HOMOTOPIES FOR RESOLUTIONS OF SKEW-HOOK SHAPES

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ABSTRACT. We present characteristic-free resolutions and splitting homotopies for the Weyl modules associated to skew-hook shapes.

RÉSUMÉ. Nous présentons des résolutions en caractéristique-libre, et des homotopies associées aux formes du type "skew-hook".

1. INTRODUCTION

The consideration of the Grothendieck ring in representation theory prompted A. Lascoux [L77] to translate the classical Giambelli determinantal expression of the Schur functions into a resolution, in characteristic zero, of the Schur modules of which these functions were the formal characters. The main idea in this translation is to determine in which dimension of the resolution the terms of the determinantal expansion are to go. Each term in the expansion corresponds to a permutation, and the length of that permutation gives the dimension of that term. Although in [L77] Lascoux dealt only with partitions, it was clear how the extension to skew shapes should be made. The precise definitions of the maps in these resolutions were given later and can be found in [A88, A92] and [Z87].

For example, consider the skew shape $(3,3,3,2,1)\,/\,(1,0,0)$. The Giambelli determinant gives the corresponding Schur function as

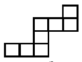
$$\begin{vmatrix} e_4 & e_6 & e_7 \\ e_2 & e_4 & e_5 \\ e_0 & e_2 & e_3 \end{vmatrix}.$$

Noting that the i^{th} elementary symmetric function e_i is the formal character of the ith exterior power Λ^i, and replacing the product of polynomials by the tensor product of the corresponding representation modules, the characteristic 0 resolution of the preceding skew shape was given as

$$0 \to \Lambda^7 \otimes \Lambda^4 \to \begin{matrix} \Lambda^6 \otimes \Lambda^5 \\ \oplus \\ \Lambda^7 \otimes \Lambda^2 \otimes \Lambda^2 \end{matrix} \to \begin{matrix} \Lambda^6 \otimes \Lambda^2 \otimes \Lambda^3 \\ \oplus \\ \Lambda^4 \otimes \Lambda^5 \otimes \Lambda^2 \end{matrix} \to \Lambda^4 \otimes \Lambda^4 \otimes \Lambda^3.$$

A completely analogous treatment can be given to the Jacobi-Trudi determinantal expansion for the Weyl module corresponding to the conjugate shape $(5,4,3)/(1,0,0)$. In this case, the entries of the matrix are the complete symmetric polynomials h_i (which are the characters of the i^{th} symmetric power in characteristic zero, but of the i^{th} divided power in arbitrary characteristic.) Thus the terms of the resolution would translate as the sums of tensor products of symmetric powers in characteristic zero, and of sums of tensor products of divided powers in a characteristic-free setting.

A number of reasons motivated the study of resolutions of Weyl modules in the characteristic-free context. From a universal algebra point of view, the modules of syzygies are of some considerable importance. But interest also was generated by questions related to $\mathbb{Z}$-forms of rational representations and to intertwining numbers (see e.g. [AB86].) ¿From the point of view of obtaining an explicit description of the syzygies, the construction of a splitting homotopy seems to be a fruitful way to proceed with these resolutions; the non-zero images of the basis elements under the homotopy usually form a basis for the syzygy modules. In [BR94] and [B98], examples of such homotopies can be found.

In this extended abstract, we construct characteristic-free resolutions of the Weyl modules associated to skew hook shapes such as . Each term of the resolution is a direct sum of tensor products of divided powers algebras. This result generalizes the characteristic-free resolutions of Weyl modules associated to hook shapes found in [B98].

Exactness is established by constructing a characteristic-free splitting homotopy; here splitting occurs over the integers. Few explicit homotopies are known, and this homotopy is new even in characteristic 0. The construction of the homotopy involves a matching between basis elements in modules at adjacent stages of the resolution. Viewed combinatorially, this matching turns out to be the sign-reversing involution appearing in the Gessel-Viennot proof [GV] of the relevant Jacobi-Trudi identity.

In Section 2 we recall some constructions from characteristic-free multilinear algebra including the Weyl (or coSchur) modules from [ABW82] and the letterplace algebras of [DRS76, GRS87]. Section 3 outlines the letterplace construction of the resolution of the nth exterior powers of a free module as described in [B98]. In Section 4, we introduce a new basis that simplifies the construction in Section 3. The construction we provide produces the resolution of a tensor product of Weyl modules having skew-hook shape as a subquotient complex (indexed by intervals in a boolean algebra) of the resolution of an exterior power representation. Finally, in Section 5 we construct the desired homotopy.

2. Constructions for characteristic-free representation theory

We review some constructions used in the sequel. The overwhelming majority of these techniques can be found either explicitly or implicitly in [ABW82, DRS76, GRS87], although the outline of the current presentation differs somewhat.

Let M be a rank m free $\mathbb{Z}$-module. The divided powers algebra $\mathcal{D}(M)$ of M is defined to be the graded Hopf algebra dual of the symmetric algebra $Sym(M^*)$ on the linear dual of M. If we fix a basis $b_1, \ldots, b_m$ for M, then $\mathcal{D}(M)$ is isomorphic to the free commutative $\mathbb{Z}$-algebra generated by the symbols $b_i^{(k)}$ for all $1 \leq i \leq m$ and all $k \geq 0$, modulo the ideal generated by the relations

$$b_i^{(j)} b_i^{(k)} = \binom{j+k}{j} b_i^{(j+k)} \qquad \text{and} \qquad b_i^{(0)} = 1\,. \tag{1}$$

The module M appears naturally as a submodule of $\mathcal{D}(M)$ and we let b_i denote $b_i^{(1)}$.

Letting each element of M have degree 1, $\mathcal{D}(M)$ is a graded algebra, its ith graded component $\mathcal{D}_i$ is a free $\mathbb{Z}$-submodule and has as basis $\{b_1^{(j_1)} \cdots b_m^{(j_m)} \mid j_1 + \cdots + j_m = i\}$.

Given two free modules, L, P^+ one can construct (see [GRS87]) a bilinear pairing $(\,|\,)$ of the divided powers algebras $\mathcal{D}(L)$ and $\mathcal{D}(P^+)$ into $\mathcal{D}(L \otimes P^+)$. We have decorated P^+ with a sign following the definition in [GRS87] which properly applies to a direct sum of free modules $P = P^+ \oplus P^-$. Elements of P^+ are said to be *positively signed* and elements of P^- are *negatively signed.* We have specialized to the case that $P^- = 0$.

In terms of bases, this pairing generalizes the permanent. In particular, suppose L and P^+ have bases $\mathcal{L}$ and $\mathcal{P}^+$ respectively. We identify the basis $\{l \otimes p \mid l \in \mathcal{L}, p \in \mathcal{P}\}$ of $L \otimes P^+$ with the set $\{(l|p) \mid l \in \mathcal{L}, p \in \mathcal{P}^+\}$ of "letterplaces." The algebra $\mathcal{D}(L \otimes P^+)$ can now be identified with the commutative associative algebra $\mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ generated by all $(l|p)$ and satisfying the relations (1) for all $b = (l|p)$. For $l_1, \ldots, l_k \in \mathcal{L}$ and $p_1, \ldots, p_k \in \mathcal{P}^+$, we have

$$(l_1 l_2 \cdots l_k | p_1 p_2 \cdots p_k) = \begin{cases} \sum_{\sigma \in S_k} (l_{\sigma(1)}|p_1) \cdots (l_{\sigma(k)}|p_k) & \text{when } k = k' \\ 0 & \text{otherwise} \end{cases}\,. \tag{2}$$

Turning to the case that $P^+ = 0$, we have $(\,|\,) : \mathcal{D}(L) \otimes \Lambda(P^-) \to \Lambda(L \otimes P^-)$. If we are given a basis $\mathcal{P}^-$ for P^-, then identifying $\Lambda(L \otimes P^-)$ with the free exterior algebra $\Lambda([\mathcal{L}|\mathcal{P}^-])$ on generators $\{(l|p) \mid l \in \mathcal{L}, p \in \mathcal{P}^-\}$, the biproduct $(\,|\,)$ still satisfies equation (2).

We note that for both $P = P^-$ and $P = P^+$, the relations (2) determine $(\,|\,)$ over $\mathbb{Z}$.

Example. Suppose we have bases $\mathcal{L} = \{a, b, c\}$, $\mathcal{P}^+ = \{\mathbf{1}, \mathbf{2}\}$, and $\mathcal{P}^- = \{1^-, 2^-\}$ for L, P^+, and P^- respectively. We have

$$(aabc|\mathbf{1222}) = 12(a|\mathbf{1})(a|\mathbf{2})(b|\mathbf{2})(c|\mathbf{2}) + 6(b|\mathbf{1})(a|\mathbf{2})(a|\mathbf{2})(c|\mathbf{2}) + 6(c|\mathbf{1})(a|\mathbf{2})(a|\mathbf{2})(b|\mathbf{2})\,,$$

so (over $\mathbb{Z}$) bilinearity implies that

$$(a^{(2)}bc|\mathbf{12}^{(3)}) = (a|\mathbf{1})(a|\mathbf{2})(b|\mathbf{2})(c|\mathbf{2}) + (b|\mathbf{1})(a|\mathbf{2})^{(2)}(c|\mathbf{2}) + (c|\mathbf{1})(a|\mathbf{2})^{(2)}(b|\mathbf{2})\,.$$

On the other hand, since $2^-2^- = 0$ in $\Lambda(P)$, we have $(aabc|1^-2^-2^-2^-) = 0$.

For the remainder of this abstract, we fix two positively signed, linearly ordered sets $\mathcal{L}$, of cardinality m, and $\mathcal{P}^+ = \{\mathbf{1} < \mathbf{2} < \cdots < \boldsymbol{n}\}$ whose elements will be the basis of free modules L and P^+ respectively. Likewise, we fix a linearly ordered, negatively signed set $\mathcal{P}^- = \{1^- < \cdots < n^-\}$ whose elements are the basis of a rank n free $\mathbb{Z}$-module P^-.

The preceding example suggests the following.

Notation. Let $\underline{u} = u_1, \ldots, u_i$ be a sequence of elements taken from some set S. For $s \in S$, define $c_s(\underline{u}) = |\{j \mid u_j = s\}|$ Further, for sequences $\underline{w}$ and $\underline{v}$ of elements in $\mathcal{L}$ and $\mathcal{P}^+$ respectively, define

$$(w_1, w_2, \ldots, w_k|v_1, v_2, \ldots, v_k) = (\prod_{l\in\mathcal{L}} l^{(c_l(\underline{w}))}| \prod_{p\in\mathcal{P}^+} p^{(c_p(\underline{v}))})\,.$$

Likewise, for sequences $\underline{w}$ and $\underline{v}$ of elements in $\mathcal{L}$ and $\mathcal{P}^-$ respectively, define

$$(w_1, w_2, \ldots, w_k|v_1, v_2, \ldots, v_k) = (\prod_{l\in\mathcal{L}} l^{(c_l(\underline{w}))}|v_1v_2\cdots v_k)\,.$$

We record the following expansion property of the biproduct.

Proposition 2.1. *Given a sequence $\underline{w}$ of elements in $\mathcal{L}$ and $v, v' \in \mathcal{D}(P^+)$, we have*

$$(\prod_{l\in\mathcal{L}} l^{(c_l(\underline{w}))}|vv') = \sum_{\underline{w_1},\underline{w_2}} (\prod_{l\in\mathcal{L}} l^{(c_l(\underline{w}_1))}|v)(\prod_{l\in\mathcal{L}} l^{(c_l(\underline{w}_2))}|v')\,,$$

where the sum is over all weakly increasing sequences $\underline{w}_1, \underline{w}_2$ such that as multisets we have $\underline{w} = \underline{w}_1 + \underline{w}_2$.

In Hopf algebras notation, one could write the preceding as $(w|vv') = \sum_{(w)}((w)_1|v)((w)_2|v')$. Indeed, this and a few closely related properties are commonly taken (see [GRS87]) as the definition of the biproduct over an arbitrary commutative ring.

Certainly the $\mathbb{Z}$-linear span of all $(w_1, \ldots, w_k|\mathbf{1}, \ldots, \mathbf{1})$ with $w_i \in \mathcal{L}$ has as basis all biproducts $(w_1, \ldots, w_k|\mathbf{1}, \ldots, \mathbf{1})$ with $w_1 \leq \cdots \leq w_k$. Further, this $\mathbb{Z}$-span is isomorphic to $\mathcal{D}(P)$ as a gl_n-representation. To make this formal over $\mathbb{Z}$, we let $U(gl_{\mathcal{L}})$ be the Kostant $\mathbb{Z}$-form of the universal enveloping algebra of the general linear group generated by all the divided powers of all $e_{a,b}$ with $a \neq b$ and by all elements $\binom{e_{a,a}}{k}$, where $a, b \in \mathcal{L}$ and $k \in \mathbb{Z}^+$. The element $e_{a,b}$ acts on $\mathcal{D}(L)$ as the letter polarization $a\frac{\partial}{\partial b}$ and on $\mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ as the polarization $\sum_{\boldsymbol{i}\in\mathcal{P}^+}(a|\boldsymbol{i})\frac{\partial}{\partial(b|\boldsymbol{i})}$. We omit the basis-free construction of this $\mathbb{Z}$-form.

More generally, one constructs $U(gl_{\mathcal{L}})$-modules as follows.

Definition 2.1. Given positive integers $0 = i_0 < i_1 < \ldots < i_k$, a sequence $w_1, \ldots, w_{i_k}$ in $\mathcal{L}$, and a sequence $v_1, \ldots, v_{i_k}$ in $\mathcal{P}^+$ (respectively $\mathcal{P}^-$), define a generalized *bitableau*

$$\left(\begin{array}{r|r} w_{i_1}, \ldots, w_1 & v_{i_1}, \ldots, v_1 \\ w_{i_2}, \ldots, w_{i_1+1} & v_{i_2}, \ldots, v_{i_1+1} \\ \cdots & \cdots \\ w_{i_k}, \ldots, w_{i_{k-1}+1} & v_{i_k}, \ldots, v_{i_{k-1}+1} \end{array}\right) = \prod_{l=1}^{k}(w_{i_l}, \ldots, w_{i_{l-1}+1} \mid v_{i_l}, \ldots, v_{i_{l-1}+1})$$

as an element of $\mathcal{D}([\mathcal{L}|\mathcal{P}])$ (respectively $\Lambda([\mathcal{L}|\mathcal{P}])$.)

In writing these bitableaux, we may choose to denote a repeated consecutive subsequence $l, \ldots, l$ of i copies of l by $l^{(i)}$.

The last shorthand for subsequences is chosen to be suggestive of identities such as $(a,b,c,d|\mathbf{4},\mathbf{3},\mathbf{1}^{(2)}) = (abcd|\mathbf{431}^{(2)})$ which latter also equals $(a,b,c,d|\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1})$.

Proposition 2.2. *If we fix positive integers $i_1 < \ldots < i_k$, and fix a sequence $v_1,\ldots,v_{i_k}$ in $\mathcal{P}^+$ (respectively $\mathcal{P}^-$) then the $\mathbb{Z}$-linear span, ranging over all sequences $w_1,\ldots,w_{i_k}$ in $\mathcal{L}$, of the bitableaux*

$$\left(\begin{array}{r|r} w_{i_1},\ldots,w_1 & v_{i_1},\ldots,v_1 \\ w_{i_2},\ldots,w_{i_1+1} & v_{i_2},\ldots,v_{i_1+1} \\ \cdots & \cdots \\ w_{i_k},\ldots,w_{i_{k-1}+1} & v_{i_k},\ldots,v_{i_{k-1}+1} \end{array}\right)$$

is a $U(gl_{\mathcal{L}})$-representation.

Note that the order of the elements in each row of the above bitableau makes no difference in the preceding definitions.

Example. If in Proposition 2.2 we let $v_{i_l} = v_{i_l-1} = \cdots = v_{i_{l-1}+1} = \boldsymbol{i_{l-1}+1}$ for all $1 \le l \le k$ where again $i_0 = 0$, then the representation constructed in that proposition is isomorphic, over $U(gl_{\mathcal{L}})$, to

$$\mathcal{D}_{i_1-i_0}(L) \otimes \cdots \otimes \mathcal{D}_{i_k-i_{k-1}}(L)\,.$$

If in Proposition 2.2 we instead had let $i_l = l$ and $v_{i_l} = 1^-$ for all $1 \le l \le k$, then the corresponding representation is isomorphic to $\Lambda^k(L)$.

Definition 2.2. If we choose the sequence $v_{i_l},\ldots,v_{i_{l-1}+1}$ to be

$$(i_l-(l-1))^-,\ldots,(i_{l-1}+1-(l-1))^-$$

then the representation given in Proposition 2.2 is by definition the *Weyl module* $K_{\lambda/\mu}$ associated to the *skew-hook* λ/μ with $\lambda_l - \mu_l = i_l - i_{l-1}$ and $\mu_l + 1 = \lambda_{l+1}$.

Notation. When $P = P^-$, it is convenient to denote a bitableau of the form given in Proposition 2.2 by $[T]$ where T is a tableau with cells filled as follows: For $i_r \ge i > i_{r-1}$, T has a cell in row r from the top and column v_i from the right; this cell contains the element w_i. Let $D(\underline{v})$ be the set of weak descents of $\underline{v}$, i.e. $D(\underline{v}) = \{1 \le i < i_k \mid v_i \ge v_{i+1}\}$. When the weak descents of $\underline{v}$ occur only in positions i_l, then $\underline{w}$ is the usual *row reading-word* of T.

For example, if in Defintion 2.2 we start with $i_0,\ldots,i_4 = 0,1,4,5,7$, then we construct the representation $K_{(4,4,2,2)/(3,1,1)}$ as the span over all sequences $w_1,\ldots,w_7$ in $\mathcal{L}$ of the bitableaux

$$\left(\begin{array}{r|r} w_1 & 1^- \\ w_4,w_3,w_2 & 3^-,2^-,1^- \\ w_5 & 3^- \\ w_7,w_6 & 4^-,3^- \end{array}\right) = \left[\begin{array}{cccc} & & & \boxed{w_1} \\ & \boxed{w_4} & \boxed{w_3} & \boxed{w_2} \\ & \boxed{w_5} & & \\ \boxed{w_7} & \boxed{w_6} & & \end{array}\right].$$

The following is a special case of the fact that the semi-standard Young tableaux of skew shape λ/μ index a $\mathbb{Z}$-basis for the Weyl module of shape λ/μ.

Proposition 2.3. *If λ/μ is a skew hook, then, following the notation of Definition 2.2, the Weyl module $K_{\lambda/\mu}$ has a $\mathbb{Z}$-basis consisting of all bitableaux*

$$\left(\begin{array}{r|r} w_{i_1},\ldots,w_1 & i_1^-,\ldots\ldots,1^- \\ \cdots & \cdots \\ w_{i_k},\ldots,w_{i_{k-1}+1} & (i_k-(k-1))^-,\ldots,(i_{k-1}+1-(k-1))^- \end{array}\right)$$

where the word $w_1,w_2,\ldots,w_{i_k}$ satisfies $w_j \ge w_{j+1}$ for $j \notin \{i_1,\ldots,i_{k-1}\}$ and $w_j < w_{j+1}$ when $j \in \{i_1,\ldots,i_{k-1}\}$. In other words, the set of strict ascents *of $w_1,w_2,\ldots,w_{i_k}$ is precisely $\{i_1,\ldots,i_{k-1}\}$.*

3. Letterplace methods for the resolution of $\Lambda^n(L)$

In this section we simplify slightly the presentation of the characteristic-free resolution of $\Lambda^n(L)$ given, via the letterplace techniques of [BR94], in [B98]. We then construct certain subquotient complexes of this resolution, corresponding to intervals in the lattice of subsets of $\{1,\dots,n-1\}$ and identify these complexes as resolutions of tensor products of Weyl modules corresponding to skew-hook shapes.

Let L be a rank m free $\mathbb{Z}$-module with basis $\mathcal{L}$. We construct a characteristic-free resolution $\mathbb{X}^n$ of $\Lambda^n(L)$ inside a certain Bar Complex [BR94] as follows.

Definition 3.1. Let A be the free algebra with divided powers on generators $\{Y_1,\dots,Y_{n-1}\}$ modulo the ideal $(Y_l^{(c)}Y_i^{(d)} - Y_i^{(d)}Y_l^{(c)} \mid \text{ if } |i-l| \geq 2)$. Let $\{x_1,\dots,x_{n-1}\}$ be a basis for another free $\mathbb{Z}$-module, and let $\Lambda = \Lambda(\{x_1,\dots,x_{n-1}\})$ be the exterior algebra generated by this module. The free product of A and Λ is known as the Bar Algebra on the algebra A and the separators $\{x_1,\dots,x_{n-1}\}$, and denoted by $\mathrm{Bar}(A;\{x_1,\dots,x_{n-1}\})$. If M is an A-module, we may consider $\mathrm{Bar}(A;\{x_1,\dots,x_{n-1}\})\otimes_A M$, and call this the free bar module of the A-module M with separators $\{x_1,\dots,x_{n-1}\}$. A boundary map is defined on $\mathrm{Bar}(A;\{x_1,\dots,x_{n-1}\})\otimes_A M$ by taking the sum of the polarizations of the x_i's to 1.

For our module, M, let us take $\mathcal{D}([\mathcal{L}|\mathcal{P}^+])$. The action of A on $\mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ is defined by letting the element Y_i act as the place polarization $D_{i+1,i} = \sum_l (l|\boldsymbol{i+1})\frac{\partial}{\partial \boldsymbol{i}}$.

Definition 3.2. In the complex $\mathrm{Bar}(A;\{x_1,\dots,x_{n-1}\})\otimes_A \mathcal{D}([\mathcal{L}|\mathcal{P}^+])$, we divide out by the subcomplex generated by the relations

$$Y_{l_1}^{(c_1)}x_{i_1}\cdots Y_l^{(d_1)}Y_{l-1}^{(d_2)}x_{l-1}\cdots Y_{l_k}^{(c_k)}x_{i_k}\otimes m \\ -\sum_\alpha Y_{l_1}^{(c_1)}x_{i_1}\cdots Y_{l-1}^{(d_2-\alpha)}x_{l-1}Y_l^{(d_1-\alpha)}\cdots Y_{l_k}^{(c_k)}x_{i_k}\otimes D_{l+1,l-1}^{(\alpha)}m$$

together with $I\otimes_A \mathcal{D}([\mathcal{L}|\mathcal{P}^+])$ where I is the ideal $(Y_l^{(c)}x_i - x_iY_l^{(c)} \mid i\neq l)$.

Define $\mathbb{X}^n$ to be the subcomplex of this quotient complex generated by all elements $b_{\{i_1,\dots,i_{k-1}\}}(\underline{w})$ for all $0=i_0<i_1<\cdots<i_{k-1}<i_k=n$ and all $w_j\in\mathcal{L}$, where $b_{\{i_1,\dots,i_{k-1}\}}(w_1,\dots,w_n)$ is defined as

$$\mathbf{Y}_{n,i_{k-1}}\mathbf{Y}_{i_{k-1},i_{k-2}}\cdots\mathbf{Y}_{i_2,i_1}\mathbf{Y}_{i_1,0}\begin{pmatrix} w_{i_1},\dots,w_1 & \mathbf{1}^{(i_1)} \\ w_{i_2},\dots,w_{i_1+1} & (\mathbf{i}_1\mathbf{+1})^{(i_2-i_1)} \\ \cdots & \cdots \\ w_{i_k},\dots,w_{i_{k-1}+1} & (\mathbf{i}_{k-1}\mathbf{+1})^{(i_k-i_{k-1})}\end{pmatrix} \tag{3}$$

with

$$\mathbf{Y}_{i,j} = Y_{i-1}x_{i-1}Y_{i-2}^{(2)}x_{i-2}\cdots x_{j+3}Y_{j+2}^{(i-j-2)}x_{j+2}Y_{j+1}^{(i-j-1)}x_{j+1}.$$

Remark. A typical generator of this complex is of the form $Y_{i_k}^{(c_k)}x_{i_k}\cdots Y_{i_1}^{(c_1)}x_{i_1}\otimes m$. The boundary of such a term is

$$(-1)^{k-1}Y_{i_k}^{(c_k)}x_{i_k}\cdots Y_{i_2}^{(c_2)}x_{i_2}\otimes Y_{i_1}^{(c_1)}m + \\ \sum(-1)^{k-\alpha}Y_{i_k}^{(c_k)}x_{i_k}\cdots Y_{i_{\alpha+1}}^{(c_\alpha+1)}x_{i_{\alpha+1}}Y_{i_\alpha}^{(c_\alpha)}Y_{i_{\alpha-1}}^{(c_{\alpha-1})}x_{i_{\alpha-1}}\cdots Y_{i_1}^{(c_1)}x_{i_1}\otimes m\,.$$

The next example shows that application of the boundary to a typical generator produces another generator. In the resolution of Λ^6, consider the term in dimension 3 of the form

$$Y_5x_5Y_4^{(2)}x_4Y_1x_1\otimes(\underline{w}_1|\mathbf{1}^{(2)})(\underline{w}_2|\mathbf{3})(\underline{w}_3|\mathbf{4}^{(3)}) = \mathbf{Y}_{6,3}\mathbf{Y}_{2,0}\otimes(\underline{w}_1|\mathbf{1}^{(2)})(\underline{w}_2|\mathbf{3})(\underline{w}_3|\mathbf{4}^{(3)}).$$

Letting $m = (\underline{w}_1|\mathbf{1}^{(2)})(\underline{w}_2|\mathbf{3})(\underline{w}_3|\mathbf{4}^{(3)})$, the boundary of this term is

$$\begin{aligned} &Y_5Y_4^{(2)}x_4Y_1x_1 \otimes m - Y_5x_5Y_4^{(2)}Y_1x_1 \otimes m + Y_5x_5Y_4^{(2)}x_4Y_1 \otimes m \\ &= Y_4^{(2)}x_4Y_1x_1Y_5 \otimes m + Y_4x_4Y_1x_1 \otimes D_{6,4}m - Y_5x_5Y_1x_1Y_4^{(2)} \otimes m - Y_5x_5Y_4^{(2)}x_4Y_1 \otimes m \end{aligned}$$

Applying the indicated place polarizations to m, gives

$$0 + \mathbf{Y}_{5,3}\mathbf{Y}_{2,0} \otimes \left(\begin{array}{c|r} \underline{w}_1 & \mathbf{1}^{(2)} \\ \underline{w}_2 & \mathbf{3} \\ \underline{w}_3 & \mathbf{64}^{(2)} \end{array} \right) - \mathbf{Y}_{6,4}\mathbf{Y}_{2,0} \otimes \left(\begin{array}{c|r} \underline{w}_1 & \mathbf{1}^{(2)} \\ \underline{w}_2 & \mathbf{3} \\ \underline{w}_3 & \mathbf{554} \end{array} \right) + \mathbf{Y}_{6,3} \otimes \left(\begin{array}{c|r} \underline{w}_1 & \mathbf{21} \\ \underline{w}_2 & \mathbf{3} \\ \underline{w}_3 & \mathbf{4}^{(3)} \end{array} \right)$$

In general if $0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = n$, then $\partial(\mathbf{Y}_{i_k,i_{k-1}} \cdots \mathbf{Y}_{i_1,i_0} \otimes m)$ equals

$$\sum_{v \in \{1,\dots,n-1\} \setminus \{i_1,\dots,i_{k-1}\}} (-1)^{r(v)} \mathbf{Y}_{i_k,i_{k-1}} \cdots \mathbf{Y}_{a_v,v} \mathbf{Y}_{v,b_v} \cdots \mathbf{Y}_{i_1,i_0} \otimes D^{(a_v - v)}_{v+1,b_v+1} m \tag{4}$$

where $b_v = \max\{i \in \{0, i_1, \dots, i_{k-1}\} \mid i < v\}$, $a_v = \min\{i \in \{i_1, \dots, i_{k-1}, n\} \mid i > v\}$, and $r(v) = |\{i > v \mid i \notin \{i_1, \dots, i_k\}\}|$.

The following result in a slightly different formulation appears in [B98]. The resolution of the exterior powers was originally described, entirely without letterplace notation, in [AB86].

Proposition 3.1. *The complex $\mathbb{X}$ is a resolution of $\Lambda^n(L)$.*

The fact that $\mathbb{X}^n$ is a complex follows from the fact that the square of any skew-derivation is 0, exactness may be proved via spectral sequences. In the sequel we provide a combinatorial proof of this result relying only on the multilinear algebra presented in the previous section.

Before constructing a characteristic-free resolution for the Weyl modules associated to the skew-hooks, we write down the decomposition of $\mathbb{X}^n$ into a direct sum of tensor products of divided powers.

Proposition 3.2. *Defining $X_{\{i_1 < \cdots < i_{k-1}\}}$ to be the $\mathbb{Z}$-linear span of all elements $b_{\{i_1,\dots,i_{k-1}\}}(w_1, \dots, w_n)$ for all sequences $w_1, \dots, w_n$ of elements in $\mathcal{L}$, gives*

$$\mathbb{X}^n = \bigoplus_{E \subseteq \{1,\dots,n-1\}} X_E$$

over $U(gl_{\mathcal{L}})$ and $X_{\{i_1 < \cdots < i_{k-1}\}} \simeq \mathcal{D}_{i_1 - 0}(L) \otimes \cdots \otimes \mathcal{D}_{n - i_{k-1}}$ over $U(gl_{\mathcal{L}})$.

The following proposition is immediate.

Proposition 3.3. *A basis for X_E is given by all $b_E(w_1, \dots, w_n)$ where $\underline{w} = w_1, \dots, w_n$ ranges over all sequences such that the strict ascents of $\underline{w}$ lie in E.*

Definition 3.3. Suppose $E_0 \subseteq E_1 \subseteq \{1, \dots, n-1\}$. We define $\mathbb{X}(E_0, E_1)$ to be the subquotient complex of $\mathbb{X}^n$ formed by taking all X_E with $E_0 \subseteq E$ in the quotient complex $\mathbb{X}/\mathbb{X}_{\not\subseteq E_1}$ of $\mathbb{X}$ by the subcomplex $\mathbb{X}_{\not\subseteq E_1}$ consisting of all X_E with $E \not\subseteq E_1$.

The image of X_E in $\mathbb{X}^n/\mathbb{X}_{\not\subseteq E_1}$ is naturally isomorphic to X_E; we continue to call it X_E.

With this definition, the complex $\mathbb{X}(\emptyset, \{1, 2, \dots, n-1\})$ is just $\mathbb{X}^n$ and the complex $\mathbb{X}(E, E)$ is just the single module X_E.

Theorem 3.4. *Let λ/μ be an n-celled shape satisfying $\lambda_{j+1} - \mu_j \in \{0, 1\}$, i.e. $K_{\lambda/\mu}$ is a tensor product of Weyl modules associated to skew shapes. Let $E_1 = \{\sum_{l=1}^{j} \lambda_l - \mu_l \mid j \in \mathbb{Z}^+\} \setminus \{n\}$, and let $E_0 = \{\sum_{l=1}^{j} \lambda_l - \mu_l \mid j \in \mathbb{Z}^+,\ \lambda_{j+1} - \mu_j = 0\} \setminus \{n\}$. The complex $\mathbb{X}(E_0, E_1)$ is exact except at X_{E_1} where its homology is $K_{\lambda/\mu}$.*

This generalizes the resolution [B98] for the hook shapes. We defer the proof to Section 5.

Assuming Theorem 3.4, we can interpret the resolution $\mathbb{X}(E_0, E_1)$ as a mapping cone.

Theorem 3.5. *Given $E_0 \subset E_1 \subseteq \{1, \ldots, n-1\}$ and $v \in E_1 \backslash E_0$, we have a short exact sequence of complexes $0 \to \mathbb{X}(E_0 \cup \{v\}, E_1) \to \mathbb{X}(E_0, E_1) \to \mathbb{X}(E_0, E_1 \backslash \{v\}) \to 0$. Further, $\mathbb{X}(E_0, E_1)$ is the mapping cone of a map $\mathbb{X}(E_0, E_1 \backslash v) \to \mathbb{X}(E_0 \cup v, E_1)$.*

Recognizing the injection in the above theorem as subset inclusion proves exactness. One may verify that $\mathbb{X}(E_0, E_1)$ is the claimed mapping cone by checking that the injection $K_{\lambda/\mu} \to K_{\kappa/\nu}$, from the homology of $\mathbb{X}(E_0, E_1 \backslash \{v\})$ at $X_{E_1 \backslash \{v\}}$ to the homology of $\mathbb{X}(E_0 \cup \{v\}, E_1)$ at E_1, lifts appropriately.

4. Bases

The preceding constructions become more tractable if we introduce a new basis for the modules X_E. This basis is a translation to Weyl modules of a special case of the "compressed" bases defined in [T00]. As before, an element of the new basis is a bitableau. Recall that the bitableau (3) defining the basis element $b_E(\underline{w})$ had $|E|+1$ rows and $\underline{w}$ was restricted to have at most $|E|$ ascents. An element $\bar{b}_E(\underline{w})$ of the new basis will still have at most $|E|$ ascents, but the tableau used to define it will have precisely as many rows as there are ascents in $\underline{w}$.

Definition 4.1. Let $E = \{i_1 < \cdots < i_{k-1}\} \subseteq \{1, \ldots, n-1\}$, let $i_k = n$, and let $\underline{w} = w_1, \ldots, w_n$ be a sequence in $\mathcal{L}$. Let $\{i_{a_1} < \cdots < i_{a_l}\}$ be the set $\{1 \leq a \leq n-1 \mid w_a < w_{a+1}\} \cup \{n\}$. Define the element $\bar{b}_E(w_1, \ldots, w_n)$ to be

$$\boldsymbol{Y}_{i_k, i_{k-1}} \cdots \boldsymbol{Y}_{i_1, 0} \left(\begin{array}{r|r} w_{i_{a_1}}, \ldots, w_1 & (\boldsymbol{i_{a_1-1}+1})^{(i_{a_1}-i_{a_1-1})} \cdots (\boldsymbol{i_1+1})^{(i_2-i_1)} \mathbf{1}^{(i_1)} \\ w_{i_{a_2}}, \ldots, w_{i_{a_1}+1} & (\boldsymbol{i_{a_2-1}+1})^{(i_{a_2}-i_{a_2-1})} \cdots (\boldsymbol{i_{a_1}+1})^{(i_{a_1+1}-i_{a_1})} \\ \cdots & \cdots \\ w_{i_{a_l}}, \ldots, w_{i_{a_{l-1}}+1} & (\boldsymbol{i_{a_l-1}+1})^{(i_{a_l}-i_{a_l-1})} \cdots (\boldsymbol{i_{a_{l-1}}+1})^{(i_{a_{l-1}+1}-i_{a_{l-1}})} \end{array} \right) \quad (5)$$

Proposition 4.1. *The module X_E has a basis consisting of all $\bar{b}_E(w_1, \ldots, w_n)$ where $w_1, \ldots, w_n$ ranges over all sequences $\underline{w}$ whose ascents lie in E.*

Since the the two bases have the same cardinality, the proof may be accomplished by first applying Proposition 2.1, to expand the elements $\bar{b}_E(w_1, \ldots, w_n)$ into the usual basis for X_E, and then applying a rewrite of the same identity to expand the generators of X_E into the basis $\bar{b}_E(w_1, \ldots, w_n)$.

Notation. As with the bitableaux forming the basis for a Weyl module $K_{\lambda/\mu}$, it will be convenient to denote the basis elements for X_E using (appropriately decorated) Young tableaux. We denote the bitableaux appearing in Equation (5) by $[\![T]\!]$ where T is defined in two steps as follows. First, we start with T being the Young tableaux of shape λ/μ where $\lambda_{l+1} = \mu_l$, $\lambda_l - \mu_l = i_{a_l} - i_{a_{l-1}}$, and whose row reading-word is $\underline{w}$. We do not draw a box around each letter. Second, we decorate T by inserting a vertical bar between the elements in positions i_j and $i_j + 1$ for all j such that $i_j \notin \{i_{a_1}, \ldots, i_{a_l}\}$.

For example, we have

$$\left(\begin{array}{r|r} w_1 & \mathbf{1} \\ w_4, w_3, w_2 & \mathbf{3}^{(2)}, \mathbf{2} \\ w_6, w_5 & \mathbf{6}, \mathbf{5} \\ w_8, w_7 & \mathbf{7}^{(2)} \end{array} \right) = \left[\!\!\left[\begin{array}{llllllll} & & & & & & & w_1 \\ & & & & w_4 & w_3 & \!\!\mid\! w_2 & \\ & & w_6 & \!\!\mid\! w_5 & & & & \\ w_8 & w_7 & & & & & & \end{array} \right]\!\!\right]$$

Alternate construction of $\mathbb{X}^n$. Alternatively, we could define $\mathbb{X}^n$ to be the linear span of

$$\left(\begin{array}{r|r} w_{i_1}, \ldots, w_1 & \mathbf{1}^{(i_1)} \\ \cdots & \cdots \\ w_{i_k}, \ldots, w_{i_{k-1}+1} & (\boldsymbol{i_{k-1}+1})^{(i_k-i_{k-1})} \end{array} \right) \quad (6)$$

in $\mathcal{D}([\mathcal{L}|\mathcal{P}])$ for all $w_j \in \mathcal{L}$ and all $0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = n$. We let $U(gl_{\mathcal{L}})$ act on $\mathbb{X}^n$ via its action on $\mathcal{D}([\mathcal{L}|\mathcal{P}])$. We define $\bar{b}_E(\underline{w})$ precisely as in Definition 4.1, simply removing the factor of $\boldsymbol{Y}_{i_k, i_{k-1}} \cdots \boldsymbol{Y}_{i_1, 1}$ appearing in the expression (5).

Let $r_{E,v} = |\{j \mid n > j > v \text{ and } j \notin E\}|$. Taking the relation

$$\partial \bar{b}_E(\underline{w}) = \sum_{v \in \{1,\ldots,n-1\} \setminus E} (-1)^{r_{E,v}} \bar{b}_{E \cup \{v\}}(\underline{w})$$

computed by equation (4) as the definition of the boundary completes the construction of $\mathbb{X}^n$. The fact that ∂ is a map of $U(gl_{\mathcal{L}})$-modules follows from the fact that the polarizations of $\mathcal{L}$ commute with the polarizations of $\mathcal{P}$.

Returning to the example following Definition 3.2, we have

$$\partial \left\| \left[_{a\,b\,c|c}{}^{b\,b}\right] \right\| = \left\| \left[_{a|b\,c|c}{}^{b\,b}\right] \right\| - \left\| \left[_{a\,b|c|c}{}^{b\,b}\right] \right\| + \left\| \left[_{a\,b\,c|c}{}^{b|b}\right] \right\| .$$

We can directly check that $\mathbb{X}^n$ is a complex by showing that for any $E \subseteq \{1,\ldots,n-1\}$, the image under ∂^2 of any basis element $\bar{b}_{E'}(\underline{w})$ is 0 in X_E. We may assume $E = E' \uplus \{v_1\} \uplus \{v_2\}$, for some $v_1, v_2 \in \{1,\ldots,n-1\}$, since otherwise the component of $\partial^2(\bar{b}_{E'}(\underline{w}))$ in X_E is automatically 0. Now we compute that the component of $\partial^2(\bar{b}_{E'}(\underline{w}))$ in X_E is the same as the component in X_E of $\partial\left((-1)^{r_{E',v_1}} \bar{b}_{E' \uplus \{v_1\}}(\underline{w}) + (-1)^{r_{E',v_2}} \bar{b}_{E' \uplus \{v_2\}}(\underline{w})\right)$, and this component is the same as the component in E of

$$((-1)^{r_{E' \uplus \{v_1\},v_2}} (-1)^{r_{E',v_1}} + (-1)^{r_{E' \uplus \{v_2\},v_1}} (-1)^{r_{E',v_2}})\, \bar{b}_{E' \uplus \{v_1,v_2\}}(\underline{w}) .$$

Assuming in the preceding notation that $v_1 \geq v_2$, we have $r_{E' \uplus \{v_1\},v_2} = r_{E',v_2} - 1$ and $r_{E' \uplus \{v_2\},v_1} = r_{E',v_1}$. So since $r_{E' \uplus \{v_1\},v_2} + r_{E',v_1} = r_{E',v_2} - 1 + r_{E' \uplus \{v_2\},v_1}$, the coefficient of $\bar{b}_{E' \uplus \{v_1,v_2\}}(\underline{w})$ above vanishes and the component of $\partial^2(\bar{b}_{E'}(\underline{w}))$ in X_E is 0.

5. The homotopy

We produce a splitting homotopy S for the resolutions $\mathbb{X}(E_0, E_1)$ of the skew hooks. This proves the exactness of these resolutions.

Definition 5.1. Given $E_0 \subseteq E_1 \subseteq \{1,\ldots,n-1\}$ we will define a $\mathbb{Z}$-linear map $S : \mathbb{X}(E_0, E_1) \to \mathbb{X}(E_0, E_1)$ by giving its action on the basis elements $\bar{b}_E(\underline{w})$ for all $E_0 \subseteq E \subseteq E_1$.

Given a basis element $\bar{b}_E(\underline{w})$, let D be the set of weak descents in $\underline{w}$, specifically $D = \{1 \leq i \leq n-1 \mid w_i \geq w_{i+1}\}$. If $(D \cap E_1) \setminus E_0 \neq \emptyset$, then let v be the minimal element of $(D \cap E_1) \setminus E_0$. We define S by

$$S(\bar{b}_E(\underline{w})) = \begin{cases} (-1)^{r_{E \setminus \{v\},v}} b_{E \setminus \{v\}}(\underline{w}) & \text{if } v \text{ exists and } v \in E \\ 0 & \text{otherwise} \end{cases} .$$

By way of example, we display the action of the homotopy (on basis elements) for the resolution $\mathbb{X}(\emptyset, \{1,3\})$ of the Weyl module $K_{(3,3,2)/(2,1,0)}$:

$$\llbracket w_5\, w_4 | w_3\, w_2 | w_1 \rrbracket \mapsto \llbracket w_5\, w_4 | w_3\, w_2\, w_1 \rrbracket \mapsto 0$$

$$\llbracket w_5\, w_4\, w_3\, w_2 | w_1 \rrbracket \mapsto -\llbracket w_5\, w_4\, w_3\, w_2\, w_1 \rrbracket \mapsto 0$$

$$\left\| \left[_{w_5\, w_4}{}^{w_3\, w_2 | w_1}\right] \right\| \mapsto \left\| \left[_{w_5\, w_4}{}^{w_3\, w_2\, w_1}\right] \right\| \mapsto 0$$

$$\left\| \left[_{w_5\, w_4 | w_3\, w_2}{}^{w_1}\right] \right\| \mapsto -\left\| \left[_{w_5\, w_4\, w_3\, w_2}{}^{w_1}\right] \right\| \mapsto 0 \qquad \left\| \left[_{w_5\, w_4}{}^{w_3\, w_2}{}^{w_1}\right] \right\| \mapsto 0$$

Necessarily, we assume that in each decorated tableau above, there is a weak descent $w_i \geq w_{i+1}$ iff we have written w_i and w_{i+1} in the same row.

Theorem 5.1. *Given $E_0 \subseteq E \subset E_1 \subseteq \{1,\ldots,n-1\}$ and $x \in X_E$, we have $S\partial(x) + \partial S(x) = x$.*

Proof. It suffices to prove the proposition when $x = \bar{b}_E(\underline{w})$ for any $\underline{w}$ with strict ascent set $\bar{D} \subseteq E$.

Case 1: Suppose v exists and $v \notin E$, so $S(\bar{b}_E(\underline{w})) = 0$. We compute

$$\partial(\bar{b}_E(\underline{w})) = \sum_{u \in E_1 \setminus E} (-1)^{r_{E,u}} \bar{b}_{E \cup \{u\}}(\underline{w}) \tag{7}$$

and observe that since the v appearing in the definition of $S(\bar{b}_F(\underline{w}))$ depends only on $\underline{w}$ and not on F, S must vanish on every term of the right-hand side except $(-1)^{r_{E,v}}\bar{b}_{E\cup\{v\}}(\underline{w})$.

Case 2: Suppose v exists and $v \in E$. Then $S(\bar{b}_E(\underline{w})) = (-1)^{r_{E\backslash\{v\},v}}\bar{b}_{E\backslash\{v\}}(\underline{w})$. In this case, the homotopy is nonzero on all terms on the right-hand side of equation (7) and we find

$$S\partial(\bar{b}_E(\underline{w})) = \sum_{u\in E_1\backslash E} (-1)^{r_{E,u}}(-1)^{r_{(E\cup\{u\})\backslash\{v\},v}}\bar{b}_{(E\cup\{u\})\backslash\{v\}}(\underline{w})\,. \tag{8}$$

On the other hand, we calculate $S(\bar{b}_E(\underline{w})) = (-1)^{r_{E\backslash\{v\},v}}b_{E\backslash\{v\}}(\underline{w})$, so

$$\partial S(\bar{b}_E(\underline{w})) = \sum_{u\in(E_1\backslash E)\cup\{v\}} (-1)^{r_{E\backslash\{v\},v}}(-1)^{r_{E\backslash\{v\},u}}b_{(E\backslash\{v\})\cup\{u\}}(\underline{w})\,. \tag{9}$$

The basis element $b_E(\underline{w})$ appears in Equation (9) with coefficient 1 when $u = v$ and is absent from Equation (8) The coefficient of any other basis element $b_{(E\backslash\{v\})\cup\{u\}}$ that could appear in $S\partial(b_E(\underline{w})) + \partial S(b_E(\underline{w}))$ vanishes for $u \neq v$. Letting $F = E\backslash\{v\}$, this amounts to applying the definition of $r_{F,u}$ to check that

$$(-1)^{r_{F,v}}(-1)^{r_{F,u}} + (-1)^{r_{F\cup\{v\},u}}(-1)^{r_{F\cup\{u\},v}} = 0\,.$$

Case 3: Finally, suppose $D\cap E_1 \subseteq E_0$, or equivalently $E_1 \subseteq E_0 \cup \bar{D}$. Then we have $S(x) = 0$. But $E_1 \supseteq E \supseteq E_0$ and $E \supseteq \bar{D}$, so $E = E_1$, where we are permitted homology.

□

Proof of Theorem 3.4. It suffices to prove that homology of $\mathbb{X}(E_0, E_1)$ is as claimed. We do this by augmenting the boundary with a map $\varepsilon : X_{E_1} \to K_{\lambda/\mu}$, defining S on $K_{\lambda/\mu}$ and proving that $S\varepsilon(x) + \partial S(x) = x$ for all $x \in X_{E_1}$.

We define ε by $\varepsilon(b_{E_1}(\underline{w})) = [T]$ where T is the tableau of shape λ/μ with row-reading word $\underline{w}$. This map is well-known to be a homomorphism of $U(gl_{\mathcal{L}})$-modules. One can verify this using superalgebra techniques by constructing ε as a product of negatively signed place polarizations. Informally, we can describe (up to sign) the action of ε as replacing each subsequence $(\boldsymbol{i_j+1})^{(i_{j+1}-i_j)}$ on the right-hand side of the bitableaux in either Expression (5) or (6) with the sequence $\lambda_j^-, \cdots, (\mu_j+2)^-, (\mu_j+1)^-$.

Let $\bar{b}_{E_1}(\underline{w})$ be a basis element of X_{E_1}, let $\bar{D} \subseteq E_1$ be the set of strict ascents of $\underline{w}$, and D its set of weak descents. Following Case (2) of the preceding proof, if there exists a minimal element v of $(D\cap E_1)\backslash E_0$ and if $\varepsilon(\bar{b}_{E_1}(\underline{w})) = 0$, then $S\varepsilon(\bar{b}_{E_1}(\underline{w})) + \partial S(\bar{b}_{E_1}(\underline{w})) = \bar{b}_{E_1}(\underline{w})$. But the existence of v guarantees that we can write $\bar{b}_{E_1}(\underline{w})$ in the following form

$$\bar{b}_{E_1}(\underline{w}) = \left(\begin{array}{c|c} \cdots & \cdots \\ \cdots & \cdots(\boldsymbol{v+1})^{(b)}(\boldsymbol{v-a+1})^{(a)}\cdots \\ \cdots & \cdots \end{array}\right),$$

for some $a, b \geq 1$. Thus, choosing m such that $v = i_m$, $\varepsilon\bar{b}_{E_1}(\underline{w})$ has two copies of λ_{m+1}^- in the displayed row. Since $\lambda_{m+1}^-\lambda_{m+1}^- = 0$, we indeed have $\varepsilon(\bar{b}_{E_1}(\underline{w})) = 0$.

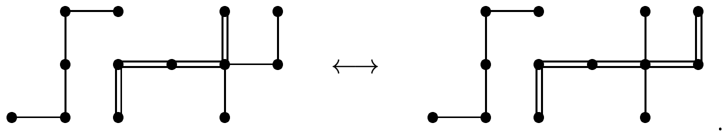
Suppose however that $(D\cap E_1)\backslash E_0$ is empty. Then we are in Case 3 of the preceding proof and so $E_1\backslash E_0 \subseteq \bar{D} \subseteq E_1$. But this is precisely the condition required to make $\varepsilon(\bar{b}_{E_1}(\underline{w}))$ the basis element $[T]$ corresponding to the semi-standard Young tableau T of shape λ/μ with row-reading word $\underline{w}$. We complete the proof by defining $S([T]) = \bar{b}_{E_1}(\underline{w})$. □

We remark that the above homotopy establishes a complete (directed) matching among the basis elements in $\mathbb{X}(E_0, E_1)$ that do not appear in $S\varepsilon(E_1)$. Since matched elements appear in adjacent stages of the resolution, their weights cancel when we compute the formal character of the homology at X_{E_1} as an alternating sum of the formal characters of the stages of the resolution. In fact, on the level of formal characters, our homotopy is the Gessel-Viennot sign reversing involution [GV]. To see this, consider the following bijection between basis elements and collections of lattice paths. Define the height of the sth horizontal step (reading from the right) of the jth lattice path from the right to be $w_{i_{j-1}+s}$.

In our previous example, the homotopy acted (in part) by

$$\left\|\left[\begin{array}{cccccc} & & \mathbf{2} & \mathbf{2} & \big| & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & & & & \end{array}\right]\right\| \longmapsto \left\|\left[\begin{array}{ccccc} & & \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & & & \end{array}\right]\right\|$$

which corresponds to the sign-reversing involution



6. Extensions

The preceding results hold if $\mathbb{Z}$ is replaced by any commutative ring $\mathfrak{r}$ with unity via minor modifications of the above constructions and proofs. Additionally, the resolution and homotopy presented above extend to the supersymmetric generalization of Schur and Weyl modules.

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