ON TREES AND CHARACTERS

AVITAL FRUMKIN, GORDON D. JAMES, AND YUVAL ROICHMAN

ABSTRACT. A new family of trees, defined in term of Young diagrams, is introduced. Values of central characters of the symmetric group are represented as a weighted enumeration of such trees. The proof involves a new decomposition theorem for representations corresponding to general shapes.
RÉSUMÉ. C'est le résumé français.

1. Central Characters

Let λ and μ be partitions of n, let S^λ be the corresponding symmetric group irreducible representation, and let χ^λ_μ be the value of its character at a conjugacy class of cycle type μ. The *central character* is defined by

$$c^\lambda_\mu := |C_\mu| \cdot \frac{\chi^\lambda_\mu}{\chi^\lambda_{(1^n)}},$$

where $|C_\mu|$ is the size of the conjugacy class of cycle type μ. It is known that the central characters of the symmetric group are integers. The goal of this paper is to give a combinatorial interpretation to the value of the central characters of cycles.

2. Young Trees

A new family of graphs is introduced. A *Young graph* is a graph whose set of vertices is a subset S of boxes in a given Young diagram $[\lambda]$, whereas there is an edge between two boxes in S iff they are in the same row or column, and there is no box of S between them. In other words, $((i_1, j_1), (i_2, j_2)) \in S^{\times 2}$ is an edge iff one the following two conditions holds :

(i) $i_1 = i_2$, and there is no j_3 such that $j_1 < j_3 < j_2$ and $(i_1, j_3) \in S$;
(ii) $j_1 = j_2$, and there is no i_3 such that $i_1 < i_3 < i_2$ and $(i_3, j_1) \in S$.

If the resulting graph is a tree we call it a *Young tree.* The *order* of a Young graph (tree) is the number of vertices in the graph.

The edges in a Young graph Γ are either horizontal or vertical (where the Young diagram is drawn in the British way; i.e. λ_1 is the length of the first upper row etc.). Denote the number of vertical edges in Γ by $\mathrm{vert}(\Gamma)$. We define the sign of Γ as

$$\mathrm{sign}(\Gamma) := (-1)^{\mathrm{vert}(\Gamma)}.$$

A *simple path* in Γ is a connected subgraph contained in one row or column. The length of the path p, denoted by $\ell(p)$, is the number of edges in p. A simple path p in a Young graph Γ is a *maximal simple path* if no simple path in Γ strictly contains p (as a subgraph). The weight of a Young tree T is the product

$$\mathrm{weight}(T) := \prod_{p \text{ is a maximal simple path in } T} \ell(p)!$$

where the product is taken over all maximal simple paths in T.

The following is the main result in [FJR].

Partially supported by the Israel Science Foundation and by internal research grants from Bar-Ilan University.

Theorem 1. *For any $k \le n$ and any partition $\lambda \vdash n$,*

$$c_k^\lambda = \sum_{\{T \mid \; T \text{ is a Young tree of order } k \text{ in } [\lambda]\}} \operatorname{sign}(T) \cdot \operatorname{weight}(T),$$

where c_k^λ is the central character of S^λ at the conjugacy class of cycles of length k.

For example, all trees of order 2 consist of one edge, which is either horizontal or vertical. The weight of all these trees is 1. Hence, c_2^λ is equal to the number of horizontal edges in the Young diagram $[\lambda]$ minus the number of vertical edges in $[\lambda]$, which is equal to $\sum\limits_i \binom{\lambda_i}{2} - \sum\limits_j \binom{\lambda_j'}{2}$, where λ_j' is the length of the j-th column. This is a well known special case of the Frobenius Trace formula. See [F,I].

3. S_n-Modules of General Shapes

An interesting family of S_n-modules is involved in the proof of Theorem 1.

Let $S \subseteq \boldsymbol{N}^2$ be a subset of lattice points of order k. We say that Q is *a standard Young tableau on* S if the points in S are replaced by the letters $1, \ldots, , k$ with no repeats, such that they increase along rows and columns. Namely,

$$Q(i_1, j_1) > Q(i_2, j_2) \quad \text{if } i_1 = i_2 \text{ and } j_1 > j_2, \quad \text{or } \; : j_1 = j_2 \text{ and } i_1 > i_2.$$

For any tableau Q of order k let Q_c be the subgroup in S_k of column preserving permutations, and let Q_r is the subgroup of row preserving permutations. Define

$$r_Q := \sum_{p \in Q_r} p.$$

$$c_Q := \sum_{q \in Q_c} \operatorname{sign}(q) q.$$

$$e_Q := r_Q \cdot c_Q = \sum_{p \in T_r, q \in T_c} \operatorname{sign}(q) pq.$$

Then e_Q is the *Young symmetrizer* of Q, and $\boldsymbol{C}[S_k]e_Q$ is a left representation of S_k. The following theorem considers the factors of this representation.

For any set $S \subseteq \boldsymbol{N}^2$ define the *height partition* $h(S) = (h_1(S), h_2(S), \ldots)$ by

$$h_i(S) := \text{ number of rows in } S \text{ with at least } i \text{ boxes }.$$

Similarly, the *width partition* $w(S) = (w_1(S), w_2(S), \ldots)$ is defined by

$$w_i(S) := \text{ number of columns in } S \text{ with at least } i \text{ boxes }.$$

Let $\preceq$ be the dominance order on partitions. Namely, $\lambda \preceq \mu$ iff for any i $\;\sum_{j=0}^i \lambda_j \le \sum_{j=0}^i \mu_j$.

Theorem 3.1. *[FJR, Theorem 5.2] For any subset $S \subseteq \boldsymbol{N}^2$ and a standard tableau Q on S the following holds:*

If the irreducible representation S^ν appears as a decomposition factor of $\boldsymbol{C}[S_n]e_Q$ then

$$h(S)' \preceq \nu \preceq w(S),$$

where $h(S)'$ is the conjugate of the height partition $h(S)$, and $w(S)$ is the width partition.

For any standard Young tableau Q on $S \subseteq \boldsymbol{N}^2$ let width(Q) be the number of nonempty columns in Q, and height(Q) be the number of nonempty rows. Let k be the cardinality of Q.

Corollary 3.2. *If*

$$\operatorname{width}(Q) + \operatorname{height}(Q) - 1 < k,$$

then no hook representation appears in the decomposition of $\boldsymbol{C}[S_k]e_Q$ into irreducible representations.

4. Proof of Theorem 1 (Sketch)

For any standard Young tableau Q let Q_c be the subgroup in S_n consisting of column preserving permutations, and Q_r be the subgroup of row preserving permutations. For any $2 \le k \le n$ let C_k be the conjugacy class of k-cycles, and let c_k^λ be the central character of S^λ at C_k. The following lemma is implicit in [Su].

Lemma 4.1. *[FJR, Corollary 2.4] For any standard Young tableau Q of shape λ, and any $2 \le k \le n$*

$$c_k^\lambda = \sum_{\{(p,q)\ \mid\ p\in Q_r, q\in Q_c, pq\in C_k\}} \mathrm{sign}(q).$$

The *support* of a permutation $\pi \in S_n$, *supp*(π), is the set of non-fixed letters under the action of π. For any permutation $\pi \in S_n$ and any standard Young tableau of order n, Q, define $\Gamma_Q(\pi)$ to be the Young graph whose set of vertices is the subset of boxes in Q, labeled by the letters of *supp*(π).

Denote by T_k^λ the set of all Young trees of order k in the Young diagram $[\lambda]$. By Lemma 4.1, in order to prove the main theorem it suffices to prove the following two complementary lemmas:

Lemma 4.2. *For any standard Young tableau Q of shape λ*

$$\sum_{\{(p,q)\ \mid\ p\in Q_r, q\in Q_c, pq\in C_k, \Gamma_Q(pq)\in T_k^\lambda\}} \mathrm{sign}(q) = \sum_{\{T\mid\ T\in T_k^\lambda\}} \mathrm{sign}(T)\cdot \mathrm{weight}(T).$$

Lemma 4.3. *For any standard Young tableau Q of shape λ*

$$\sum_{\{(p,q)\ \mid\ p\in Q_r, q\in Q_c, pq\in C_k, \Gamma_Q(pq)\not\in T_k^\lambda\}} \mathrm{sign}(q) = 0.$$

Lemma 4.2 is proved by induction on k (for a detailed proof see [FJR, Section 4]). Lemma 4.3 is proved by applying the decomposition theorem, descibed in Section 3.

5. Proof of Lemma 4.3 (Sketch)

For any subset S of boxes in a Young diagram $[\lambda]$ let width(S) be the number of nonempty columns in S, and height(S) be the number of nonempty rows.

Lemma 5.1. *[FJR, Lemma 6.1] Let S be a subset in a Young diagram, whose Young graph $\Gamma(S)$ is connected. Then $\Gamma(S)$ is not a tree iff*

$$\mathrm{width}(S) + \mathrm{height}(S) - 1 < |S|.$$

The following lemma is well known.

Lemma 5.2. *[Sa, Lemma 4.10.3] Let λ be a partition of k, which is not of hook shape. Then*

$$\chi_k^\lambda = 0,$$

where χ_k^λ is the value of the character of S^λ at cycles of length k.

Combining Corollary 3.2, Lemma 5.1 and Lemma 5.2 completes the proof of Lemma 4.3. For further details see [FJR, Section 6].

Remark. For a different approach for computing symmetric group central charcters see [GPS]. Thanks to the anonymous referee for this reference.

References

[BL] W. M. Benyon and G. Lusztig, *Some numerical results on the characters of exceptional Weyl groups.* Math. Proc. Cambridge Philos. Soc. 84 (1978), 417–426.

[FJR] A. Frumkin, G. D. James and Y. Roichman, *On trees and characters*, preprint.

[F] G. Frobenius, *Über die Charaktere der Symmetrischen Gruppe.* Berliner Berichte (1901).

[GPS] A. Goupil, D. Poulalhon and G. Schaeffer, *Central characters and conjugacy classes of the symmetric group, or On some conjectures of J. Katriel.* Formal Power Series and Algebraic Combinatorics (Moscow 2000), 238–249, Springer, Berlin, 2000.

[I] R. E. Ingram, *Some characters of the symmetric group.* Proc. A.M.S. 1 (1950), 358-369.

[JK] G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group.* Encyclopedia of Math. and its Appl. 16, Addison-Wesley, 1981.

[Sa] B. E. Sagan, *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions.* Wadsworth & Brooks/Cole, CA, 1991.

[Su] M. Suzuki, *The values of irreducible characters of the symmetric group.* The Arcata Conference on Representations of Finite Groups, AMS Proceedings of Symposia in Pure Mathematics, Vol. 47 - Part 2, 1987, 317–319.

Department of Mathematics and Computer Science, Tel-Aviv University, Ramat-Aviv, Israel.
E-mail address: `frumkin@math.tau.ac.il`

Department of Mathematics, Imperial College of Science, Technology and Medicine, 180 Queen's Gate, London SW7 2BZ, England.
E-mail address: `g.james@ic.ac.uk`

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel.
E-mail address: `yuvalr@math.biu.ac.il`