NUMERATOR FORMULAE EXPANSIONS FOR AFFINE KAC-MOODY ALGEBRAS

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Abstract. The character of each integrable highest weight irreducible representation of an affine Kac-Moody algebra is expressed as the ratio of a numerator to a denominator, each of which may itself be expressed as an infinite sum of characters of irreducible representations of a maximal simple Lie subalgebra. The denominator expansions are the celebrated Macdonald identities. In the case of all seven infinite series of affine Kac-Moody algebras that are indexed by their rank, these identities are rendered in a rank-independent way by making use of particular coset representatives of the relevant affine Weyl group with respect to its finite Weyl subgroup. These same coset representatives are then used to construct new numerator expansions.
Résumé. The character of each integrable highest weight irreducible representation of an affine Kac-Moody algebra is expressed as the ratio of a numerator to a denominator, each of which may itself be expressed as an infinite sum of characters of irreducible representations of a maximal simple Lie subalgebra. The denominator expansions are the celebrated Macdonald identities. In the case of all seven infinite series of affine Kac-Moody algebras that are indexed by their rank, these identities are rendered in a rank-independent way by making use of particular coset representatives of the relevant affine Weyl group with respect to its finite Weyl subgroup. These same coset representatives are then used to construct new numerator expansions.

1. Affine Kac-Moody algebras

A matrix $A=(a_{ij})_{i,j\in I}$ is said to be a generalised Cartan matrix (GCM) if: (i) $a_{ii}=2$; (ii) $a_{ij}\leq 0$ for $i\neq j$; (iii) $a_{ij}=0$ implies $a_{ji}=0$; (iv) $a_{ij}\in\mathbb{Z}$. The associated Kac-Moody algebra $\mathbf{g}(A)$ is a complex Lie algebra with generators e_i, f_i, h_i for $i\in I$ satisfying the relations

$$\begin{array}{ll} [h_i,h_j]=0, & [e_i,f_j]=\delta_{ij}h_i, \\ [h_i,e_j]=a_{ij}e_j, & [h_i,f_j]=-a_{ij}f_j, \\ (\mathrm{ad}\,e_i)^{1-a_{ij}}e_j=0 \text{ for } i\neq j, & (\mathrm{ad}\,f_i)^{1-a_{ij}}f_j=0 \text{ for } i\neq j, \end{array}$$

where $(\mathrm{ad}\,x)y=[x,y]$ for all $x,y\in\mathbf{g}(A)$.

If A is indecomposable and all its principal minors are positive then $\mathbf{g}(A)$ is finite and isomorphic to one of the simple Lie algebras $\mathbf{g}_0$. On the other hand if A is again indecomposable but $\det A=0$ and all the proper principal minors of A are positive then there exists a vector δ with positive integer components such that $A\delta=0$ and $\mathbf{g}(A)$ is infinite-dimensional and isomorphic to one of the affine Kac-Moody algebras $\mathbf{g}$.

The classification of all complex simple Lie algebras $\mathbf{g}_0$, due to Cartan and Killing, takes the form:

$$A_\ell;\quad B_\ell;\quad C_\ell;\quad D_\ell;\quad E_6;\quad E_7;\quad E_8;\quad F_4;\quad G_2.$$

while that of the affine Kac-Moody algebras $\mathbf{g}$, due to Kac and Moody, takes the form:

$$\begin{array}{l} A_\ell^{(1)};\quad B_\ell^{(1)};\quad C_\ell^{(1)};\quad D_\ell^{(1)};\quad E_6^{(1)};\quad E_7^{(1)};\quad E_8^{(1)};\quad F_4^{(1)};\quad G_2^{(1)}; \\ A_{2\ell}^{(2)};\quad A_{2\ell-1}^{(2)};\quad D_{\ell+1}^{(2)};\quad E_6^{(2)};\quad D_4^{(3)}. \end{array}$$

A simple Lie algebra $\mathbf{g}_0$ of rank ℓ is said to be naturally embedded in an affine Kac-Moody algebra $\mathbf{g}$ of the same rank if there exist index sets $I=\{0,1,2,\ldots,\ell\}$ and $I_0=\{1,2,\ldots,\ell\}$ such that $\mathbf{g}=\mathbf{g}(A)$ with $A=(a_{ij})_{i,j\in I}$ and $\mathbf{g}_0=\mathbf{g}(A_0)$ with $A_0=(a_{ij})_{i,j\in I_0}$. The natural embeddings of the form $\mathbf{g}_0 \subset \mathbf{g}$ arising from the conventional index sets are the following:

$$\begin{array}{lllll} A_\ell \subset A_\ell^{(1)}; & B_\ell \subset B_\ell^{(1)}, A_{2\ell}^{(2)}, D_{\ell+1}^{(2)}; & C_\ell \subset C_\ell^{(1)}, A_{2\ell-1}^{(2)}; & D_\ell \subset D_\ell^{(1)}; & \\ E_6 \subset E_6^{(1)}; & E_7 \subset E_7^{(1)}; & E_8 \subset E_8^{(1)}; & F_4 \subset F_4^{(1)}, E_6^{(2)}; & G_2 \subset G_2^{(1)}, D_4^{(3)}. \end{array}$$

Some notational preliminaries are required. Let the affine Kac-Moody algebra $\mathbf{g}$ have Cartan subalgebra $\mathbf{h}$, with dual $\mathbf{h}^*$, positive roots Δ^+, simple roots $\Pi = \{\alpha_i \in \mathbf{h}^* | i \in I\}$ and imaginary root $\delta \in \mathbf{h}^*$. Let the simple Lie algebra $\mathbf{g}_0$ have Cartan subalgebra $\mathbf{h}_0$, with dual $\mathbf{h}_0^*$, positive roots Δ_0^+ and simple roots $\Pi_0 = \{\alpha_i \in \mathbf{h}_0^* | i \in I_0\}$. The affine Weyl group W of $\mathbf{g}$ is generated by $\{s_i | i \in I\}$ and the finite Weyl group W_0 of $\mathbf{g}_0$ is generated by $\{s_i | i \in I_0\}$, where $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ for all $i \in I$ and $\lambda \in \mathbf{h}^*$ with $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ji}$. The length $\ell(w)$ of the element $w \in W$ is the minimal value of t such that $w = s_{i_1} s_{i_2} \cdots s_{i_t}$ with $i_j \in I$ for $j = 1, 2, \ldots, t$. Turning to the weight spaces of $\mathbf{g}$ and $\mathbf{g}_0$, the sets of dominant integral weights of $\mathbf{g}$ and $\mathbf{g}_0$ are given by $P^+ = \{\lambda \in \mathbf{h}^* \,|\, \lambda_i = \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}^+ \text{ for all } i \in I\}$ and $P_0^+ = \{\lambda \in \mathbf{h}_0^* \,|\, \lambda_i = \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}^+ \text{ for all } i \in I_0\}$, respectively, and the sets of fundamental weights by $\{\Lambda_i \in P^+ \,|\, \langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij} \text{ for all } i, j \in I\}$ and $\{\omega_i \in P_0^+ \,|\, \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \text{ for all } i, j \in I_0\}$, with $\Lambda_i = \omega_i + (c_i^\vee / c_0^\vee) \Lambda_0$ for $i \in I_0$ where $c_i^\vee$ for $i \in I$ are the comarks of $\mathbf{g}$. In terms of $\mathbf{h}^* = \mathbf{h}_0^* \oplus (\mathbb{C}\Lambda_0 + \mathbb{C}\delta)$ an arbitrary weight λ may be expressed in either of the two forms

$$\lambda = \sum_{i=0}^{\ell} \lambda_i \Lambda_i - d(\lambda)\delta = \overline{\lambda} + \frac{L(\lambda)}{c_0^\vee} \Lambda_0 - d(\lambda)\delta, \tag{1}$$

where λ is of level $L(\lambda) = \sum_{i \in I} c_i^\vee \lambda_i$ and depth $d(\lambda)$, and $\overline{\lambda} \in \mathbf{h}_0^*$ is the restriction of $\lambda \in \mathbf{h}^*$ to $\mathbf{h}_0^*$ defined by $\overline{\lambda} = \sum_{i \in I_0} \lambda_i \omega_i$. In a standard Euclidean basis $\overline{\lambda} = \sum_{i=1}^{\ell} \overline{\lambda_i} \epsilon_i$ for $\mathbf{g}_0 = B_\ell$, C_ℓ or D_ℓ, and $\overline{\lambda} = \sum_{i=1}^{\ell+1} \overline{\lambda_i} \epsilon_i$ for $\mathbf{g}_0 = A_\ell$ with $\sum_{i=1}^{\ell+1} \epsilon_i = 0$. Finally, it is convenient to introduce $\rho = \sum_{i \in I} \Lambda_i$, so that $\langle \rho, \alpha_i^\vee \rangle = 1$ for all $i \in I$, $L(\rho) = \sum_{i \in I} c_i^\vee$, and $\overline{\rho} = \rho_0 = \sum_{i \in I_0} \omega_i$ with $\langle \rho_0, \alpha_i^\vee \rangle = 1$ for all $i \in I_0$.

With this notation, the Kac-Weyl character formula for $\mathbf{g}$ takes the form [5]:

$$\mathrm{ch}\, V^\lambda = \sum_{w \in W} \mathrm{sgn}\,(w) e^{w(\lambda+\rho)-\rho} \Big/ \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}\,(\alpha)}. \tag{2}$$

where $\mathrm{sgn}\,(w) = (-1)^{\ell(w)}$ and $\mathrm{mult}\,(\alpha)$ is the multiplicity of the root α. This is a direct generalisation of Weyl's character formula

$$\mathrm{ch}\, V_0^\lambda = \sum_{w \in W_0} \mathrm{sgn}\,(w) e^{w(\lambda+\rho_0)-\rho_0} \Big/ \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha}). \tag{3}$$

2. A character formula for affine Kac-Moody algebras

The first step is to re-write the Kac-Weyl character formula 2 for $\mathbf{g}$ in terms of characters of $\mathbf{g}_0$ in accordance with the following prescription.

Proposition 1. [4] *For each dominant integral highest weight λ of $\mathbf{g}$ of level $L(\lambda)$ and depth $d(\lambda)$*

$$\mathrm{ch}\, V^\lambda = \frac{M^\lambda}{M} = \frac{e^{L(\lambda)\Lambda_0} \; \sum_{w_s \in W(I_0)} \mathrm{sgn}\,(w_s) e^{-d(w_s \cdot \lambda)\delta} \; \mathrm{ch}\, V_0^{\overline{w_s \cdot \lambda}}}{\sum_{w_s \in W(I_0)} \mathrm{sgn}\,(w_s) e^{-d(w_s \cdot 0)\delta} \; \mathrm{ch}\, V_0^{\overline{w_s \cdot 0}}}, \tag{4}$$

where $W(I_0)$ is a set of right coset representatives of the Weyl group W of $\mathbf{g}$ with respect to the Weyl subgroup W_0 of $\mathbf{g}_0$, while $w_s \cdot \lambda = w_s(\lambda + \rho) - \rho$ and $w_s \cdot 0 = w_s(\rho) - \rho)$ for all $w_s \in W(I_0)$.

Proof. Divide the numerator and denominator of the character formula of $\mathbf{g}$ by the denominator of the character formula of $\mathbf{g}_0$. Then write $w \in W$ in the form $w = w_0 w_s$ with $w_0 \in W_0$ and $w_s \in W(I_0)$, and express $w_0 w_s(\lambda + \rho) - \rho$ and $w_0 w_s(\rho) - \rho$ in terms of Λ_0 and δ, recalling the notation of 1 whereby $\overline{\mu}$ is used to denote the restriction of $\mu \in \mathbf{h}^*$ to $\mathbf{h}_0^*$. It is also necessary to use the fact that $w_0(\rho) - \rho = w_0(\rho_0) - \rho_0$. ∎

To proceed further it is helpful to make a particular choice of right coset representatives of W with respect to W_0. Such a choice is based on the following.

Lemma 1. [8, 1, 9] *Let*

$$W(I_0) = \{w \in W \,|\, \Phi_w \subseteq \Delta^+ \backslash \Delta_0^+\} \quad \textit{where} \quad \Phi_w = \{\alpha \in \Delta^+ \,|\, w^{-1}(\alpha) \in \Delta^-\}.$$

Then $W(I_0)$ is a complete set of right coset representatives of W with respect to W_0 such that for all dominant integral $\lambda \in \mathbf{h}^*$ and $w_s \in W(I_0)$ the restriction $\overline{w_s \cdot \lambda}$ of the weight $w_s \cdot \lambda = w_s(\lambda+\rho) - \rho$ from $\mathbf{h}^*$ to $\mathbf{h}_0^*$ is itself dominant integral.

This choice is just what is required to ensure that all the weights $\overline{w_s \cdot \lambda}$ and $\overline{w_s \cdot 0}$ specifying irreducible representations of $\mathbf{g}_0$ in the numerator and denominator of Proposition 1 are dominant integral.

It should be noted that the denominator of $\mathrm{ch}\, V^\lambda$ in Proposition 1 takes the form

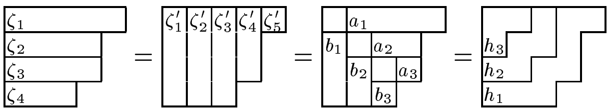
$$M = M^0 = \prod_{\alpha \in \Delta^+ \backslash \Delta_0^+} (1 - e^{-\alpha})^{\mathrm{mult}\,(\alpha)} = \sum_{w_s \in W(I_0)} \mathrm{sgn}\,(w_s) e^{-d(w_s \cdot 0)\delta}\ \mathrm{ch}\, V_0^{\overline{w_s \cdot 0}}. \tag{5}$$

This is one way of expressing the Macdonald identities [10] which can be made explicit for each affine Kac-Moody algebra $\mathbf{g}$ through the identification of $w_s \in W(I_0)$ and the determination of $w_s \cdot 0 = w_s(\rho) - \rho$. ¿From these can be read off not only $\mathrm{sgn}\,(w_s) = (-1)^{\ell(w_s)}$ but also both $\overline{w_s \cdot 0}$ and $d(w_s \cdot 0)$.

3. Partitions, Young diagrams and characters of simple Lie algebras

Before describing the required identification of $w_s \in W(I_0)$ and the determination of $w_s(\rho) - \rho$ for any particular affine Kac-Moody algebra $\mathbf{g}$, it is convenient to introduce the partitions and associated Young diagrams which are used both to specify the elements w_s themselves and their action on ρ.

Let P be the set of all partitions, including 0. Each partition $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{\ell(\zeta)}) \in P$ of weight $|\zeta|$ and length $\ell(\zeta)$ specifies a Young diagram F^ζ with row lengths ζ_i. The column lengths ζ_j' define the conjugate partition ζ'. In Frobenius notation $\zeta = \binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$, where F^ζ consists of p nested hooks or wrapped continuous strips of lengths $h_k = a_k + b_k + 1$:

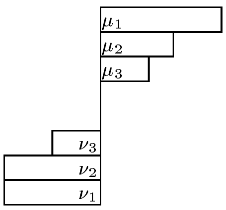
$F^\zeta =$ 

If $\ell(\zeta) \leq \ell$ then F^ζ specifies characters of the simple Lie groups,

$$A_\ell : \mathrm{ch}\, V_0^{\{\zeta\}}, \quad B_\ell : \mathrm{ch}\, V_0^{[\zeta]}, \quad C_\ell : \mathrm{ch}\, V_0^{\langle\zeta\rangle}, \quad D_\ell : \mathrm{ch}\, V_0^{[\zeta]},$$

each of highest weight $\sum_{i=1}^{\ell(\zeta)} \zeta_i \epsilon_i$. In the case of D_ℓ, if $\ell(\zeta) = \ell$ then $\mathrm{ch}\, V_0^{[\zeta]} = \mathrm{ch}\, V_0^{[\zeta]+} + \mathrm{ch}\, V_0^{[\zeta]-}$. For $\ell(\zeta) > \ell$ it is necessary to invoke certain modification rules [2, 7]. The corresponding characters are either null or $\pm$ a standard character specified by some $\xi \in P$ with $\ell(\xi) \leq \ell$.

By the same token each pair of partitions μ and ν specifies a composite Young diagram $F^{\mu;\overline{\nu}}$:



If $\ell(\mu) + \ell(\nu) \leq \ell + 1$ then $F^{\mu;\overline{\nu}}$ specifies the character $\mathrm{ch}\, V_0^{\{\mu;\overline{\nu}\}}$ of an irreducible representation $V_0^{\{\mu;\overline{\nu}\}}$ of A_ℓ of highest weight $\sum_{i=1}^{\ell(\mu)} \mu_i \epsilon_i - \sum_{j=1}^{\ell(\nu)} \nu_j \epsilon_{\ell-j+2}$. This time if $\ell(\mu) + \ell(\nu) > \ell + 1$ then modificaton rules [2, 7] must again be invoked.

It has been shown that the required set of coset representatives $W(I_0)$ may be determined recursively through the use of the following:

Lemma 2. [6] *Let* $W(I_0) = \cup_{d=0}^{\infty} W_d(I_0)$ *with* $W_d(I_0) = \{w_s \in W(I_0)\,|\,\ell(w_s) = d\}$*. Then, if* $w = w's_j$ *and* $\ell(w) = \ell(w') + 1 = d + 1$*, then* $w \in W_{d+1}(I_0)$ *if and only if* $w' \in W_d(I_0)$ *and* $w'(\alpha_j) \in \Delta^+ \backslash \Delta_0^+$*.*

The recursion is started by noting that $W_0(I_0) = \{1\}$. It is then easy to see first that $W_1(I_0) = \{s_0\}$ and then $W_2(I_0) = \{s_0 s_j \,|\, \langle \alpha_j, \alpha_0^\vee \rangle = a_{j0} < 0\}$. Proceeding recursively we can identify all $w_s \in W(I_0)$ and then calculate $w_s \cdot 0 = w_s(\rho) - \rho$, as required. We obtain the following results, either inductively with respect to the depth d in the case of each pair $\mathbf{g} \supset \mathbf{g}_0$, or quite independently by Schur function methods.

Proposition 2. [7, 3] *For each of the seven sequences of natural embeddings of a simple Lie algebra* $\mathbf{g}_0$ *in an affine Kac-Moody algebra* $\mathbf{g}$ *that are indexed by their rank* ℓ*, the expansions of the denominator functions* $M = \prod_{\alpha \in \Delta^+ \backslash \Delta_0^+} (1 - e^{-\alpha})^{\mathrm{mult}\,(\alpha)}$ *take the form:*

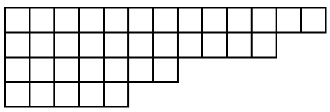
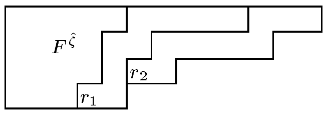
$$\begin{array}{ll}
A_\ell^{(1)} \supset A_\ell : & M = \sum_{\zeta \in F} \, (-1)^{|\zeta|} \; e^{-|\zeta|\,\delta} \; \mathrm{ch}\, V_0^{\{\zeta;\overline{\zeta'}\}}; \\
B_\ell^{(1)} \supset B_\ell : & M = \sum_{\alpha \in A} \, (-1)^{|\alpha|/2} \; e^{-(|\alpha|/2)\delta} \; \mathrm{ch}\, V_0^{[\alpha]}; \\
C_\ell^{(1)} \supset C_\ell : & M = \sum_{\gamma \in C} \, (-1)^{|\gamma|/2} \; e^{-(|\gamma|/2)\delta} \; \mathrm{ch}\, V_0^{\langle\gamma\rangle}; \\
D_\ell^{(1)} \supset D_\ell : & M = \sum_{\alpha \in A} \, (-1)^{|\alpha|/2} \; e^{-(|\alpha|/2)\delta} \; \mathrm{ch}\, V_0^{[\alpha]}; \\
A_{2\ell}^{(2)} \supset B_\ell : & M = \sum_{\gamma \in C} \, (-1)^{|\gamma|/2} \; e^{-(|\gamma|/2)\delta} \; \mathrm{ch}\, V_0^{[\gamma]}; \\
A_{2\ell-1}^{(2)} \supset C_\ell : & M = \sum_{\alpha \in A} \, (-1)^{|\alpha|/2} \; e^{-(|\alpha|/2)\delta} \; \mathrm{ch}\, V_0^{\langle\alpha\rangle}; \\
D_{\ell+1}^{(2)} \supset B_\ell : & M = \sum_{\varepsilon \in E} \, (-1)^{(|\varepsilon|+p)/2} \; e^{-|\varepsilon|\,\delta} \; \mathrm{ch}\, V_0^{[\varepsilon]}.
\end{array}$$

where in Frobenius notation

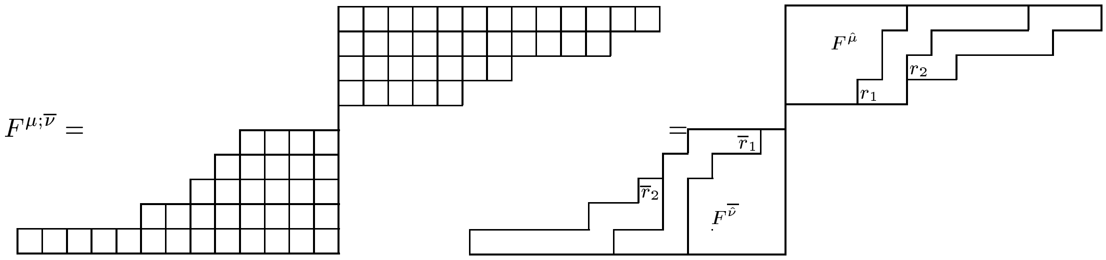
$$\begin{array}{rcl}
A & = & \left\{ \alpha \in P \mid \alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1{+}1 & a_2{+}1 & \cdots & a_p{+}1 \end{pmatrix} \right\}; \\
C & = & \left\{ \gamma \in P \mid \gamma = \begin{pmatrix} b_1{+}1 & b_2{+}1 & \cdots & b_p{+}1 \\ b_1 & b_2 & \cdots & b_p \end{pmatrix} \right\}; \\
E & = & \left\{ \varepsilon \in P \mid \varepsilon = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix} \right\}; \\
F & = & \left\{ \zeta \in P \mid \zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix} \right\}.
\end{array}$$

It should be stressed that the characters appearing in these expressions are universal characters defined for all values of the rank ℓ. However, as indicated earlier it will in general be necessary to invoke modification rules to interpret them in terms of standard characters for particular values of ℓ. As a result of applying these modification rules it is found that the above generic forms of the Macdonald identities may be recast in terms of certain core diagrams supplemented by the addition of continuous boundary strips of fixed length. This supplementation is made precise as follows.

The partition ζ is said to have L-core $\hat{\zeta}$ if the Young diagram $F^{\hat{\zeta}}$ is obtained from that of F^{ζ} by the removal of all possible continuous boundary strips of length N in such a way that after each removal the diagram remains of regular shape with its rows defining a partition. Constraining this notion still further we say that $\zeta \equiv \hat{\zeta} (\mathrm{mod}\, L)$ if and only if F^{ζ} may be obtained from $F^{\hat{\zeta}}$ by the addition of a sequence of s continuous boundary strips of length L that start in rows $r_1, r_2, \ldots, r_s$ and end in each case in row 1, with $r_1 \geq r_2 \geq \cdots \geq r_s$. In such a case we set $r = r_1 + r_2 + \cdots + r_s$. Typically, for $L = 10$ and $\zeta = (13, 11, 7, 5)$ we have $\hat{\zeta} = (5, 4, 4, 3)$ and $F^{13,11,7,5,4}$ is obtained from $F^{5,4,4,3}$ by the addition of $s = 2$ continuous boundary strips of length $L = 10$ starting in rows $r_1 = 4$ and $r_2 = 3$, as illustrated by:

$F^{\zeta} =$  $=$ 

Generalising this to the case of composite Young diagrams we say that $\mu;\overline{\nu} \equiv \hat{\mu};\overline{\hat{\nu}} (\mathrm{mod}\, L)$ if and only if $F^{\mu;\overline{\nu}}$ may be obtained from $F^{\hat{\mu};\overline{\hat{\nu}}}$ by the addition of a sequence of s pairs of continuous boundary strips each of length L that start in rows $r_1, r_2, \ldots, r_s$ and $\overline{r}_1, \overline{r}_2, \ldots, \overline{r}_s$ of each part of the composite diagram and end in the first row of the corresponding part, counting rows from the top in F^{μ} and from the bottom in $F^{\overline{\nu}}$. It is required that $r_1 \geq r_2 \geq \cdots \geq r_s$ and $\overline{r}_1 \geq \overline{r}_2 \geq \cdots \geq \overline{r}_s$. In such a case we set $r = r_1 + r_2 + \cdots + r_s$ and $\overline{r} = \overline{r}_1 + \overline{r}_2 + \cdots + \overline{r}_s$. Typically, for $L = 10$ and $\mu;\overline{\nu} = (13, 11, 7, 5; \overline{4, 5, 6, 8, 13})$ we have $\hat{\mu};\overline{\hat{\nu}} = (5, 4, 4, 3; \overline{1, 3, 4, 4, 4})$ and $F^{13,11,7,5;\overline{4,5,6,8,13}}$ is obtained from $F^{5,4,4,3;\overline{1,3,4,4,4}}$ by the addition of $s = 2$ pairs of continuous boundary strips of length $L = 10$ starting in rows $r_1 = 4, r_2 = 3$ and $\overline{r}_1 = 5, \overline{r}_2 = 3$, as illustrated by:



The modification rules appropriate to non-standard characters of the classical simple Lie algebras A_ℓ, B_ℓ, C_ℓ and D_ℓ are such that Proposition 2 may be reformulated as:

Proposition 3. *For each of the seven sequences of natural embeddings of a simple Lie algebra* $\mathbf{g}_0$ *in an affine Kac-Moody algebra* $\mathbf{g}$ *that are indexed by their rank* ℓ*, the expansions of the denominator functions* $M = \prod_{\alpha\in\Delta^+\setminus\Delta_0^+}(1-e^{-\alpha})^{\mathrm{mult}\,(\alpha)}$ *take the form:*

$$A_\ell^{(1)} \supset A_\ell : \qquad M = \sum_{\substack{\mu;\overline{\nu}\equiv\zeta;\overline{\zeta}'\,\mathrm{mod}\,(\ell+1),\,\zeta\in F\\ \ell(\mu)+\ell(\nu),\,\ell(\zeta)+\ell(\zeta')\leq\ell+1}} (-1)^{|\zeta|+r+\overline{r}+s}\ e^{-C_2(\mu;\overline{\nu})\delta}\mathrm{ch}\, V_0^{\{\mu;\overline{\nu}\}};$$

$$B_\ell^{(1)} \supset B_\ell : \qquad M = \sum_{\substack{\mu\equiv\alpha\,\mathrm{mod}\,(2\ell-1),\,\alpha\in A\\ \ell(\mu),\ell(\alpha)\leq\ell}} (-1)^{|\alpha|/2+r}\ e^{-C_2[\mu]\delta}\ \mathrm{ch}\, V_0^{[\mu]};$$

$$C_\ell^{(1)} \supset C_\ell : \qquad M = \sum_{\substack{\mu\equiv\gamma\,\mathrm{mod}\,(2\ell+2),\,\gamma\in C\\ \ell(\mu),\ell(\gamma)\leq\ell}} (-1)^{|\gamma|/2+r+s}\ e^{-C_2\langle\mu\rangle\delta}\ \mathrm{ch}\, V_0^{\langle\mu\rangle};$$

$$D_\ell^{(1)} \supset D_\ell : \qquad M = \sum_{\substack{\mu\equiv\alpha\,\mathrm{mod}\,(2\ell-2),\,\alpha\in A\\ \ell(\mu),\ell(\alpha)\leq\ell}} (-1)^{|\alpha|/2+r}\ e^{-C_2[\mu]\delta}\ \mathrm{ch}\, V_0^{[\mu]};$$

$$A_{2\ell}^{(2)} \supset B_\ell : \qquad M = \sum_{\substack{\mu\equiv\gamma\,\mathrm{mod}\,(2\ell+1),\,\gamma\in C\\ \ell(\mu),\ell(\gamma)\leq\ell}} (-1)^{|\gamma|/2+r}\ e^{-(C_2[\mu]/C_2[2])\delta}\ \mathrm{ch}\, V_0^{[\mu]};$$

$$A_{2\ell-1}^{(2)} \supset C_\ell : \qquad M = \sum_{\substack{\mu\equiv\alpha\,\mathrm{mod}\,(2\ell),\,\alpha\in A\\ \ell(\mu),\ell(\alpha)\leq\ell}} (-1)^{|\alpha|/2+r+s}\ e^{-(C_2\langle\mu\rangle/C_2\langle 1^2\rangle)\delta}\ \mathrm{ch}\, V_0^{\langle\mu\rangle};$$

$$D_{\ell+1}^{(2)} \supset B_\ell : \qquad M = \sum_{\substack{\mu\equiv\varepsilon\,\mathrm{mod}\,(2\ell),\,\varepsilon\in E\\ \ell(\mu),\ell(\varepsilon)\leq\ell}} (-1)^{(|\varepsilon|+p)/2+r}\ e^{-(C_2[\mu]/C_2[1])\delta}\ \mathrm{ch}\, V_0^{[\mu]}.$$

where C_2 *is the second order Casimir operator of the appropriate simple Lie algebra* $\mathbf{g}_0$.

It should be noted that the strip lengths L, specified by the $\mathrm{mod}\,(L)$ conditions associated with each summation are all of the form $L = L(\rho) = h^\vee$, the dual Coxeter number, except in the case of $C_\ell^{(1)} \supset C_\ell$ for which $L = 2L(\rho) = 2h^\vee$.

4. Numerator expansions

Since $M^\lambda = \sum_{w_s\in W(I_0)} M_s^\lambda$ with $M_s^\lambda = \mathrm{sgn}\,(w_s)\ e^{L(\lambda)\Lambda_0}\ e^{-d(w_s\cdot\lambda)\delta}\ \mathrm{ch}\, V_0^{\overline{w_s\cdot\lambda}}$, we can evaluate M^λ if we can determine $w_s\cdot\lambda = w_s(\lambda+\rho)-\rho$ for each $w_s \in W(I_0)$. To do this it is convenient to introduce certain numberings of the various Young diagrams that we have encountered. Such a numbering involves placing entries in each box of the appropriate diagram. In each case we denote by η_{ij} the entry to be placed in the box in the ith row and jth column. With certain constraints on ℓ, it is found that η_{ij} is specified for each $\mathbf{g}$ by means of the formulae given in the following tabulation, which also includes illustrative examples:

$\mathbf{g} \supset \mathbf{g}_0$	ℓ	η_{ij}	$w_s \in W(I_0)$	
$A_\ell^{(1)} \supset A_\ell$	$\ell = 9$	$\begin{cases} i-j & \text{if } i \geq j \\ \ell+i-j+1 & \text{if } i < j \end{cases}$	$w_\zeta =$ $s_0s_1s_2s_3s_9s_8s_7s_6$ $s_0s_1s_2s_9s_8$ $s_0s_1s_9$	$\begin{array}{ccccccccc} &&&&0&9&8&7&6\\ &&&&1&0&9&8&\\ &&&&2&1&0&9&\\ &&&&3&2&1&&\\ &&&6&&&&&\\ &9&8&7&&&&&\\ 1&0&9&8&&&&&\\ 2&1&0&9&&&&&\\ 3&2&1&0&&&&& \end{array}$
$B_\ell^{(1)} \supset B_\ell$ $D_\ell^{(1)} \supset D_\ell$ $A_{2\ell-1}^{(1)} \supset C_\ell$	$\ell \geq 6$	$\begin{cases} i-j & \text{if } i > j+1 \\ j+1 \bmod (2) & \text{if } i = j, j+1 \\ j-1+1 & \text{if } i < j \end{cases}$	$w_\alpha =$ $s_0s_2s_3s_4$ s_1s_2	$\begin{array}{cccc} 0&2&3&4\\ 0&1&2&\\ 2&1&&\\ 3&2&&\\ 4&&& \end{array}$
$C_\ell^{(1)} \supset C_\ell$ $A_{2\ell}^{(2)} \supset B_\ell$	$\ell \geq 4$	$\begin{cases} i-j & \text{if } i \geq j \\ j-i+1 & \text{if } i < j \end{cases}$	$w_\gamma =$ $s_0s_1s_2s_3$ s_0s_1	$\begin{array}{ccccc} 0&0&1&2&3\\ 1&0&0&1&\\ 2&1&&&\\ 3&&&& \end{array}$
$D_{\ell+1}^{(2)} \supset B_\ell$	$\ell \geq 5$	$\begin{cases} i-j & \text{if } i \geq j \\ j-i-1 & \text{if } i < j \end{cases}$	$w_\varepsilon =$ $s_0s_1s_2s_3s_4$ $s_0s_1s_2$	$\begin{array}{ccccc} 0&1&2&3&4\\ 1&0&1&2&\\ 2&1&&&\\ 3&2&&&\\ 4&&&& \end{array}$

In these illustrative examples w_s has been denoted by w_ζ, w_α, w_γ or w_ε, in line with their corresponding contributions to the expansions of $M = \sum_{w_s \in W(I_0)} M_{w_s}$ given in Proposition 2. In each example the elements w_s have been displayed as a product in which each line of generators $s_i s_j \cdots s_k$ is associated in a rather obvious way with one of the nested hooks of the corresponding numbered diagram. Apart from the first case for which $\ell = 9$, these numberings are indeed typical provided that ℓ is sufficiently large. It should be noted that the numbering given in the case $A_\ell^{(1)}$ applies to the F^ζ part of $F^{\zeta;\overline{\zeta'}}$, with the numbering of the $F^{\overline{\zeta'}}$ part determined by taking the conjugate of the numbered form of F^ζ. The significance of these numberings lies in the following:

Conjecture 4. *Let $M = \sum_{w_s \in W(I_0)} M_s$ with $M_s = \mathrm{sgn}\,(w_s)\; e^{-d(w_s\cdot 0)\delta}\; \mathrm{ch}\, V_0^{\overline{w_s\cdot 0}}$, and let w_ζ, w_α, w_γ and w_ε be precisely those $w_s \in W(I_0)$ that give rise to the contributions M_{w_s} to M appearing in Proposition 2 that are labelled by the partitions ζ, α, γ and ε, respectively. Then, in those cases for which the corresponding characters $\mathrm{ch}\, V_0^{\overline{w_s\cdot 0}}$ are standard, without the necessity of recourse to modification rules, $w_s \cdot 0$ can be expressed in the form:*

$$\begin{aligned} w_\zeta(\rho) - \rho &= \textstyle\sum_{(i,j)\in F^\zeta} \; (\epsilon_i - \epsilon_{\ell-j+2} - \delta), \\ w_\alpha(\rho) - \rho &= \textstyle\sum_{(i,j)\in F^\alpha} \; (\epsilon_i - \frac{1}{2}\delta), \\ w_\gamma(\rho) - \rho &= \textstyle\sum_{(i,j)\in F^\gamma} \; (\epsilon_i - \frac{1}{2}\delta), \\ w_\varepsilon(\rho) - \rho &= \textstyle\sum_{(i,j)\in F^\varepsilon} \; (\epsilon_i - \delta), \end{aligned}$$

where the summations are carried out over all (i,j) specifying the position of a box in the ith row and jth column of the relevant Young diagram. Correspondingly, if $\lambda \in \mathbf{h}^*$ is dominant integral with $\lambda = \sum_{i=0}^{\ell} \lambda_i \Lambda_i$, then with the given numberings $w_s(\lambda) - \lambda$ takes the form:

$$\begin{array}{rcl} w_\zeta(\lambda) - \lambda &=& \sum_{(i,j)\in F^\zeta} \ \lambda_{\eta_{ij}} (\epsilon_i - \epsilon_{\ell-j+2} - \delta), \\ w_\alpha(\lambda) - \lambda &=& \sum_{(i,j)\in F^\alpha} \ \lambda_{\eta_{ij}} (\epsilon_i - \frac{1}{2}\delta), \\ w_\gamma(\lambda) - \lambda &=& \sum_{(i,j)\in F^\gamma} \ \lambda_{\eta_{ij}} (\epsilon_i - \frac{1}{2}\delta), \\ w_\varepsilon(\lambda) - \lambda &=& \sum_{(i,j)\in F^\varepsilon} \ \lambda_{\eta_{ij}} (\epsilon_i - \delta). \end{array}$$

The first part of this Conjecture that applies to the evaluation of $w_s \cdot 0 = w_s(\rho) - \rho$ is just a re-statement of that part of Proposition 2 that involves contributions to M that involve standard characters. The second part of the Conjecture that applies to $w_s(\lambda) - \lambda$ is based on a very large number of specific calculations together with a proof of the fact that in passing from $w'_s \in W(I_0)$ to $w_s = w'_s s_k$ any contribution to $w_s(\rho) - \rho$ is simply scaled by a factor of λ_k to give the corresponding contribution to $w_s(\lambda) - \lambda$. However, it remains to be shown that the scaling factors are precisely those specified by the given numberings. In the case of $A_\ell^{(1)}$ the validity of the numbering scheme has been proved by induction [3]. The inductive arguments appropriate to the cases of $C_\ell^{(1)}$, $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ are very similar, but not yet complete, while those of $B_\ell^{(1)}$, $D_\ell^{(1)}$ and $A_{2\ell-1}^{(2)}$ are markedly more complicated, especially that of $D_\ell^{(1)}$.

The use of these results in deriving numerator expansions is illustrated in the case of $A_\ell^{(1)}$ with $\ell = 9$ by considering the action of the coset representative $w_s = s_0 s_1 s_2 s_3 s_9 s_8 s_0 s_1 s_9$ of length $\ell(w_s) = 9$ for which $w_s(\rho) - \rho = 3\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + \epsilon_4 - 2\epsilon_7 - 3\epsilon_8 - 5\epsilon_9 - 9\delta$. This serves to define the partition ζ such that

$$w_s(\rho) - \rho = w_\zeta(\rho) - \rho = \sum_{i=1}^{\ell(\zeta)} \zeta_i \epsilon_i - \sum_{j=1}^{\ell(\zeta')} \zeta'_j \epsilon_{\ell-j+2} - |\zeta|\delta. \tag{6}$$

In fact $\zeta = \binom{2\,1}{3\,1} = (3321) \in F$. The corresponding composite Young diagram and its numbering along with its contribution to the denominator expansion are given by

$$F^{\zeta;\overline{\zeta}'} = \begin{array}{cccc|ccc} & & & & 0 & 9 & 8 \\ & & & & 1 & 0 & 9 \\ & & & & 2 & 1 & \\ & & & & 3 & & \\ & & & & & & \\ & & 9 & 8 & & & \\ & 1 & 0 & 9 & & & \\ 3 & 2 & 1 & 0 & & & \end{array} \quad \text{and} \quad M_{w_s} = -e^{-9\delta} \, \mathrm{ch}\, V_0^{\{3321;\overline{234}\}},$$

where the sign factor is just $\mathrm{sgn}\,(w_s) = (-1)^{\ell(w_s)} = (-1)^{|\zeta|} = (-1)^9$ and the power of $e^{-\delta}$ is $\ell(w_s) = |\zeta| = 9 = C_2\{\zeta;\overline{\zeta}'\}$.

To obtain the corresponding contribution $M_{w_s}^\lambda$ to the numerator expansion in the case of some dominant integral highest weight λ, first note that:

$$w_s \cdot \lambda = w_s(\lambda + \rho) - \rho = \lambda + (w_s(\rho) - \rho) + (w_s(\lambda) - \lambda). \tag{7}$$

Thus we add three contributions: one from λ itself corresponding in general to a composite Young diagram of the form $F^{\mu;\overline{\nu}}$, one obtained, as above, from the row lengths of $F^{\zeta;\overline{\zeta}'}$, and one obtained by stretching the rows of $F^{\zeta;\overline{\zeta}'}$ by factors of $\lambda_{\eta_{ij}}$.

For example, in the case $\lambda = 3\Lambda_0 + \Lambda_1 + 2\Lambda_2 + \Lambda_8 + 2\Lambda_9 = 9\Lambda_0 + 3\epsilon_1 + 2\epsilon_2 - \epsilon_9 - 3\epsilon_{10}$, for which $\overline{\lambda} = (\mu;\overline{\nu}) = (3,3;\overline{1,3})$, the complete stretched diagram, its numbering and contribution to the numerator expansion are given by:

$$\begin{array}{cccccccccccccccccccccc}
 & & & & & & & & & & & & \mu & & & 0&0&0&0&9&9&9&8&8\\
 & & & & & & & & & & & & & & 1&1&0&0&0&0&9&9&9\\
 & & & & & & & & & & & & 2&2&2&1&1\\
 & & & & & & & & & & & & 3\\
 & & & & & & & 9&9&9&8&8\\
 & & & 1&1&0&0&0&0&9&9&9\\
3&2&2&2&1&1&0&0&0&0& & \overline{\nu}
\end{array}
\quad\text{and}\quad M^{\lambda}_{w_s}=e^{9\Lambda_0}\,e^{-24\delta}\,\mathrm{ch}\,V_0^{\{12,11,5,1;\overline{5,10,13}\}}.$$

It is to be noted that the nested hook or wrapped strip structure is preserved. The power of e^{Λ_0} is $L(\lambda)$ and that of $e^{-\delta}$ is $(L(\rho)/L(\lambda+\rho))(C_2\{\sigma;\overline{\tau}\}-C_2\{\overline{\lambda}\})$ with $(\sigma;\overline{\tau})=(12,11,5,1;\overline{5,10,13})$ and $\overline{\lambda}=(\mu;\overline{\nu})=(3,2;\overline{1,3})$.

Similarly for $C_\ell^{(1)}$ with $\ell\geq 4$, the coset representative $w_s=s_0s_1s_2s_3s_0s_1$ of length $\ell(w_s)=6$ is such that $w_s(\rho)-\rho=5\epsilon_1+4\epsilon_2+2\epsilon_3+\epsilon_4-6\delta$. This serves to define $\gamma\in C$ such that

$$w_s(\rho)-\rho=w_\gamma(\rho)-\rho=\sum_{i=1}^{\ell(\gamma)}\gamma_i\epsilon_i-(|\gamma|/2)\delta. \tag{8}$$

In this case $\gamma=\binom{4\,2}{3\,1}=(5421)$. Correspondingly, the Young diagram F^γ and its numbering, along with its contribution to the denominator expansion are given by

$$F^\gamma=\begin{array}{|c|c|c|c|c|}\hline 0&0&1&2&3\\\hline 1&0&0&1\\\cline{1-4} 2&1\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad\text{and}\quad M_{w_s}=e^{-6\delta}\,\mathrm{ch}\,V_0^{\langle 5421\rangle}.$$

Once again, the sign factor is just $(-1)^{\ell(w_s)}=(-1)^{|\gamma|/2}=(-1)^6$, and the power of $e^{-\delta}$ is $\ell(w_s)=|\gamma|/2=6=C_2\langle\gamma\rangle$.

To obtain the corresponding contribution to $M^\lambda_{w_s}$, first note that once again we need to add three contributions as in 7: one from $\overline{\lambda}$, the restricted part of λ, one obtained as above from the row lengths of F^γ, and one obtained by stretching these rows by factors of $\lambda_{\eta_{ij}}$.

By way of example, if $\lambda=2\Lambda_0+3\Lambda_1+2\Lambda_3=7\Lambda_0+5\epsilon_1+2\epsilon_2+2\epsilon_3$, for which $\overline{\lambda}=(5,2,2)$, then the complete stretched diagram, its numbering and its contribution to the numerator expansion take the form:

$$\begin{array}{cccccccccccccccccccc}
\overline{\lambda} & & & & & 0&0&0&0&0&0&0&1&1&1&1&2&3&3&3\\
 & & 1&1&1&1&0&0&0&0&0&0&0&1&1&1&1\\
 & & 2&1&1&1&1\\
3&3&3
\end{array}
\quad\text{and}\quad M^{\lambda}_{w_s}=e^{7\Lambda_0}\,e^{-36\delta}\,\mathrm{ch}\,V_0^{\langle 19,16,7,3\rangle}.$$

Once again the nested hook or wrapped strip structure is preserved. The power of e^{Λ_0} is the level $L(\lambda)$ and the power of $e^{-\delta}$ is $(L(\rho)/L(\lambda+\rho))(C_2\langle\mu\rangle-C_2\langle\overline{\lambda}\rangle)$ with $\mu=(19,16,7,3)$ and $\overline{\lambda}=(5,2,2)$.

All of these results may be generalised to cases involving the use of modification rules. These arise in the numerator expansions just as they did in generalising the denominator expansions from those given in Proposition 2 to those of Proposition 3. In the numerator case the rules regarding the construction of the relevant diagrams in terms of core contributions and added strips of length L are very much as before. Now however the stretching by way of numberings also stretches L itself from $L(\rho)$ to $L(\lambda+\rho)$. In addition the numberings themselves require extension to the right in a periodic manner.

Typically, for $\ell=4$, these numberings are exemplified by:

$A_\ell^{(1)} \supset A_\ell$ with $\ell = 4$:

0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	...
1	0	4	3	2	1	0	4	3	2	1	0	4	3	2	...
2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	...
3	2	1	0	4	3	2	1	0	4	3	2	1	0	4	...
4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	...

$B_\ell^{(1)} \supset B_\ell$ with $\ell = 4$:

0	2	3	4	3	2	(0	1)	2	3	4	3	2	(0	1)	...
(0	1)	2	3	4	3	2	(1	0)	2	3	4	3	2	(1	...
2	(1	0)	2	3	4	3	2	(0	1)	2	3	4	3	2	...
3	2	(0	1)	2	3	4	3	2	(1	0)	2	3	4	3	...

$C_\ell^{(1)} \supset C_\ell$ with $\ell = 4$:

0	0	1	2	3	4	4	3	2	1	0	0	1	2	3	...
1	0	0	1	2	3	4	4	3	2	1	0	0	1	2	...
2	1	0	0	1	2	3	4	4	3	2	1	0	0	1	...
3	2	1	0	0	1	2	3	4	4	3	2	1	0	0	...

$D_\ell^{(1)} \supset D_\ell$ with $\ell = 4$:

0	2	(3	4)	2	(0	1)	2	(3	4)	2	(0	1)	2	(3	...
(0	1)	2	(4	3)	2	(1	0)	2	(4	3)	2	(1	0)	2	...
2	(1	0)	2	(3	4)	2	(0	1)	2	(3	4)	2	(0	1)	...
3)	2	(0	1)	2	(4	3)	2	(1	0)	2	(4	3)	2	(1	...

$A_{2\ell}^{(2)} \supset B_\ell$ with $\ell = 4$:

0	0	1	2	3	4	3	2	1	0	0	1	2	3	4	...
1	0	0	1	2	3	4	3	2	1	0	0	1	2	3	...
2	1	0	0	1	2	3	4	3	2	1	0	0	1	2	...
3	2	1	0	0	1	2	3	4	3	2	1	0	0	1	...

$A_{2\ell-1}^{(2)} \supset C_\ell$ with $\ell = 4$:

0	2	3	4	4	3	2	(0	1)	2	3	4	4	3	2	...
(0	1)	2	3	4	4	3	2	(1	0)	2	3	4	4	3	...
2	(1	0)	2	3	4	4	3	2	(0	1)	2	3	4	4	...
3	2	(0	1)	2	3	4	4	3	2	(1	0)	2	3	4	...

$D_{\ell+1}^{(2)} \supset B_\ell$ with $\ell = 4$:

0	1	2	3	4	3	2	1	0	1	2	3	4	3	2	...
1	0	1	2	3	4	3	2	1	0	1	2	3	4	3	...
2	1	0	1	2	3	4	3	2	1	0	1	2	3	4	...
3	2	1	0	1	2	3	4	3	2	1	0	1	2	3	...

In these numbering grids the bracketed pairs of numbers such as (0 1) indicate their interchangeability, in certain circumstances, in specifying the numbering of any particular Young diagram corresponding to some $w_s \in W(I_0)$. This specification is normally accomplished, as indicated in the earlier $A_\ell^{(1)}$ and $C_\ell^{(1)}$ examples, by superposing the relevant Young diagram associated with each term of the denominator expansion directly onto the numbering grid. The leftmost box in the top row of the diagram, whether standard or composite, must be placed over an entry 0 in the topmost line of the numbering grid. Then the entries lying in all boxes of the diagram as a result of this superposition define the amounts by which the diagrams must be stretched along each row in order to write down the corresponding contribution to the numerator expansion. Periodically extended numbering grids of the above type, but generalised to the relevant value of ℓ, accomodate all the diagrams arising in Proposition 3. However, care must be taken in correctly assigning which of the two bracketed pairs of numbers is to be exploited in the case of those diagrams which overlap just one but not both of that pair. The case of $D_\ell^{(1)}$ is further complicated by the fact that composite diagrams arise involving just a final, ℓth row extending to the left rather than the right.

It should be added that it is further conjectured that the numbering grids may also be used to obtain the coset representatives $w_s \in W(I_0)$ themselves. To date this been verified only for the case $A_\ell^{(1)}$.

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