PERMUTATION SHAPES AND THE BIN PACKING PROBLEM

MARGUERITE A. EISENSTEIN-TAYLOR

ABSTRACT. We define a partition of the unit hypercube into polytopes. These polytopes correspond to arrangements of a sequence of objects into bins in the NEXT FIT solution to the one-dimensional bin packing problem. The probability that an arrangement appears is given by the volume of the corresponding polytope in the partition. We show that this volume can be calculated as the solution to a problem of permutation enumeration and apply this technique to analyze the behavior of the NEXT FIT algorithm.

RÉSUMÉ. Nous décrivons une partition du hypercube de l'unité en polytopes. Ces polytopes correspondent aux arrangements d'une séquence d'objets en huches selon la solution de l'algorithme "NEXT FIT" au problème 1-dimensionel de l'emballage des huches. La probabilité qu'un arrangement apparaît est le volume du polytope correspondant dans la partition. Nous montrons qu'on peut calculer ce volume comme solution d'un problème de l'énumeration des permutations, et appliquons cette technique afin d'analyser le comportement de l'algorithme "NEXT FIT".

1. INTRODUCTION

We consider the one-dimensional bin packing problem, in which objects of varying sizes are placed in equal capacity bins so that the sum of the sizes of the objects in a bin does not exceed the bin's capacity. Equivalently, if the objects correspond to a set of items $\{1,\ldots,m\}$, where i has size x_i, and k is the capacity of a bin, then for any bin B, $\sum_{i\in B} x_i \le k$.

We are concerned with the bin arrangements derived from the greedy "NEXT FIT" packing algorithm. Under this algorithm, objects are placed in order into one bin until the next object will not fit. That bin is never revisited, and the next object starts a new bin. In particular, suppose bins have capacity k and NEXT FIT has already placed objects 1 through $i-1$, with i starting a new bin. Then NEXT FIT places objects i through i' in a single bin precisely when we have the inequalities

$$\sum_{j=i}^{i'} x_j \le k \qquad \text{and} \qquad \sum_{j=i}^{i'+1} x_j > k \ . \tag{1}$$

We take the capacity of a bin to be $k=1$. We consider sequences of objects with sizes chosen uniformly at random from the interval $(0,1]$.

In this abstract, we consider each sequence of objects as a point $(x_1,\ldots,x_m)$ in the unit m-hypercube (minus some boundary) $(0,1]^m$. For each distinct arrangement A of objects into bins, we form the polytope $P_A \subset (0,1]^m$ described by the inequalities in Equation (1). Then the probability that m objects will yield the bin arrangement A is precisely the volume of P_A.

Similarly, given any permutation $\sigma \in S_m$, we can construct a polytope $P_\sigma \subset [0,1]^m$. We take the inequalities $y_{\sigma(i)} < y_{\sigma(i+1)}$ for $\sigma(i) < \sigma(i+1)$ and $y_{\sigma(i)} > y_{\sigma(i+1)}$ for $\sigma(i) > \sigma(i+1)$. We then introduce a piecewise linear, volume preserving map ψ, which takes each P_A bijectively to an appropriate P_σ. Thus we can calculate the probability of a particular bin arrangement occuring by counting permutations according to their patterns of runs.

The polytopes P_σ derived from permutations are special cases of the order polytopes Stanley discusses in [12]. The polytopes P_A are similar to but distinct from the chain polytopes in [12].

As an example, we recover via permutation enumeration results obtained probabilistically by Hofri ([7], [8]) giving the generating function, expectation, and variance for the number of bins used by NEXT FIT.

We conclude with some remarks on generalizations and similar problems.

2. Definitions and Preliminaries

Definition 1. A (NEXT FIT) bin arrangement of m objects into n bins is a composition of m into n parts.

Note that if the ith part of the composition is equal to j, then the ith bin will contain j objects.

Definition 2. A polytope is a bounded finite intersection of closed and open half spaces.

For the purposes of this abstract, we do not insist that polytopes be closed.

Definition 3. Let A be the bin arrangement of m elements into n bins, where the lth bin contains objects $i_l, i_l+1, \ldots, i_{l+1}-1$. Define a polytope P_A by the inequalities

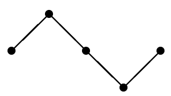
$$\sum_{j=i_l}^{i_{l+1}-1} x_j \leq 1 \quad \text{and} \quad \sum_{j=i_l}^{i_{l+1}} x_j > 1$$

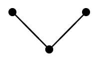
together with the inequalities $0 < x_j \leq 1$.

The following is immediate.

Proposition 1. *If* NEXT FIT *places a sequence of m objects in a bin arrangement A, then the m-tuple formed by its weights $(x_1, \ldots, x_m)$ is a point in the polytope P_A. Thus for any m,* NEXT FIT *partitions $(0,1]^m$ into polytopes corresponding to the different bin arrangements. The volume of each polytope is precisely the probability that a random sequence of objects will yield the corresponding bin arrangement under* NEXT FIT.

Definition 4. We define a *shape* to be a word in the letters a and d. We draw a shape by representing a as a northeast step and d as a southeast step. For instance, the shapes $adda$ and da are drawn as follows.

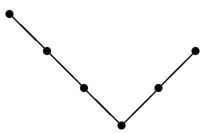


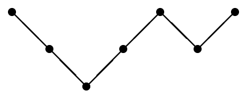


We associate a shape to each permutation in the obvious way.

Definition 5. Define a function Sh which takes permutations of m integers to shapes of length $m-1$. If $\sigma \in S_m$ has a descent in the ith position, then the ith letter of $\mathrm{Sh}(\sigma)$ is d. If σ has an ascent in the ith position, then the ith letter of $\mathrm{Sh}(\sigma)$ is a.

For example, if σ_1 is 632145 and σ_2 is 4213756, then $\mathrm{Sh}(\sigma_1)$ and $\mathrm{Sh}(\sigma_2)$ are $dddaa$ and $ddaada$:





The function Sh is certainly not injective: Sh also takes 654123 to the first shape. However, it is easy to show inductively that Sh is surjective.

We will write shapes in the form $d^{j_1}a^{j_2}d^{j_3}\cdots a^{j_{n-1}}d^{j_n}$ (for n odd) or $d^{j_1}a^{j_2}d^{j_3}\cdots a^{j_{n-2}}d^{j_{n-1}}a^{j_n}$ (for n even). We permit j_1 to be any non-negative integer, and require $j_2, \ldots, j_n$ to be positive integers. For example, $adda$ is $d^0a^1d^2a^1$, $\mathrm{Sh}(\sigma_1)$ is d^3a^2, and $\mathrm{Sh}(\sigma_2)$ is $d^2a^2d^1a^1$. This sequence $(j_1, \ldots, j_n)$ uniquely determines the shape.

Definition 6. Let s be a shape with $m-1$ letters. Define a polytope $P_s \subset [0,1]^m$ by the following inequalities:

(1) $y_i > y_{i+1} \geq 0$ if the ith letter of s is d
(2) $y_i < y_{i+1} \leq 1$ if the ith letter of s is a
(3) $0 \leq y_1 < 1$

Remark 1. The disjoint union of the polytopes P_A is all of $(0,1]^m$. However, the disjoint union of the polytopes P_s is certainly not all of $[0,1]^m$. The hyperplanes of the form $y_i = y_{i+1}$ are missing, as is some of the boundary. Nonetheless, to calculate the volume of P_s for a particular shape s, it is irrelevant which inequalities forming the facets are strict and which are weak. We merely need to count the number of permutations $\sigma \in S_m$ such that $\mathrm{Sh}(\sigma) = s$, then divide by $m!$.

3. Bin Arrangements and Permutation Shapes

We define a function ψ that will relate bin arrangements and shapes.

Definition 7. Let A be a bin arrangement with m elements. Define a function ψ_A from P_A to $[0,1]^m$ with $\psi_A(x_1,\ldots,x_m) = (y_1,\ldots,y_m)$ as follows:

$$y_j = \begin{cases} 1 - \sum_{i=b}^{j} x_i, & \text{if } j \text{ is in an odd-numbered bin;} \\ \sum_{i=b}^{j} x_i, & \text{if } j \text{ is in an even-numbered bin;} \end{cases}$$

where b is the first element in the bin that contains j. For odd bins, y_i is just the remaining space left in the bin, and for even bins, it is just the total amount in the bin so far.

Proposition 2. *Given the bin arrangement A, ψ_A is an affine linear, volume preserving map.*

Proof. We show that ψ_A can be represented by an $(m+1)\times(m+1)$ unitriangular matrix N_A if we introduce dummy variables $x_0 = y_0 = 1$. Then N_A is invertible with determinant ± 1.

The first row of N_A is $[1,0,\ldots,0]$. Now consider the $j+1$st row of N_A. Suppose b is the first element in the bin containing j. If this is an odd-numbered bin, then $y_j = 1 - \sum_{i=b}^{j} x_i$ and hence the $j+1$st row of N_A will start with 1, then contain -1 in the $(b+1)$st through $(j+1)$st positions, with zero everywhere else. If this is an even-numbered bin, then $y_j = \sum_{i=b}^{j} x_i$ and hence the $j+1$st row of N_A will contain 1's in the $(b+1)$st through $(j+1)$st positions, with zero everywhere else. So N_A is lower triangular with ± 1 on the main diagonal. □

Definition 8. Given $m > 0$, we will define a piecewise affine linear, volume preserving map ψ on $(0,1]^m$ into $[0,1]^m$. Divide $(0,1]^m$ into polytopes P_A for each bin arrangement A of m elements. We define ψ by setting $\psi(x_1,\ldots,x_m) = \psi_A(x_1,\ldots,x_m)$ for $(x_1,\ldots,x_m) \in P_A$.

Proposition 3. *Given the bin arrangement A, the points in $\psi(P_A)$ satisfy the following inequalities.*

(1) $y_j > y_{j+1} \geq 0$ *if $j+1$ is in an odd-numbered bin*
(2) $y_j < y_{j+1} \leq 1$ *if $j+1$ is in an even bin*
(3) $0 \leq y_1 < 1$

Proof. Suppose $j+1$ is in an odd bin. Then y_{j+1} measures the space left in the bin, so $y_{j+1} \geq 0$. If j is in the same bin, then there is less space left after adding in the size of $j+1$, so $y_j > y_{j+1}$. If j was the last element in the previous (necessarily even-numbered) bin, y_j is the total amount in that bin and $1 - y_j$ the space left. Since $j+1$ is in the next bin, it clearly did not fit. So $1 - y_j < x_{j+1}$, which rewrites to $y_j > 1 - x_{j+1} = y_{j+1}$.

Suppose $j+1$ is in an even bin. Then similarly to before, y_{j+1} is the total in the bin after including $j+1$, so $y_{j+1} \leq 1$. If j is in the same bin, there is of course more in the bin after including $j+1$, so $y_j < y_{j+1}$. If j is in the previous bin, y_j measures the space left in that bin. But this is less than $x_{j+1} = y_{j+1}$, since $j+1$ did not fit. Hence $y_j < y_{j+1}$. □

We will present a combinatorial interpretation for bin arrangements using the following construction of marking shapes. For each adjacent pair ad and da, we mark the second letter with a dot: $a\dot{d}$ and $d\dot{a}$. If the first letter in a shape is an a, we also give it a dot. For example, we will write $d^0a^1d^1a^3d^2a^1d^2$ as $\dot{a}\dot{d}\dot{a}aa\dot{d}d\dot{a}\dot{d}d$ and $d^3a^4d^1a^2d^2a$ as $ddd\dot{a}aaa\dot{d}\dot{a}a\dot{d}d\dot{a}$. Note this marking is merely a bookkeeping device and adds no additional information to the shape.

Definition 9. Let s be a (marked) shape containing $m-1$ letters. We define a function, $\psi'_s : P_s \to \mathbf{R}^m$ with $\psi'_s(y_1,\ldots,y_m) = (x_1,\ldots,x_n)$ as follows:

$$x_i = \begin{cases} 1-y_i, & \text{if the } (i-1)\text{st letter of } s \text{ is } \dot{d}, \text{ or if } i=1; \\ y_{i-1}-y_i, & \text{if the } (i-1)\text{st letter of } s \text{ is } d; \\ y_i, & \text{if the } (i-1)\text{st letter of } s \text{ is } \dot{a}; \\ y_i - y_{i-1}, & \text{if the } (i-1)\text{st letter of } s \text{ is } a. \end{cases}$$

Theorem 4. *Let A be the bin arrangement of m elements whose n bins contain $j_1, j_2, \ldots, j_n$ elements respectively. If n is odd, let s be the shape $d^{j_1-1}a^{j_2}\cdots d^{j_n}$. If n is even, let s be the shape $d^{j_1-1}a^{j_2}d^{j_3}\cdots a^{j_n}$. Then ψ bijectively takes P_A to P_s, and $\psi_A^{-1} = \psi'_s$.*

Proof. First, the inequalities described in Proposition 3 are precisely those that define P_s. Hence $\psi(P_A) \subseteq P_s$.

Now we show the inclusion $\psi'_s(P_s) \subseteq P_A$. Write $j'_k = j_1 + j_2 + \cdots + j_k$. We take j'_0 to be 0. Consider the sums

$$\sum_{i=j'_{k-1}+1}^{j'_k} x_i \qquad \text{and} \qquad \sum_{i=j'_{k-1}+1}^{j'_k+1} x_i \ .$$

For k odd (in other words, j_k is the exponent of a d), the first sum becomes

$$(1 - y_{j'_{k-1}+1}) + (y_{j'_{k-1}+1} - y_{j'_{k-1}+2}) \cdots + (y_{j'_k-1} - y_{j'_k}) = 1 - y_{j'_k} \ .$$

Since $0 \le y_{j'_k} < 1$, we get that $\sum_{i=j'_{k-1}+1}^{j'_k} x_i \le 1$.

Since k is odd, we know the j'_kth letter of S is an $\dot{a}$ and $y_{j'_k} < y_{j'_k+1}$. Hence we know that $x_{j'_k+1} = y_{j'_k+1}$, and the second sum is $1 - y_{j'_k} + y_{j'_k+1} > 1$. So $\sum_{i=j'_{k-1}+1}^{j'_k+1} x_i > 1$.

The calculations for the case where k is even (j_k is the exponent of an a) are similar, and once again yield the inequalities

$$\sum_{i=j'_{k-1}+1}^{j'_k} x_i \le 1 \qquad \text{and} \qquad \sum_{i=j'_{k-1}+1}^{j'_k+1} x_i > 1 \ .$$

It is easy to show that $\psi'_s(P_s) \subseteq (0,1]^m$. Hence $\psi'_s(P_s) \subseteq P_A$.

Now a simple check shows that $\psi'_s = \psi_A^{-1}$. □

Instead of finding the volume of a polytope P_A, Theorem 4 allows us to count permutations, a much easier problem.

Given a permutation π, the number of bins in the bin arrangement that $\mathrm{Sh}(\pi)$ represents is determined by the maximal increasing and decreasing runs in π as follows.

Corollary 5. *Let π be a permutation and consider the first run of π to be necessarily decreasing. Then the number of items in the first bin is the length of the leading decreasing run. Subsequently, if the ith maximal run has length j, the ith bin contains $j-1$ items.*

Corollary 6. *If $\psi(P_A) = P_s \subset [0,1]^m$, then the probability that the bin arrangement A occurs is $\frac{1}{m!}$ times the number of permutations π of $[m]$ such that $\mathrm{Sh}(\pi) = s$.*

In a shape the links ad and da correspond to peaks and valleys, hence the following.

Corollary 7. *Given a permutation π, define p_π to be the number of peaks, v_π to be the number of valleys, and a_π to be 0 if $\pi(1) > \pi(2)$ and 1 if $\pi(1) < \pi(2)$. Then the number of bins in the bin arrangement represented by π is $1 + p_\pi + v_\pi + a_\pi$.*

4. The generating function

Though we can find the probability that a particular bin arrangement occurs, one is usually only interested in the total number of bins, and not in precisely which objects are in which bins. We will use Corollaries 5 and 6 to calculate this generating function.

This calculation will require the following decomposition of a permutation into disjoint alternating decreasing and increasing runs.

Proposition 8. *Any permutation of* $[2m]$ *can be split uniquely into a sequence of disjoint alternating decreasing and increasing runs, each of maximal even (possibly zero) length. Any permutation of* $[2m+1]$ *can be similarly split into alternating increasing and decreasing runs of maximal even length following an initial decreasing run of maximal odd length.*

For example, consider $\sigma_1 = 4\,3\,2\,1\,7\,6\,5\,8\,9\,10$. This splits into the decreasing run $4\,3\,2\,1$, an increasing run of length zero, the decreasing run $7\,6$, and the increasing run $5\,8\,9\,10$. If we take $\sigma_2 = 4\,3\,2\,1\,7\,6\,5\,8\,9$, we split it into runs $4\,3\,2$, $1\,7$, $6\,5$ and $8\,9$. We split $\sigma_3 = 4\,5\,6\,7\,3\,2\,1$ into 4 (a decreasing run of length 1), $5\,6$ and $7\,3\,2\,1$.

While this proposition can be proved using an easy inductive argument, we will instead employ a construction that we will later need.

Proof. Let σ be a permutation of $[2m]$. We will mark $\mathrm{Sh}(\sigma)$ by drawing for each i a line through the $2i$th letter of $\mathrm{Sh}(\sigma)$ unless the $2i-1$st, $2i$th, and $2i+1$st letters are all the same.

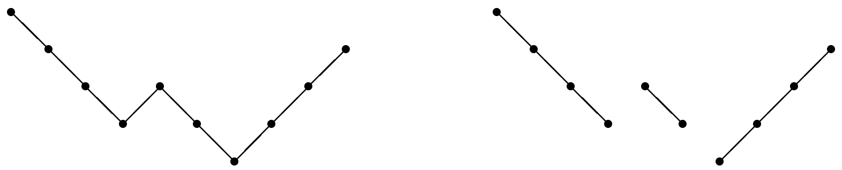
This results in dividing $\mathrm{Sh}(\sigma)$ into subwords of the form a^j and d^j (where j is odd), separated by $\cancel{a}$'s and $\cancel{d}$'s. Under this division, a subword a^j consisting of the $k+1$st through $k+j$th letters of $\mathrm{Sh}(\sigma)$ corresponds to the increasing run $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+j+1)$. Similarly, a subword d^j consisting of the $k+1$st through $k+j$th letters of $\mathrm{Sh}(\sigma)$ corresponds to the decreasing run $\sigma(k+1) > \sigma(k+2) > \cdots > \sigma(k+j+1)$.

If two consecutive separated subwords are both of the form a^j, we say there is a decreasing sequence of length zero between them; if they are both of the form d^j, we say there is an intervening increasing sequence of length zero. If $\mathrm{Sh}(\sigma)$ starts with an a, we say there is an initial decreasing run of length zero.

The construction for permutations of odd length is similar. Let τ be a permutation of $[2m+1]$. We mark $\mathrm{Sh}(\tau)$ by drawing a line through the $2i+1$th letter of $\mathrm{Sh}(\tau)$ (for $i \geq 1$) unless the $2i$th, $2i+1$st and $2i+2$nd letters are all the same. We also draw a line through the first letter, unless both it and the second letter are d's. Similar to before, this results in subwords of the form d^j and a^j separated by $\cancel{d}$'s and $\cancel{a}$'s. The first subword has an even exponent, the rest odd exponents. If the first letter of the marked $\mathrm{Sh}(\tau)$ is $\cancel{a}$ or $\cancel{d}$, the exponent of the first subword is zero and the initial decreasing run of $\tau(1)$ has length 1. Otherwise, we form the increasing and decreasing sequences as we did before. $\square$

Remark 2. Note that the 'maximality' does not refer to the runs being as long as possible, merely to being as long as possible under the condition that all runs are even length. Take for example a permutation σ' whose shape is $daaad$. It certainly has an increasing run of length 4: $\sigma'(2) < \sigma'(3) < \sigma'(4) < \sigma'(5)$. However, in order to insure all runs are of even length, the algorithm above results in both $\sigma'(2)$ and $\sigma'(5)$ being part of decreasing runs.

To illustrate this construction, consider σ_1 from above. We mark $\mathrm{Sh}(\sigma_1) = dddaddaaa$ as $ddd\cancel{a}d\cancel{d}aaa$. The first subword d^3 means that $\sigma_1(1)\,\sigma_1(2)\,\sigma_1(3)\,\sigma_1(4) = 4\,3\,2\,1$ is the first decreasing run. Since the next subword is d, there is an intervening empty increasing run followed by the decreasing run $\sigma_1(5)\,\sigma_1(6) = 7\,6$. Finally the last subword a^3 results in the increasing run $5\,8\,9\,10$. We represent this pictorially by leaving blank the northeast step for the $\cancel{a}$ and the southeast step for the $\cancel{d}$ as follows.



For σ_2, we write $\mathrm{Sh}(\sigma_2) = dddaddaa$ as $dd\not{d}a\not{d}d\not{a}a$.

For σ_3 we have $aaaddd$ and hence $\not{a}a\not{a}ddd$.

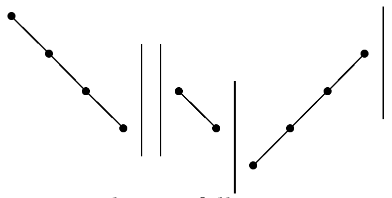
Definition 10. A barred permutation of m letters is a permutation which can have lines (or bars) drawn in the $m+1$ spaces between, before, and after the letters.

For example, $||3\,4|2\,5|||1$ and $3\,4\,2||||5\,1||$ are both barred permutations of the set $\{1, 2, 3, 4, 5\}$ with 6 bars. See [9] for more on barred permutations.

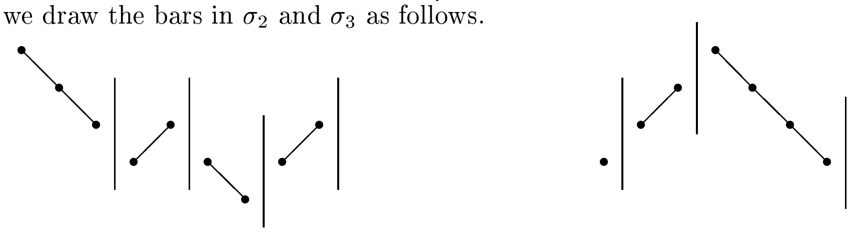
We will be counting barred permutations that have additional restrictions on where the bars may appear.

Definition 11. Given a permutation of $[m]$, we decompose it into alternating decreasing and increasing runs as in Proposition 8. Following the proof of Proposition 8, we place a bar after each run, in the space described by the $\not{a}$ or $\not{d}$ that marks the end of that run's corresponding subword. (An increasing run of length zero has its bar in the position of the $\not{a}$ that splits the surrounding d^i subwords, and an empty decreasing run's bar is in the position given by the $\not{d}$ splitting the surrounding a^i subwords. The bar for the last run is in the mth space.) We define $P_{m,n}$ to be the number of permutations of $[m]$ that have n such bars, and we write $P_m(y) = \sum_n P_{m,n} y^n$.

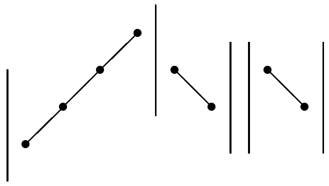
In our earlier examples, σ_1 would have two bars in the fourth place (one for the initial decreasing run described by d^3, and one for the empty increasing run), a bar in the sixth place, and a bar at the end in the tenth place. We write it $4\,3\,2\,1||7\,6|5\,8\,9\,10|$ and draw it as shown below.



Similarly, we draw the bars in σ_2 and σ_3 as follows.



If a permutation of $[2m]$ starts with an ascent, there will be a bar in the 0th position, corresponding to the initial empty decreasing run. We show the placement of the bars for $\sigma_4 = 3\,4\,5\,8\,7\,1\,6\,2$.



Remark 3. We note that for m even, the bars are in the even spaces, and that for m odd they are in the odd spaces.

Proposition 9. *The number of permutations of $[m]$ whose shape corresponds to a bin arrangement with n bins is $P_{m,n}$.*

Proof. Each maximal increasing or decreasing run has exactly one bar associated with it. If the run is of length at least 3, then at least part of it will form one of the runs described in Proposition 8, and hence is guaranteed a bar. If the run is length 2 (in other words, it is described by a single a or d in the shape) then either it forms a run in Proposition 8, or it becomes an $\not a$ surrounded by d's or a $\not d$ surrounded by a's. In either case, the construction gives it a bar. Finally, an initial ascent (the first decreasing run is of length 1) gets a bar as well: in the 0th place for m even and in the first place for m odd.

¿From Corollary 5, since we assume the first run is a decreasing one, this is the number of bins. ☐

Proposition 10. *We have*

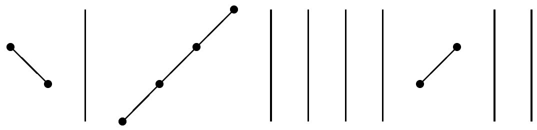
$$(1+y)\sum_{m=0}^{\infty}\frac{P_{2m}(y)}{(1-y^2)^{m+1}}\,\frac{x^{2m}}{(2m)!}=\frac{1}{1-y\cosh x} \tag{2}$$

and

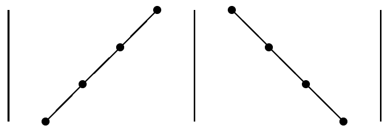
$$(1+y)\sum_{m=0}^{\infty}\frac{P_{2m+1}(y)}{(1-y^2)^{m+1}}\,\frac{x^{2m+1}}{(2m+1)!}=\frac{y\sinh x}{1-y\cosh x}\ . \tag{3}$$

Proof. For each equation, the left and right sides describe different ways of constructing certain barred permutations. We will let a barred permutation of $[m]$ with n bars have the weight $y^n\frac{x^m}{m!}$.

For the right side of Equation (2), we write alternating even-length (labeled) decreasing and increasing sequences, with a bar after each one. We will illustrate them as shapes of the form d^i and a^i. For example, we may have sequences of lengths 2, 4, 0, 0, 0, 2, and 0, as in the picture below.



Sequences of lengths 0,4, and 4 would result in the following picture.

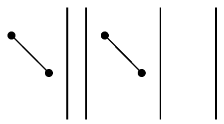
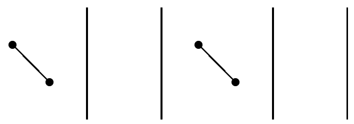


(Whether there is an ascent or descent between any two sequences depends on the precise choice of labeling. No assumption is made on this in the above pictures.) If the sum of the lengths of the sequences is m, the labels will be drawn without repetition from $[m]$. This results in a factor of $(1-y\cosh x)^{-1}$.

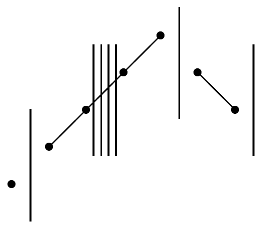
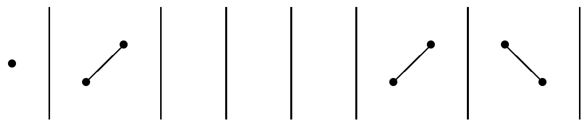
For the right side of Equation (3), we start by drawing an odd-length decreasing sequence followed by a bar, and then continue alternating even-length increasing and decreasing sequences, once again placing a bar after each sequence. We label as before, which gives us $(y \sinh z)(1 - y \cosh x)^{-1}$.

On the left side of both Equation (2) and (3), the barred permutations discussed in Definition 11 give the factors $P_{2m}(y)$ and $P_{2m+1}(y)$ respectively. We then permit any number of additional optional pairs of bars in each of the $m+1$ even spaces for Equation (2), and in each of the $m+1$ odd spaces for Equation (3), for a factor of $(1-y^2)^{m+1}$. Finally, in each expression we wish to allow any positive number of bars, not just an odd number, at the end of a permutation. Hence we multiply each summation by a factor of $1 + y$. $\square$

For example, consider the barred permutation $2\,1||4\,3||$. Since Definition 11 places bars in $2\,1\,4\,3$ to form $2\,1||4\,3|$, we obtain $2\,1||4\,3||$ on the left side of (2) by permitting an additional bar at the end. On the right, we get it from choosing the labels appropriately for the sequences of lengths 2,0,2, and 0.

Similarly, Definition 11 places bars in $1\,2\,3\,4\,7\,6\,5$ to form $1|2\,3\,4\,7|6\,5|$. We get $1|2\,3||||4\,7|6\,5|$ from the left side of (3) by allowing an extra two pairs of bars in the 3rd space. On the right side we take sequences of lengths 1,2,0,0,0,2, and 2 with the appropriate labeling.

This analysis is similar to one in [5, ch 7].

Theorem 11. *We have*

$$\sum P_{m,n} y^n \frac{x^m}{m!} = \frac{1 - y^2 + \sqrt{1-y^2}\; y \,\sinh(x\sqrt{1-y^2})}{(1+y)(1 - y\cosh(x\sqrt{1-y^2}))}\ . \tag{4}$$

Proof. We replace x with $x\sqrt{1-y^2}$, multiply Equation (2) by $(1-y^2)(1+y)^{-1}$ and (3) by $\sqrt{1-y^2}(1+y)^{-1}$, and add. $\square$

This generating function differs from the one Hofri derives by probabilistic methods in [8] by $1-y$; in his base case, the empty sequence requires a bin, while we consider it not to need one.

There are other ways to apply Theorem 4 to calculate the generating function (4). We can apply the methods of permutation enumeration developed in [6, section 4.2] to count permutations according to their maximal increasing runs. We can also combine and apply techniques discussed in [10, pp. 241-243], [5, ch 4 and 6] and [2] to count permutations according to their explicit pattern of run lengths, appropriately weighted. This last method can be used in some of the generalizations discussed in the last section of this abstract. See [3] for details.

5. An application to calculating the expectation and variance

Definition 12. We define the following random variables on permutations of $[m]$. Let $Z(\pi)$ be the number of bins in the bin arrangement corresponding to $\psi^{-1}(P_{\mathrm{Sh}(\pi)})$. For $1 \leq i \leq m-1$, define

$$X_i(\pi) = \begin{cases} 1, & \text{for } \pi(i) > \pi(i+1); \\ 0, & \text{for } \pi(i) < \pi(i+1). \end{cases}$$

The following proposition restates Corollary 7 in terms of these random variables.

Proposition 12. *For $m \geq 3$,*

$$Z = 1 + (1 - X_1) + \sum_{|i-j|=1} X_i(1 - X_j) \ . \tag{5}$$

To calculate the expected number of bins under NEXT FIT we need only find $\mathrm{E}(Z)$, the expectation of Z. This is simple to compute following the above formulation.

Theorem 13. *Given $m \geq 2$ objects, the expected number of bins they use under* NEXT FIT *is $\frac{2m}{3} + \frac{1}{6}$. (If $m = 1$, the expected number is 1.)*

Proof. In the last summand of (5), for $(i,j) = (b, b+1)$ and $(i,j) = (b+1, b)$ the shapes *da* and *ad* each occur with probability $\frac{1}{3}$ for each $b = 1, \ldots, m-2$. So $\sum_{|i-j|=1} X_i(1-X_j)$ contributes $\frac{2(m-2)}{3}$ to the expectation, and $1 + (1 - X_1)$ contributes $\frac{3}{2}$. □

To calculate the variance, we first write Z^2 in terms of the X_i:

$$Z^2 = 2Z - 2X_1 + X_1^2 - 2X_1 \sum_{|i-j|=1} X_i(1-X_j) \\ + 2 \sum_{|i-j|=1} X_i(1-X_j) + \sum_{\substack{|i-j|=1 \\ |k-l|=1}} X_i(1-X_j)X_k(1-X_l) \ . \tag{6}$$

The variance is equal to $\mathrm{E}(Z^2) - \mathrm{E}(Z)^2$. The calculation of $\mathrm{E}(Z^2)$ requires a bit of work, but is nonetheless straightforward. It is necessary to examine each term of (6) and calculate the probability that each specified subword occurs in a permutation's shape. For example, if we choose $(i,j,k,l) = (b, b+1, b+3, b+2)$ in the last summand, the resulting subword *daad* occurs with probability $\frac{11}{120}$ for each $b = 1, \ldots, m-4$.

Theorem 14. *Given m objects, the variance of the number of bins they use under* NEXT FIT *is*

$$V(Z) = E(Z^2) - E(Z)^2 = \begin{cases} 8m/45 - 13/180, & \text{if } m \geq 4; \\ 17/36, & \text{if } m = 3; \\ 1/4, & \text{if } m = 2; \\ 0, & \text{if } m = 1. \end{cases}$$

6. The dual bin packing problem and other generalizations.

In [1], Assman, Johnson, Kleitman, and Leung consider the related "dual bin packing" problem, where bins must be filled to at least a given threshold k. In other words, for a bin B to be full, the inequality $\sum_{i \in B} x_i \geq k$ must hold. Similarly to NEXT FIT, the DUAL NEXT FIT algorithm places objects in order into a bin until that bin is full. In particular, objects i through i' fill a bin of threshold k when the following inequalities hold:

$$\sum_{j=i}^{i'-1} x_j < k \qquad \text{and} \qquad \sum_{j=i}^{i'} x_j \geq k \ .$$

We define polytopes corresponding to dual bin arrangements and calculate their volumes. For the threshold $k=1$ case, we obtain the previously unknown generating function

$$A(x,y,z)=\frac{1+z(e^x-1)}{1-y[(x-1)e^x+1]}$$

where $x^m y^n z^p$ is the weight given to a sequence of m objects that fills n bins and has p non-full bins at the end. For the case where the threshold is some positive integer k, we obtain the generating function

$$A_k(x,y,z)=\frac{1+z(g_k(x)-1)}{1-y[(x-1)g_k(x)+1]}$$

where we define $g_k(x)$ by

$$\frac{1}{1-te^{x(1-t)}}=\sum_{k=0}^{\infty} g_k(x)\ t^k\ .$$

We also obtain generating functions for the threshold 1 and k cases when we give the object sizes a uniform discrete distribution.

We define a map from the polytope corresponding to a dual bin arrangement to a polytope corresponding to a particular set of permutation shapes. Under this map, it turns out that a dual bin arrangement is equivalent to the hook factorization of permutations defined in [4]. This provides another derivation of the generating function for derangement numbers.

Finally, the map ψ of Section 3 of this abstract generalizes to the case when we permit bins to have capacity k in the bin packing problem. We can describe the resulting permutation enumeration formulation, and calculate the generating function for $k=2$.

Acknowledgments

This work forms part of the author's Ph.D. thesis [3]; she would like to thank her thesis advisor Ira Gessel for his unwavering support and many helpful suggestions.

References

[1] S.F. Assman, D.S. Johnson, D.J. Kleitman, and J. Y.-T. Leung, *On a dual version of the one-dimensional bin packing problem*, Journal of Algorithms **5** (1984), 502–522.
[2] L. Carlitz, R. Scoville and T. Vaughan, *Enumeration of pairs of sequences by rises, falls, and levels*, Manuscripta Math. **19** (1976).
[3] Marguerite Eisenstein-Taylor, *Polytopes, Permutations, and the Bin Packing Problems* Ph.D. Thesis, Brandeis University, 2001.
[4] Ira Gessel, *A coloring problem*, American Mathematical Monthly **98** (1991), 530–533.
[5] Ira Gessel, *Generating Functions and Enumeration of Sequences*, Ph.D. Thesis, Massachusetts Institute of Technology, 1977.
[6] I.P. Goulden and D.M. Jackson, *Combinatorial Enumeration*, John Wiley & Sons, New York, 1983.
[7] Micha Hofri, *Probabilistic Analysis of Algorithms*, Springer-Verlag, New York-Berlin, 1987.
[8] Micha Hofri, *A probabilistic analysis of the next-fit bin packing algorithm*, Journal of Algorithms **5** (1984), 547–556.
[9] Paul Klingsburg and Cynthia Schmalzried, *Barred permutations*, Congr. Numer. **95** (1993), 153-161.
[10] Richard P. Stanley, *Enumerative Combinatorics Volume I*, Cambridge University Press, New York, 1997.
[11] Richard P. Stanley, *Eulerian partitions of a unit hypercube*, Higher Combinatorics (M. Aigner, ed.), 1977.
[12] Richard P. Stanley, *Two Poset Polytopes*, Discrete and Computational Geometry **1** (1986), 9-23.

Department of Mathematics, Brandeis University, Waltham, Massachusetts 02454-9100