e. there exists a sequences of vector spaces and linear maps $V_0 \to V_1 \to \cdots \to V_n$ such that $\dim V_i = \operatorname{rank} E_i$ and $\operatorname{rank}(V_i \to V_j) = r_{ij}$ for all $i < j$. When this is the case, the expected (and maximal possible) codimension of the quiver variety Ω_r in X is the number $d(r) = \sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$.

With Fulton we have proved [5] that when Ω_r has this codimension, then its cohomology class is given by the formula

$$[\Omega_r] = \sum_{|\mu| = d(r)} c_\mu(r)\, s_{\mu_1}(E_1 - E_0)\cdots s_{\mu_n}(E_n - E_{n-1}) \in H^*(X)\,.$$

Here the sum is over all sequences $\mu = (\mu_1,\dots,\mu_n)$ of n partitions μ_i, such that the sum of the weights of these partitions, $|\mu| = \sum |\mu_i|$ is equal to the expected codimension $d(r)$. If E and F are vector bundles over X and λ is a partition, then the class $s_\lambda(F - E)$ denotes the double Schur polynomial for λ evaluated on the Chern roots of F and E. The coefficients $c_\mu(r)$ are integers which are uniquely determined by the statement of the formula. They can be calculated by an explicit combinatorial algorithm which is based on computations in the ring of symmetric functions.

Surprisingly these coefficients appear to be non-negative. A conjecture in [5] states that each coefficient $c_\mu(r)$ counts the number of sequences of semistandard Young tableaux satisfying certain properties. This conjecture has been proved when the sequence $E_\bullet$ contains at most four vector bundles [1].

Now define $G_\lambda(F - E) = \sum_{\sigma,\tau} d^\lambda_{\sigma\tau}\, G_\sigma(F) \cdot G_{\tau'}(E^\vee) \in K^\circ X$, where τ' denotes the conjugate partition of τ. We then obtain a very similar formula for the structure sheaf of the quiver variety Ω_r [4]:

Theorem 3.1. *If Ω_r has its expected codimension, then its structure sheaf is given by a formula*

$$[\mathcal{O}_{\Omega_r}] = \sum_{|\mu| \geq d(r)} c_\mu(r)\, G_{\mu_1}(E_1 - E_0) \cdots G_{\mu_n}(E_n - E_{n-1}) \in K^\circ X\,.$$

The most noticeable change is that all Schur polynomials in the cohomology formula have been replaced by stable Grothendieck polynomials. Furthermore, we need to include sequences of partitions μ for which the sum of the weights $|\mu|$ is larger than the codimension $d(r)$. However, the formula still has finitely many terms.

The coefficients $c_\mu(r)$ in Theorem 3.1 are again integers determined by the geometry, and they agree with the cohomology coefficients when $|\mu| = d(r)$. Regarding the signs of these more general coefficients we pose:

Conjecture 3.2. *The coefficients $c_\mu(r)$ of have alternating signs, i.e. $(-1)^{|\mu|-d(r)}\, c_\mu(r) \geq 0$.*

The structure constants $c^\nu_{\lambda\mu}$ of Γ are all special cases of the quiver coefficient $c_\mu(r)$. More generally, the coefficients obtained when a stable Grothendieck polynomial is expanded as a linear combination of the power series G_λ are special cases [2, 4]. Lascoux has proved [10] that the later coefficients have alternating signs, which confirms our conjecture in this special case. Another case where Conjecture 3.2 is known to be true is when the quiver variety Ω_r is a variety of complexes, i.e. $r_{ij} = 0$ for $j - i \geq 2$. In fact, all coefficients $c_\mu(r)$ are ± 1 or zero in this case.

We will now describe an algorithm which computes the coefficients $c_\mu(r)$ of Theorem 3.1. This is the same as the algorithm that computes the coefficients of the cohomology formula, except that the ring of symmetric functions has been replaced with Γ.

Start by arranging the rank conditions $r = \{r_{ij}\}$ in a *rank diagram*:

$$\begin{array}{ccccccccc}
E_0 & \to & E_1 & \to & E_2 & \to & \cdots & \to & E_n \\
r_{00} & & r_{11} & & r_{22} & & \cdots & & r_{nn} \\
& r_{01} & & r_{12} & & \cdots & & r_{n-1,n} & \\
& & r_{02} & & \cdots & & r_{n-2,n} & & \\
& & & \ddots & & & & & \\
& & & & r_{0n} & & & &
\end{array}$$

In this diagram, replace each small triangle of numbers

$$\begin{array}{ccc}
r_{i,j-1} & & r_{i+1,j} \\
& r_{ij} &
\end{array}$$

by a rectangle R_{ij} with $r_{i+1,j} - r_{ij}$ rows and $r_{i,j-1} - r_{ij}$ columns.

$$R_{ij} = \underbrace{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}}_{r_{i,j-1} - r_{ij}} \; r_{i+1,j} - r_{ij}$$

These rectangles are arranged in a *rectangle diagram*:

$$\begin{array}{ccccccc}
R_{01} & & R_{12} & & \cdots & & R_{n-1,n} \\
& R_{02} & & \cdots & & R_{n-2,n} & \\
& & \ddots & & & & \\
& & & R_{0n} & & &
\end{array}$$

The combinatorial data contained in the rank conditions $r = \{r_{ij}\}$ is very well represented by this diagram. First of all, the rank conditions can occur if and only if the rectangles always get shorter when one travels south-east, while they get narrower when one travels south-west. Furthermore, the expected codimension $d(r)$ is equal to the sum of the areas of the rectangles R_{ij}. Finally, the algorithm that computes the coefficients $c_\mu(r)$ depends only on the rectangle diagram.

We will proceed by constructing an element P_r in the n'th tensor power of Γ, so that

$$P_r = \sum_\mu c_\mu(r)\, G_{\mu_1} \otimes \cdots \otimes G_{\mu_n} \in \Gamma^{\otimes n}.$$

This is done by induction on n. When $n = 1$ (corresponding to a sequence of two vector bundles), the rectangle diagram has only one rectangle $R = R_{01}$. In this case we set

$$P_r = G_R \in \Gamma^{\otimes 1},$$

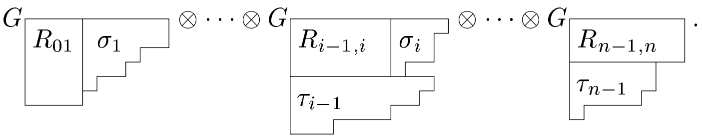
where R is identified with the partition for which it is the Young diagram.

If $n \geq 2$, we let $\bar{r}$ denote the bottom n rows of the rank diagram. Then $\bar{r}$ is a valid set of rank conditions, so by induction we can assume that

$$P_{\bar{r}} = \sum_\mu c_\mu(\bar{r})\, G_{\mu_1} \otimes \cdots \otimes G_{\mu_{n-1}} \tag{1}$$

is a well defined element of $\Gamma^{\otimes n-1}$. Now P_r is obtained from $P_{\bar{r}}$ by replacing each basis element $G_{\mu_1} \otimes \cdots \otimes G_{\mu_{n-1}}$ in (1) with the sum

$$\sum_{\substack{\sigma_1,\ldots,\sigma_{n-1}\\ \tau_1,\ldots,\tau_{n-1}}} \left(\prod_{i=1}^{n-1} d^{\mu_i}_{\sigma_i \tau_i} \right)$$



This sum is over all partitions $\sigma_1, \ldots, \sigma_{n-1}$ and $\tau_1, \ldots, \tau_{n-1}$ such that σ_i has fewer rows than $R_{i-1,i}$ and the coproduct structure constant $d^{\mu_i}_{\sigma_i \tau_i}$ of Γ is non-zero. A diagram consisting of a rectangle $R_{i-1,i}$ with (the Young diagram of) a partition σ_i attached to its right side, and τ_{i-1} attached beneath should be interpreted as the sequence of integers giving the number of boxes in each row of this diagram.

As suggested by the picture, it may happen that such a sequence of integers is not a partition. To handle this possibility, we define a stable Grothendieck polynomial associated to any sequence of integers as follows. If I and J are arbitrary sequences of integers and $p < q$, we set

$$G_{I,p,q,J} = \sum_{k=p+1}^{q} G_{I,q,k,J} - \sum_{k=p+1}^{q-1} G_{I,q-1,k,J} \ \in \ \Gamma.$$

By repeating this rule, $G_{I,p,q,J}$ is eventually expanded as a linear combination of the basis elements G_λ, and the result is independent of the order in which the expansion is done [4].

4. Combinatorics of Γ

In this last section we will discuss some of the properties of Γ. We will start by comparing Γ to the ring of symmetric functions Λ.

First, the fact that the Schur function s_λ is the term of lowest degree in G_λ immediately implies that Λ is the associated graded bialgebra of Γ by the filtration defined in the introduction. Furthermore, the completions of Λ and Γ isomorphic, since both are equal to the set of all symmetric power series in $\mathbb{Z}[[x_1, x_2, \ldots]]$.

Despite from these facts, Γ and Λ are not isomorphic as bialgebras themselves. The reason is that Γ does not have an antipode which makes it a Hopf algebra. Recall that an antipode is a ring homomorphism $S : \Gamma \to \Gamma$ such that for any non-empty partition ν we have

$$\sum_{\lambda,\mu} d^{\nu}_{\lambda\mu}\, S(G_\lambda) \cdot G_\mu = 0\,.$$

Taking $\nu = (1)$ we get $S(G_1) \cdot 1 + 1 \cdot G_1 - S(G_1) \cdot G_1 = 0$ which implies that $G(t) = t^{-1}$ where $t = 1 - G_1$. Since t^{-1} is not an element of Γ, this shows that no antipode can exist. However, Γ is not far from being a Hopf algebra. In fact, if we let Γ_t denote the localization generated by Γ and t^{-1}, then Γ_t is a Hopf algebra [3].

Given a partition λ, let $\bar\lambda$ denote the partition obtained by removing the top row of the Young diagram of λ. From the Littlewood-Richardson rule of Theorem 2.2 we then obtain the following Pieri formula for the coproduct in Γ.

Corollary 4.1. *Let λ and μ be partitions such that λ contains μ and λ/μ is a horizontal strip. Then for any $k \geq 0$ we have*

$$d^{\lambda}_{\mu,(k)} = (-1)^{k-|\lambda/\mu|} \binom{r(\mu/\bar\lambda)}{k - |\lambda/\mu|}$$

where $r(\mu/\bar\lambda)$ is the number of rows in $\mu/\bar\lambda$. If λ/μ is not a horizontal strip then $d^{\lambda}_{\mu,(k)} = 0$.

In particular we obtain the formula

$$G_\lambda(x_1, \dots, x_p) = \sum_{\mu,k} (-1)^{k-|\lambda/\mu|} \binom{r(\mu/\bar\lambda)}{k - |\lambda/\mu|} G_\mu(x_1, \dots, x_{p-1})\, x_p^k$$

where the sum is over all integers $k \geq 0$ and all partitions $\mu \subset \lambda$ such that λ/μ is a horizontal strip. This provides a practical way to compute the polynomial $G_\lambda(x_1, \dots, x_p)$.

As another consequence of Corollary 4.1, notice that when λ/μ is a horizontal strip then

$$\sum_{k\geq 0} d^{\lambda}_{\mu,(k)} = \sum_{k \geq 0} (-1)^{k-|\lambda/\mu|} \binom{r(\mu/\bar\lambda)}{k - |\lambda/\mu|} = \begin{cases} 1 & \text{if } \mu = \bar\lambda; \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$G_\lambda(1, x_1, x_2, \dots) = \sum_{\mu,k} d^{\lambda}_{\mu,(k)}\, G_\mu(x) \cdot 1^k = G_{\bar\lambda}(x)\,.$$

In particular, if we let $\phi_p : \Gamma \to \Gamma$ be the linear map which maps G_λ to G_μ where $\mu = (\lambda_{p+1}, \lambda_{p+2}, \dots)$ is obtained by removing the top p rows of λ, then ϕ_p is a ring homomorphism. Being used to calculating in the ring of symmetric functions, we find this somewhat surprising. Similarly we have $\Delta\, \phi_p = (\phi_i \otimes \phi_j)\, \Delta$ whenever $i + j = p$. This implies lots of identities among the structure constants $c^{\nu}_{\lambda\mu}$ and $d^{\nu}_{\lambda\mu}$ of Γ. For example, if $\emptyset \neq \mu \subset \lambda$ are partitions and $p \geq \ell(\lambda)$, then by comparing the coefficients of $G_\mu \otimes 1$ in $\Delta\, \phi_p\, G_\lambda = (1 \otimes \phi_p)\, \Delta\, G_\lambda$ we obtain the identity $\sum_\tau d^{\lambda}_{\mu\tau} = 0$. The map ϕ_p is related to the proper pushforward map on Grothendieck groups from a Grassmann bundle to its base variety. See [4] for details about this.

J. Stembridge has recently classified all multiplicity free products of Schur functions [13]. We will say that a product $G_\lambda \cdot G_\mu$ is multiplicity free if all constants $c^{\nu}_{\lambda\mu}$ are ± 1 or zero. Stembridge's result then has the following analogue for stable Grothendieck polynomials.

Proposition 4.2. *The product $G_\lambda \cdot G_\mu$ is multiplicity free if and only if both partitions λ and μ are rectangles or one of them is a single box or empty.*

Regarding the structure of Γ as an abstract ring, we conjecture:

Conjecture 4.3. *(a) Any stable polynomial G_λ can be written as a polynomial in the elements G_R for rectangular partitions R contained in λ.*

(b) The elements $\{\dots, G_3, G_2, G_1, G_{(1,1)}, G_{(1,1,1)}, \dots\}$ corresponding to partitions with only one row or one column are algebraically independent.

Part (a) of this conjecture has been verified for all partitions λ of weight at most 9. For (b), if we define the degree of a monomial in the G_k and $G_{(1^\ell)}$ to be the total number of boxes in the partitions of the factors, then all such monomials of degree up to 9 are linearly independent. Furthermore, it is not hard to prove that for any integer $k \geq 2$, the elements $\{G_k, G_1, G_{(1,1)}, G_{(1,1,1)}, \dots\}$ are algebraically independent. Namely, if one uses the lexicographic order on partitions, then the monomials in these elements all have a different maximal partition λ for which the coefficient of G_λ is non-zero.

The conjecture has some interesting consequences for the structure of Γ. If (a) is true, then Γ_t is generated by the the elements G_k, $G_{(1^\ell)}$ in addition to t^{-1}. To see this, notice that if $R = (q)^p$ is a rectangular partition with at least two rows and two columns, and $\lambda = (q^{p-1}, q-1)$ is the partition obtained by removing the box in the corner of R, then

$$t \cdot G_R = G_1\, G_\lambda - G_{\lambda+(1)} - G_{\lambda,(1)} + G_{\lambda+(1),(1)}\,.$$

Using (a) this shows that $t \cdot G_R$ can be written as a polynomial in G_{q+1}, $G_{(1^{p+1})}$, and the elements $G_{\tilde{R}}$ for rectangles $\tilde{R}$ which are strictly contained in R. This shows that G_R is in the ring generated by the elements G_k, $G_{(1^\ell)}$, and t^{-1} by induction on the size of R.

However, if (b) is true then the elements G_k and $G_{(1^\ell)}$ do not generate Γ as a ring. In fact, the identity

$$t \cdot G_{(2,2)} = G_1\, G_2 + G_1\, G_{(1,1)} - G_2\, G_{(1,1)} - (G_1)^3$$

implies that $G_{(2,2)}$ can't be written as a polynomial in these elements if they are algebraically independent.

Geometrically, the fact that Γ is a commutative and cocommutative bialgebra implies that $\operatorname{Spec}\Gamma$ is an Abelian semigroup scheme. The existence of an antipode on Γ_t means that the dense open subset $\operatorname{Spec}\Gamma_t$ is a group scheme. Furthermore, if Conjecture 4.3 is true then this open subset looks like an infinite dimensional affine space with a hyperplane removed.

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