COMBINATORIAL CONSTRUCTIONS OF SCHUBERT POLYNOMIALS: A UNIFIED APPROACH

EXTENDED ABSTRACT

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ABSTRACT. This paper investigates the connections between several combinatorial structures for the construction of Schubert polynomials. We designate certain strand diagrams known as rc-graphs as the main structure. The other structures in the literature to which we refer are: increasing labeled chains in Bruhat order on the symmetric group, balanced labelings of the diagram of a permutation, Kohnert diagrams, and semistandard Young tableaux.

RÉSUMÉ. Dans cet article nous étudions les connexions entre plusieures structures combinatoires pour la construction des polynômes de Schubert. Nous considerons certains diagrammes de lignes dénommés rc-graphes les structures principalles. Les autres structures considerées sont: certaines chaînes étiquetées croissantes pour l'ordre de Bruhat sur le groupe symétrique, les étiquetages équilibrés du diagramme d'une permutation, les diagrammes de Kohnert et les tableaux semistandard.

1. INTRODUCTION

Schubert polynomials are polynomials with positive integer coefficients in n variables $x_1, \ldots, x_n$, which are indexed by permutations in the symmetric group S_n. They satisfy a stability property, namely the Schubert polynomials for S_n coincide with certain Schubert polynomials for S_{n+1} under the standard embedding of S_n in S_{n+1}. As n ranges over the positive integers, we obtain an infinite collection of polynomials which turns out to form a basis of $\mathbb{Z}[x_1, x_2, \ldots]$. Schubert polynomials are important because they are natural representatives for Schubert classes in the cohomology of the variety of complete flags in $\mathbb{C}^n$, which can also be viewed as SL_n/B, with B the corresponding Borel subgroup of upper or lower triangular matrices. Recall that the cohomology ring is just the quotient $\mathbb{Z}[x_1, \ldots, x_n]/I_n$, where I_n is the ideal generated by symmetric polynomials in $x_1, \ldots, x_n$ with no constant term. Schur polynomials, which form a basis of the algebra of symmetric polynomials, are special cases of Schubert polynomials; furthermore, skew Schur polynomials and flagged (skew) Schur polynomials are also special cases.

Schubert polynomials were defined by Lascoux and Schützenberger in [12] and studied by them in a series of papers. The combinatorics related to Schubert polynomials, which generalizes the combinatorics of Young tableaux related to Schur polynomials, is quite complex, and thus attracted the interest of many people in algebraic combinatorics (see the references, including the surveys in [16, 8, 17]). Several combinatorial constructions of Schubert polynomials appeared in the literature in the last decade, together with many generalizations of Schubert polynomials (such as generalizations to other Lie groups, to the K-theory and the quantum cohomology of flag varieties etc.). While all these generalizations appeared, the connections between the existing structures for the construction of Schubert polynomials themselves were not fully understood. In this paper we attempt to elucidate most of the connections not yet understood.

We designate certain strand diagrams introduced in [2] and called *rc-graphs* as the main combinatorial structure for the construction of Schubert polynomials. The other structures in the literature to which we refer are: increasing labeled chains in Bruhat order on the symmetric group [3], balanced labelings of the diagram of a permutation [6], Kohnert diagrams [10, 19, 20], and semistandard Young tableaux [13, 14].

2. RC-Graphs

RC-graphs are combinatorial objects associated to permutations w in the symmetric group S_n. We let n and w be fixed throughout, unless otherwise specified, and we use the one-line notation for permutations $w = w_1 \dots w_n$. RC-graphs can be defined as certain subsets of

$$\{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \;:\; i+j \le n\}\,.$$

Given an arbitrary such subset of pairs R, we can define a linear order on it by setting

$$(i,j) \le (i',j') \iff (i<i') \text{ or } (i=i' \text{ and } j \ge j')\,.$$

Let (i_k, j_k) be the kth pair in this linear order. Consider the sequences

$$d(R) := (i_1+j_1-1, i_2+j_2-1, \dots)\,, \qquad c(R) := (i_1, i_2, \dots)\,.$$

We call R an rc-graph associated to w if $d(R)$ is a reduced decomposition of w; in this case, $c(R)$ is called a *compatible sequence* with $d(R)$. We denote the collection of rc-graphs associated to a permutation w by $\mathcal{R}(w)$, and the permutation corresponding to a given rc-graph R by $w(R)$.

We can represent an rc-graph R graphically as a planar history of the inversions of w (see Example 2.2). To this end, we draw n strands going up and to the right, such that the ith strand starts at position $(i,1)$ and ends at position $(1,w_i)$; here the positions are numbered as in a matrix, whence the first index is the row index, and the second one is the column index. The rule for constructing the strand diagram is the following: two strands meeting at position (i,j) cross if (i,j) is in R; otherwise, they avoid each other at that position.

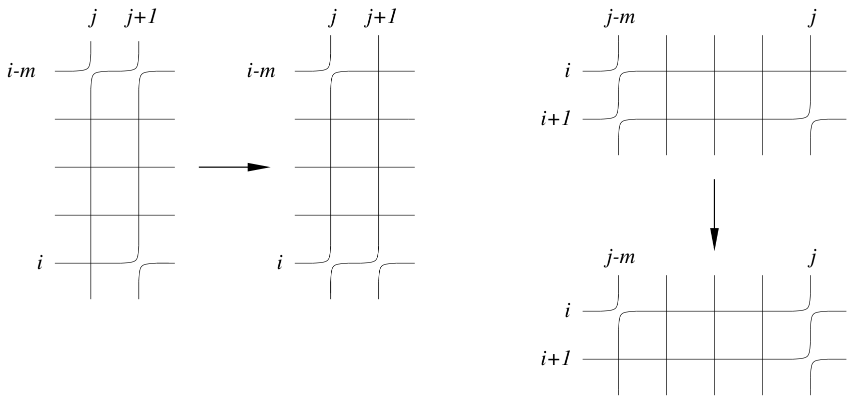
Given an rc-graph R, we define

$$x(R) := \prod_{(i,j)\in R} x_i\,.$$

According to [4, 7, 2], the Schubert polynomial $\mathfrak{S}_w$ indexed by w can be expressed as

$$\mathfrak{S}_w = \sum_{R\in\mathcal{R}(w)} x(R)\,. \tag{2.1}$$

Bergeron and Billey [2] defined certain moves on rc-graphs, which they called *ladder* and *chute* moves. These are the moves L_{ij} and C_{ij} of the following type:



Note that L_{ij} moves the crossing in position (i,j) to some position $(i-m, j+1)$; we will denote it by $(i,j) \to (i-m, j+1)$.

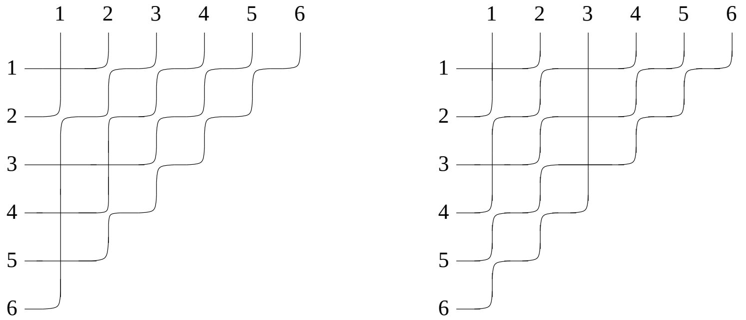
Recall that a permutation w is determined by its *code* $c(w)$, which is the sequence $(c_1, \dots, c_{n-1})$ defined by $c_i = |\{j \,:\, j>i \text{ and } w_j < w_i\}|$. Let us also recall the rc-graphs $R_{\mathrm{bot}}(w)$ and $R_{\mathrm{top}}(w)$

defined by

$$R_{\rm bot}(w) := \{(i,j)\ :\ j \le c_i(w)\}\,, \qquad R_{\rm top}(w) := \{(i,j)\ :\ i \le c_j(w^{-1})\}\,.$$

Bergeron and Billey showed that $\mathcal{R}(w)$ is the transitive closure of $\{R_{\rm bot}(w)\}$ under ladder moves L_{ij} which move the rightmost crossing in a row (i.e., (i,l) is not in the rc-graph to which L_{ij} is applied if $l > j$); we call these moves, simply, *L-moves.* There is a dual property, namely that $\mathcal{R}(w)$ is the transitive closure of $\{R_{\rm top}(w)\}$ under chute moves C_{ij} which move the bottom crossing in a row (i.e., (k,j) is not in the rc-graph to which C_{ij} is applied if $k > i$).

Example 2.2. Here are the graphical representations of two rc-graphs associated to the permutation $w = 215463$; the first is just $R_{\rm bot}(w)$.



It is sometimes useful to have a standard way of obtaining a given rc-graph R for w from $R_{\rm bot}(w)$ via L-moves. This can be done by applying the algorithm below, based on the theorem following it.

Algorithm 2.3.
set $R' := R_{\rm bot}(w)$*;*
for i *from* 1 *to* $n-1$ *do*
  for j *from* $n-i$ *downto* 1 *do*
    if $(i,j) \in R \setminus R'$ *then*
      find the position (k,l) *where the strands of* R' *avoiding each other at* (i,j) *cross;*
      set $R' := R' \cup \{(i,j)\} \setminus \{(k,l)\}$*;*
    fi;
  od;
od.

Proposition 2.4. *In the above algorithm, each crossing at* (k,l) *is moved into position* (i,j) *by a sequence of L-moves.*

Example 2.5. The standard sequence of L-moves for the second rc-graph in Example 2.2 is $(4,1) \to (2,2)$, $(2,2) \to (1,3)$, $(3,2) \to (2,3)$, $(5,1) \to (4,2)$, $(4,2) \to (3,3)$.

3. Increasing Labeled Chains in Bruhat Order

Let us recall from [3] another combinatorial construction of Schubert polynomials, in terms of labeled chains in the k-Bruhat order. The *k-Bruhat order* (denoted $\le_k$) is the suborder of the Bruhat order on the symmetric group S_n with the following covering relations:

$$v \le_k w\,,\ \ l(w) - l(v) = 1 \quad \Longleftrightarrow \quad v^{-1}w = t_{ij}\,,\ \ i \le k < j\ ,$$

where $l(w)$ denotes the number of inversions in w, and t_{ij} denotes the transposition of the elements i, j. We say that a pair $\binom{k}{l}$ is an admissible labeling of the covering relation $v < w$ in Bruhat order if the following conditions are satisfied:

$$v \le_k w \qquad \text{and} \qquad l = j,\ \text{where}\ wv^{-1} = t_{ij},\ i < j\,. \tag{3.1}$$

Note that all the constructions below can be done after replacing the condition $l = j$ with $l = i$ in (3.1). We now consider labeled chains $v_0 < v_1 < \ldots < v_m$ in the Bruhat order with every $v_i < v_{i+1}$ a covering relation having an admissible labeling. We call a labeled chain increasing if its labels are (strictly) increasing in lexicographic order (this is defined on pairs by $\binom{k}{l} \leq \binom{k'}{l'}$ if and only if $k < k'$, or $k = k'$ and $l \leq l'$). Let us denote by $\mathcal{C}(v, w)$ the set of increasing labeled chains from v to w. If C is such a chain, and $\binom{k_1}{l_1} < \ldots < \binom{k_m}{l_m}$ are its labels, we define

$$x(C) := \left(\prod_{i=1}^{n-1} x_i^{n-i} \right) / \left(\prod_{i=1}^{m} x_{k_i} \right) ;$$

it is easy to see that all x_i's appear with nonnegative exponents. The following result was proved in [3]:

$$\mathfrak{S}_w = \sum_{C \in \mathcal{C}(w, w_0)} x(C) \,, \tag{3.2}$$

where w_0 is the longest permutation in S_n.

Comparing (2.1) with (3.2) suggests that there should be a natural bijection between $\mathcal{R}(w)$ and $\mathcal{C}(w, w_0)$. Indeed, such a bijection exists, and is presented below. Consider an rc-graph R in $\mathcal{R}(w)$, and construct the sequence of rc-graphs $R = R_0, R_1, \ldots, R_{l(w_0)-l(w)}$ by scanning the rows of R from left to right and top to bottom, and by adding all the missing pairs $(k, j) \notin R$, $k + j \leq n$. It is easy to see that every R_i is indeed an rc-graph, and that $w = w(R_0) < w(R_1) < \ldots < w(R_{l(w_0)-l(w)}) = w_0$ is a chain in Bruhat order. If the pair (k, j) is added to R_i, we can also check that $\binom{k}{l}$ is an admissible labeling of $w(R_i) < w(R_{i+1})$, where l is defined by (3.1). Furthermore, the obtained labeled chain, which we denote by $C(R)$, is increasing. We can prove the following result.

Theorem 3.3. *The map $R \mapsto C(R)$ from $\mathcal{R}(w)$ to $\mathcal{C}(w, w_0)$ is a bijection satisfying $x(C(R)) = x(R)$.*

We call a bijection with the above property a monomial preserving bijection. The inverse map is obtained easily, by letting each covering relation $v \leq_k w$ labeled $\binom{k}{l}$ in an increasing labeled chain correspond to a cross in position (k, i) in an rc-graph, where $wv^{-1} = t_{ij}$.

Example 3.4. Below we present an example of an rc-graph and the associated increasing labeled chain.

$$\begin{array}{ccc} \cdot & + & + \\ \cdot & \cdot & \\ + & & \end{array} \qquad [1,4,3,2] \xrightarrow{\binom{1}{4}} [4,1,3,2] \xrightarrow{\binom{2}{2}} [4,2,3,1] \xrightarrow{\binom{2}{3}} [4,3,2,1]$$

In [15] we extend the point of view presented above by defining the *skew Schubert polynomial* $\mathfrak{S}_{w/v}$ in terms of increasing labeled chains from the permutation v to the permutation w. We show that this polynomial expands in the basis of Schubert polynomials with coefficients equal to certain structure constants for the multiplication of Schubert polynomials.

We end by mentioning the connection with the insertion algorithm for rc-graphs in [9], which bijectively proves Pieri's formula. Recall that this formula is a rule for multiplying a Schubert polynomial by a complete homogeneous symmetric polynomial $h_i(x_1, \ldots, x_k)$. Given an rc-graph R with $x(R) = x_1^{\alpha_1} \ldots x_{n-1}^{\alpha_{n-1}}$, the chain $C(R)$ can be obtained by successive insertions into R corresponding to multiplications of $x(R)$ by the monomials $x_k^{n-k-\alpha_k}$ in $h_{n-k-\alpha_k}(x_1, \ldots, x_k)$, for $k = 1, \ldots, n-1$. Note that, in this special case, the insertion algorithm reduces to the older one in [2].

4. Balanced Labelings

Recall that the *diagram* $D(w)$ of the permutation w is the set

$$D(w) := \{(i, w_j) \,:\, i < j,\ w_i > w_j\} \subseteq [n] \times [n]\,,$$

where $[n] := \{1, \ldots, n\}$. Fomin et al. [6] present a construction of Schubert polynomials in terms of *balanced column-strict labelings* of the diagram of a permutation which satisfy a certain flag condition. A labeling of the cells of the diagram $D(w)$ is called balanced if it satisfies the following condition: if one rearranges the labels within any hook of $D(w)$ (i.e., its corner is a cell of $D(w)$) such that they weakly increase from right to left and from top to bottom, then the corner label remains unchanged. A balanced labeling is called column-strict if no column contains two equal entries. The flag condition mentioned above is the requirement that all labels in row i are at most i.

In [6] a bijection was constructed from $\mathcal{R}(w)$ to column-strict balanced labelings of the diagram $D(w)$ satisfying the flag condition above. The construction of the balanced labeling corresponding to an rc-graph R for w is based on the reduced decomposition $d(R) = (a_1, a_2, \ldots)$ and the compatible sequence $c(R) = (i_1, i_2, \ldots)$. The labeling is constructed as follows: the cell in row p and column q is labeled i_r if the adjacent transposition $s_{a_r} = (a_r, a_r + 1)$ in $d(R)$ transposes q and w_p, where $q < w_p$. Note that the number of labels i is equal to the number of crossings in row i of the rc-graph; this means that the bijection preserves the monomials associated with the combinatorial objects. For instance, the balanced labeling of the diagram of 215463 corresponding to the second rc-graph in Example 2.2 is the following.

	1	2	3	4	5	6
1	1					
2						
3			2	3		
4			1			
5			3			
6						

We now present an easy construction of the inverse bijection, from column-strict balanced labelings T of $D(w)$ satisfying the flag condition to rc-graphs $R = R(T)$ for w. The construction of $R(T)$ is described by the following algorithm.

Algorithm 4.1.
for $k = 1, \ldots, n$ *do*
  start at the left end $(k, 1)$ *of strand* k*;*
  while not at the upper end of strand k *do*
    let (i, j) *be the current position;*
    if a previous strand crosses position (i, j) *horizontally then*
      cross position (i, j) *vertically;*
    else
      if $(i, j) \neq (k, 1)$ *then avoid position* (i, j) *fi;*
      move to the right past a number of lattice points
          equal to the number of labels i *in the* k*th row of* T*;*
      avoid the current position;
    fi;
  od;
od.

5. Kohnert Diagrams

Kohnert [10] conjectured that Schubert polynomials can be constructed in terms of certain diagrams, known at present as *Kohnert diagrams.* These are subsets of $[n] \times [n]$ which can be obtained from the diagram of a permutation by certain moves. A *Kohnert move* (or, simply, *K-move*) on a Kohnert diagram picks the rightmost square in any row i, and moves it up to the first nonoccupied square directly above it. Since a K-move is uniquely determined by the corresponding row i, we use sequences of numbers to specify sequences of K-moves. Let $\mathcal{K}(w)$ denote the closure of $\{D(w)\}$ under all possible K-moves. Kohnert's conjecture says that

$$\mathfrak{S}_w = \sum_{K\in\mathcal{K}(w)} x(K)\,, \tag{5.1}$$

where $x(K) := \prod_{(i,j)\in K} x_i$.

Winkel claimed that he proved this conjecture in [19], and recently came up with a shorter proof in [20]. However, these are not transparent proofs. The first one is a very involved induction, while the second, although shorter, is still based on an induction procedure which is not very illuminating. In fact, Winkel himself recognizes in the second paper that his shorter proof is "done by an exhaustive study of cases, which establishes correctness *without leading to a deeper understanding*". As opposed to Winkel's two proofs, Bergeron and Billey's proof in [2] of the fact that L-moves generate all rc-graphs associated to a given permutation is straightforward. In this paper, we show how to relate K-moves and L-moves. As a byproduct, we get a bijection between $\mathcal{R}(w)$ and $\mathcal{K}(w)$ which preserves the associated monomials.

The main difficulty in our approach is the impossibility of finding a perfect correspondence between K-moves and L-moves (see for example the sequence of L-moves in Example 2.5 and the sequence of K-moves in Example 5.3). The way around this difficulty is to group certain L-moves together, and view this group as just one move. We define an *L^*-move* as a sequence of L-moves on an rc-graph of the form

$$(i-k_1,j)\to(i-k_0,j+1),\ (i-k_2,j)\to(i-k_1,j+1),\ \ldots,\ (i,j)\to(i-k_p,j+1)\,.$$

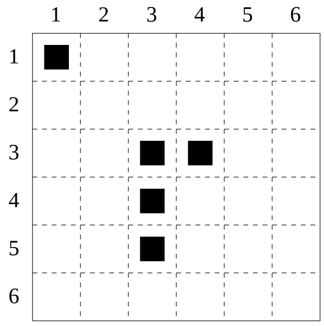
Since an L*-move is uniquely determined by the corresponding pair $(i, i-k_0)$, we use sequences of pairs to denote sequences of L*-moves.

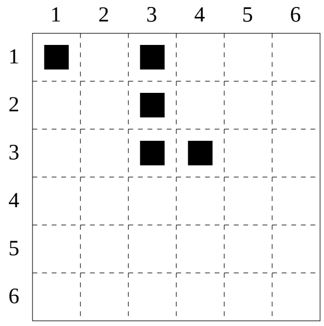
Theorem 5.2. *Consider the smallest lexicographic sequence $(k_1,\ldots,k_r)$ specifying a sequence of K-moves for generating an arbitrary Kohnert diagram K from the diagram $D(w)$ of the permutation w; assume that the ith move is from row k_i to row l_i. Then the sequence $((k_1,l_1),\ldots,(k_r,l_r))$ specifies a sequence of L^*-moves for constructing an rc-graph $R=R(K)$ from $R_{\rm bot}(w)$. Furthermore, the map $K\mapsto R(K)$ is a bijection between $\mathcal{K}(w)$ and $\mathcal{R}(w)$ preserving the associated monomials.*

The inverse map, from $\mathcal{R}(w)$ to $\mathcal{K}(w)$, is more complicated to construct. Essentially, given an rc-graph R, we identify the crosses which are moved into their positions by L-moves from row k to row l with $k-l>1$. It turns out that the set of these crosses does not depend on the particular sequence of L-moves considered. The most difficult part is to figure out an order in which these crosses are moved into their positions, plus the corresponding sequences of L-moves, such that successive L-moves can be grouped into L*-moves corresponding to K-moves; the mentioned L-moves are not related to the standard ones in Algorithm 2.3.

Example 5.3. Below are shown the diagram of the permutation $w = 215463$, corresponding to the first rc-graph $R_{\rm bot}(w)$ in Example 2.2, and the Kohnert diagram corresponding to the second rc-graph in the mentioned example via the bijection above. The smallest lexicographic sequence specifying K-moves for constructing the second Kohnert diagram from the first one is $(4,2,5,4)$. It is easy to see that these K-moves correspond to L*-moves for obtaining the second rc-graph in

Example 2.2 from the first one.





Recall that a permutation is called *Grassmannian* if it has a unique descent (that is, a unique position d for which $w_d > w_{d+1}$). It is well-known that the Schubert polynomial indexed by such a permutation w is just the Schur polynomial $s_{\lambda(w)}(x_1, \ldots, x_d)$, where $\lambda(w)$ is the shape (in English notation) $(w_d - d, \ldots, w_1 - 1)$. If the permutation w is Grassmannian, the bijection in Theorem 5.2 can be defined explicitly in a nice way. To do this, let us denote by $s_1(i,j) < s_2(i,j)$ the rows where the two strands crossing at (i,j) in an rc-graph start.

Corollary 5.4. *If w is a Grassmannian permutation, then the bijection between $\mathcal{R}(w)$ and $\mathcal{K}(w)$ in Theorem 5.2 maps every rc-graph R to the Kohnert diagram*

$$\{(i, w_{s_2(i,j)}) \;:\; (i,j) \in R\}\,.$$

Unfortunately, the bijection above does not work in general, although it is a monomial preserving bijection between rc-graphs and a different set of diagrams than Kohnert diagrams.

6. Semistandard Young Tableaux

It was shown in [4] how to partition the set of rc-graphs $\mathcal{R}(w)$, such that the rc-graphs in a block of this partition are in a natural bijection with certain semistandard Young tableaux. In order to describe this partition, we need to recall the *Edelman-Greene insertion* algorithm [5].

The insertion of an element a into a tableaux T is done, as usual, by inserting a into row 1 (using the algorithm below), then by inserting the "bumped" entry, if any, into row 2, and so on, until there is no "bumped" entry. The insertion of a into a row is done as follows. If a and $a+1$ are in the current row, then the row is left unchanged, and the "bumped" entry is $a+1$. Otherwise, a is inserted in the place of the smallest element larger than a, if any, and the mentioned element is "bumped"; if no element is larger than a, we add a to the right end of the current row.

Given an rc-graph R, one applies the Edelman-Greene insertion to the reduced decomposition $d(R) = (a_1, a_2, \ldots)$, and obtains a tableau $P(R)$. A "recording tableau" $Q(R)$ analogous to the one in the usual Robinson-Schensted correspondence can be defined by using the entries of the compatible sequence $c(R) = (i_1, i_2, \ldots)$. Indeed, one starts with an empty recording tableau Q_0; if after k insertions we have the pair of tableaux (P_k, Q_k), and the insertion of a_{k+1} produces the tableau P_{k+1}, then we place i_{k+1} into Q_k such that the resulting tableau has the same shape as P_{k+1}. It is easy to show that the transposed tableau $Q(R)^T$ is a semistandard Young tableau. On the other hand, the map $R \mapsto Q(R)^T$ clearly preserves the corresponding monomials, that is, $x(R) = x^{c(Q(R))}$; here $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots$, and $c(T)$ is the *content* of the tableau T, i.e., the composition whose ith part is the number of occurences of i in T.

We use the following notation:

$$\mathcal{P}(w) := \{P(R) \;:\; R \in \mathcal{R}(w)\}\,, \qquad \mathcal{Q}(w)^T := \{Q(R)^T \;:\; R \in \mathcal{R}(w)\}\,,$$

$$\mathcal{R}(w,P) := \{R \in \mathcal{R}(w) \ : \ P(R) = P\}\,.$$

It makes sense to call the sets $\mathcal{R}(w,P)$ nilplactic classes of rc-graphs.

Lascoux and Schützenberger [11] defined certain operators traditionally denoted by e_r and f_r on words with letters $\{1,2,\ldots\}$. These operators are also known as *coplactic operations*, since they act on coplactic classes (that is, classes of words with the same recording tableau). In order to define them, we recall the notion of *r-pairing* of a word u, which is just a set of indexed pairs (called r-pairs) (u_i,u_j) such that $i<j$, $u_i=r+1$, and $u_j=r$. View each $r+1$ (resp. r) as a left (resp. right) paranthesis, ignoring the other letters of u. The r-pairs of u are precisely the matched parantheses. The subword of unpaired r's and $r+1$'s is necessarily of the form $r^s(r+1)^t$. Define $e_r(u)$ to be the word obtained by changing the leftmost unpaired $r+1$ to r if $t>0$; otherwise, define it to be the empty tableau $\emptyset$. Similarly, $f_r(u)$ is defined by changing the rightmost unpaired r to $r+1$ if $s>0$. The operators e_r and f_r act on semistandard Young tableaux, upon identification of the tableaux with their row words.

We now define similar operators on rc-graphs; note, in passing, that they were defined on Kohnert diagrams in [1], but their properties were not studied, since the only goal was to derive a combinatorial formula for Schubert polynomials. The idea is to concentrate on rows r and $r+1$ in the graphical representation of an rc-graph R, and to perform an analogous procedure to r-pairing on the columns containing just one cross in the mentioned rows. Mark every column containing a cross in row r (resp. $r+1$) as a left (resp. right) paranthesis. Define $e_r(R)$ to be the rc-graph obtained by performing an inverse chute move to the rightmost unpaired cross in row $r+1$, if such a cross exists; otherwise, define it to be the empty rc-graph $\emptyset$. Similarly, define $f_r(R)$ to be the rc-graph obtained by performing a chute move to the leftmost unpaired cross in row r, if such a cross exists and such a move is possible; otherwise, define it to be the empty rc-graph. Note that in the definitions above the inverse chute move is always possible if the corresponding cross exists, whereas the chute move is not possible (assuming the existence of the corresponding cross) precisely when all positions to the left of the cross to be moved are occupied by crosses.

Example 6.1. The tableaux $P(R)$, $Q(R)^T$, and $e_2(Q(R)^T)$ corresponding to the second rc-graph R in Example 2.2 are the following. The rc-graph $e_2(R)$ is obtained by moving the cross of R in position (3,1) to position (2,2).

$$\begin{array}{ccc} 1 & 3 & 5 \\ 3 & 4 & \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 2 & 3 \\ 3 & \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & \end{array}$$

We proved that the action of the operators e_r and f_r on rc-graphs is compatible with that on semistandard Young tableaux. Moreover, by analogy with the fact that these operators act on coplactic classes (of tableaux), we proved that they act on nilplactic classes of rc-graphs. In order to give the precise statement of the results, we need a slight modification $\widetilde{f}_r$ of the operator f_r on tableaux. Namely, we define the action of $\widetilde{f}_r$ on $\mathcal{Q}(w)^T$ to coincide with that of f_r, except when $f_r(Q(R)^T)$ is not in $\mathcal{Q}(w)^T$, in which case we set $\widetilde{f}_r(Q(R)^T) := \emptyset$.

Theorem 6.2. (1) *The following diagram commutes, together with the corresponding diagram for* e_r*:*

$$\begin{array}{ccc} \mathcal{R}(w) & \xrightarrow{\ f_r\ } & \mathcal{R}(w)\cup\{\emptyset\} \\ {\scriptstyle Q^T}\big\downarrow & & \big\downarrow{\scriptstyle Q^T} \\ \mathcal{Q}(w)^T & \xrightarrow{\ \widetilde{f}_r\ } & \mathcal{Q}(w)^T\cup\{\emptyset\} \end{array}$$

(2) *We have that* $P(f_r(R)) = P(R)$ *for any rc-graph* R*, provided that* $f_r(R)\neq\emptyset$*.*

At this point we need to recall some concepts from [14, 18]. First, recall the *Coxeter-Knuth* or *nilplactic* equivalence on reduced decompositions, on which the Edelman-Greene insertion [5] is based. This is the symmetric transitive closure of the relations

$$\mathbf{a}i(i+1)i\mathbf{b} \sim \mathbf{a}(i+1)i(i+1)\mathbf{b}\,, \qquad \mathbf{a}xzy\mathbf{b} \sim \mathbf{a}zxy\mathbf{b}\,, \qquad \mathbf{a}yxz\mathbf{b} \sim \mathbf{a}yzx\mathbf{b}\,,$$

where $x < y < z$. Second, recall the *left nil key* $K_{-}^{0}(T)$ of a tableau T. This is the tableau of the same shape as T whose jth column is the first column of any skew tableau in the nilplactic class of T with the following properties: (1) its sequence of column lengths is a rearrangement of the corresponding sequence for T; (2) its first column has the same length as the jth column of T.

Let us now define a partial order on the set of rc-graphs $\mathcal{R}(w,P)$. We set $R \preceq R'$ if and only if $Q(R) \geq Q(R')$, where $\geq$ means entrywise comparison. We have the following result.

Theorem 6.3. *The poset $(\mathcal{R}(w,P), \preceq)$ has a maximum $R_{\mathrm{top}}(w,P)$ and a minimum $R_{\mathrm{bot}}(w,P)$. These are uniquely determined by*

$$Q(R_{\mathrm{top}}(w,P))^T = U(\lambda(P^T))\,, \qquad Q(R_{\mathrm{bot}}(w,P))^T = K_{-}^{0}(P^T)\,.$$

Here $\lambda(P^T)$ denotes the shape of P^T, and $U(\lambda)$ is the tableau of shape λ with all entries in row i equal to i.

Clearly, the rc-graphs $R_{\mathrm{top}}(w)$ and $R_{\mathrm{bot}}(w)$ are among the rc-graphs $R_{\mathrm{top}}(w,P)$ and $R_{\mathrm{bot}}(w,P)$. On the other hand, it is not difficult to show that any rc-graph in $\mathcal{R}(w,P)$ can be obtained from $R_{\mathrm{top}}(w,P)$ by successively applying operators f_r, but not all rc-graphs can be obtained from $R_{\mathrm{bot}}(w,P)$ by applying operators e_r.

Our results immediately imply some classical results about *key polynomials*, which are thus derived in a more natural framework. Key polynomials (for the symmetric group) are certain polynomials in several variables indexed by compositions α, which interpolate between monomials and Schur polynomials; from a representation theoretic point of view, they are certain "partial" characters of the irreducible representations of $GL_n(\mathbb{C})$. More precisely, the key polynomial κ_α is defined by

$$\kappa_\alpha := \pi_{u(\alpha)}(x^{\lambda(\alpha)})\,.$$

Here $u(\alpha)$ is the permutation of minimum length which realizes the rearrangement $\lambda(\alpha)$ of α in weakly decreasing order via the right action $\alpha w = (\alpha_{w_1}, \alpha_{w_2}, \ldots)$. On the other hand, π_w is the isobaric divided difference operator on $P = \mathbb{Z}[x_1, x_2, \ldots]$, defined by

$$\pi_w := \pi_{a_1} \ldots \pi_{a_p}\,, \qquad \pi_r := \partial_r \circ x_r\,, \qquad \partial_r := \frac{1 - s_r}{x_r - x_{r+1}}\,;$$

here $(a_1, \ldots, a_p)$ is any reduced decomposition of w, s_r is the adjacent transposition $(r, r+1)$, 1 is the identity on P, x_r denotes multiplication by x_r, and a permutation w acts on P by $w\,x_i = x_{w_i}$.

The results about key polynomials we are able to rederive in a more natural way are: (1) the expansion of a Schubert polynomial as a positive sum of key polynomials; (2) a combinatorial formula for key polynomials in terms of rc-graphs. Essentially, the rc-graphs in every class $\mathcal{R}(w,P)$ correspond to monomials in a certain key polynomial. We present the precise formulation of the mentioned results below.

Theorem 6.4. [13, 18] *We have that*

$$\mathfrak{S}_w = \sum_{P \in \mathcal{P}(w)} \kappa_{c(K_{-}^{0}(P^T))}\,.$$

Theorem 6.5. [18] *Given a composition α, and any semistandard Young tableau P in $\mathcal{P}(w)$, for some permutation w, with $c(K_{-}^{0}(P^T)) = \alpha$, we have*

$$\kappa_\alpha = \sum_{R \in \mathcal{R}(w,P)} x(R)\,.$$

The main idea for deriving the above results from our results in Theorems 6.2 and 6.3 is to use combinatorial versions of the isobaric divided difference operators. Given a tableau T, recall that $\pi_r(T)$ was defined in [14] to be the set of tableaux $\{T, f_r(T), \ldots, f_r^s(T)\}$, if the subword of unpaired r's and $r+1$'s is of the form r^s; otherwise it is defined to be the empty set. More generally, one can define π_r on a set of tableaux, by applying it to every tableau in the set, and by taking the union of the sets. In general, the definition of π_w on a tableau depends on the choice of a reduced word for w. However, it was shown in [14] that this does not happen in the case of $\pi_{u(\alpha)}$ applied to $U(\lambda(\alpha))$. The result is a set of tableaux $\mathcal{T}(\alpha)$, which was described combinatorially in [14] using the concept of *key* (of a tableau). Moreover, it was shown that

$$\kappa_\alpha = \sum_{T \in \mathcal{T}(\alpha)} x^{c(T)} \,. \tag{6.6}$$

We can now define π_r in a similar way on rc-graphs, and lift the previous formula for κ_α to the context of rc-graphs using Theorems 6.2 and 6.3.

We end by mentioning a connection with representation theory. Using the operators e_r and f_r on words, one can endow the coplactic classes of words with the structure of a colored graph, known as *crystal graph*; these graphs arise in the representation theory of quantum groups. We are currently investigating the role of the corresponding operators for rc-graphs in representation theory.

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