A SELF-DUAL POSET ON OBJECTS COUNTED BY THE CATALAN NUMBERS AND A TYPE-B ANALOGUE

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Abstract. We introduce two partially ordered sets, P_n^A and P_n^B, of the same cardinalities as the type-A and type-B noncrossing partition lattices. The ground sets of P_n^A and P_n^B are subsets of the symmetric and the hyperoctahedral groups, consisting of permutations which avoid certain patterns. The order relation is given by (strict) containment of the descent sets. In each case, by means of an explicit order-preserving bijection, we show that the poset of restricted permutations is an extension of the refinement order on noncrossing partitions. Several structural properties of these permutation posets follow, including self-duality and the strong Sperner property. We also discuss posets Q_n^A and Q_n^B similarly associated with noncrossing partitions.

Résumé On introduit deux ensembles partiellement ordonnés, P_n^A et P_n^B, ayant le même nombre d'éléments que les réseaux des partitions non-croisées de type A et B. Les ensembles sous-jacents de P_n^A et P_n^B sont des sous-ensembles des groupes symétriques et hyperoctahedraux, formés par des permutations qui évitent de certains motifs. L'ordre partiel est défini par l'inclusion des ensembles de déscents. Nous prouvons que P_n^A et P_n^B ont la propriété Sperner, et ils ont des anti-automorphismes. Nous étudions aussi les ensembles partiellement ordonnés Q_n^A et Q_n^B des partitions non-croisées de type A et B, définis d'une façon semblable.

1. Introduction

There are more than 150 different objects enumerated by Catalan numbers; [12] contains an extensive list of such combinatorial objects and their properties. Two of the most carefully studied ones are noncrossing partitions and permutations avoiding a 3-letter pattern.

A partition π of the set $[n]:=\{1,2,\cdots,n\}$, having blocks $\beta_1,\beta_2,\cdots,\beta_k$, is called *noncrossing* if there are no four elements $1\le a<b<c<d\le n$ so that $a,c\in\beta_i$ and $b,d\in\beta_j$ for some distinct blocks β_i and β_j. The set of noncrossing partitions of $[n]$ constitutes a lattice under the refinement order, where $\pi<\nu$ if each block of ν is a union of blocks of π. An investigation of properties of this lattice was initiated by Kreweras [4], and continued by several authors, e.g., [2], [5], [8]. We denote the lattice of noncrossing partitions of $[n]$ as NC_n^A, since it is a subposet, indeed, a sub-meet-semilattice, of the intersection lattice associated with the type-A hyperplane arrangement in $\mathbf{R}^n$ (which consists of the hyperplanes with equations $x_i=x_j$, for $1\le i<j\le n$). For our purposes, recall from [4] that the poset NC_n^A is ranked with rank function $\mathrm{rk}(\pi)=n-\mathrm{bk}(\pi)$, where $\mathrm{bk}(\pi)$ denotes the number of blocks of the partition π, rank-symmetric and rank-unimodal with rank sizes given by the Narayana numbers $\left(\frac{1}{n}\binom{n}{k}\binom{n}{k+1}\right)_{0\le k<n}$. Furthermore, it is self-dual (see [4], [8]) and has the strong Sperner property (see [8]; that is, for every k, the maximum cardinality of the union of k antichains is the sum of the k largest rank-sizes).

A permutation $\sigma=\sigma_1\sigma_2\cdots\sigma_n$ of $[n]$, or, in what follows, an n-permutation, is called *132-avoiding* if there are no three positions $1\le a<b<c\le n$ so that $\sigma_a<\sigma_c<\sigma_b$. Classes of restricted permutations avoiding other patterns are defined similarly.

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We introduce the partially ordered set P_n^A whose elements are the 132-avoiding n-permutations, ordered by $\sigma < \rho$ if $\mathrm{Des}(\sigma) \subset \mathrm{Des}(\rho)$, where Des denotes the descent set of a permutation. One can think of P_n^A as a Boolean algebra of rank $n-1$ in which each element S is replicated as many times as there are 132-avoiding permutations with S as the descent set. We show that this poset of restricted permutations is an extension of the lattice of noncrossing partitions NC_n^A by exhibiting a natural order-preserving bijection from the dual order $(NC_n^A)^*$ to the poset P_n^A. This yields the fact that P_n^A has the same rank-generating function as NC_n^A. It then follows that P_n^A is rank-unimodal, rank-symmetric and strongly Sperner. We also prove that P_n^A is itself a self-dual poset.

We also present type-B analogues of these results. The notion of a type-B noncrossing partition of $[n]$ is that first considered by Montenegro [5], systematically studied by Reiner [6]. Type-B noncrossing partitions of $[n]$ form a lattice, NC_n^B, which shares naturally a variety of properties of NC_n^A. In particular, NC_n^B is a rank-unimodal, self-dual, strongly Sperner poset. We define a poset P_n^B into which NC_n^B can be embedded via an order-preserving bijection, with properties analogous to those obtained for type-A. The parallel between the type-A and type-B cases includes the fact that the poset P_n^B is defined in terms of pattern-avoiding elements of the hyperoctahedral group (or signed permutations), ordered by containment of the descent set. The relevant pattern restriction is the simultaneous avoidance of the patterns 21 and $\overline{2}\,\overline{1}$.

2. The type-A case

2.1. **A bijection and its properties.**

It is not difficult to find a bijection from the set of noncrossing partitions of $[n]$ onto that of 132-avoiding n-permutations. Here we exhibit and analyze the structure of such a bijection, f, which will serve as the main tool in proving the results of this section. To avoid confusion, integers belonging to a partition will be called *elements*, while integers belonging to a permutation will be called *entries*. An n-permutation will always be written in the one-line notation, $p = p_1p_2\cdots p_n$, with $p_i = p(i)$ denoting its ith entry.

Let $\pi \in NC_n^A$. We construct the 132-avoiding permutation $p = f(\pi)$ corresponding to it as follows. Let k be the largest element of π which is in the same block of π as 1. Put the entry n of p in the kth position, i.e., set $p_k = n$. As p is 132-avoiding, this implies that the entries larger than $n-k$ are on the left of n in p, and the entries smaller than or equal to $n-k$ are on the right of n. Delete k from π and apply this procedure recursively, with obvious minor adjustments, to the restrictions of π to the sets $\{1,\ldots,k-1\}$ and $\{k+1,\ldots,n\}$, which are also noncrossing partitions. Namely, if j is the largest element in the same block as $k+1$, we set $p_j = n-k$, so that the restriction π_1 of π to $\{k+1,k+2,\ldots,n\}$ yields a 132-avoiding permutation of $\{1,2,\ldots,n-k\}$ placed on the right of n in $p = f(\pi)$. Similarly, if in the restriction π_2 of π to the set $\{1,2,\ldots,k-1\}$ the largest element in the same block as 1 is equal to j, we set $p_j = n-1$. Thus, recursively, π_2 yields a 132-avoiding permutation which we realize on the set $\{n-k+1, n-k+2, \ldots, n-1\}$ and we place it to the left of n in $p = f(\pi)$. In other words, with a slight abuse of notation, $f(\pi)$ is the concatenation of $f(\pi_2)$, n, and $f(\pi_1)$, where $f(\pi_2)$ permutes the set $\{n-k+1, n-k+2, \cdots, n-1\}$ and $f(\pi_1)$ permutes the set $[n-k]$.

To see that this is a bijection note that we can recover the maximum of the block containing the element 1 from the position of the entry n in p, and then proceed recursively.

Example 1. If $\pi = (\{1,4,6\},\{2,3\},\{5\},\{7,8\})$, then $f(\pi) = 64573812$.

Definition 1. *Let $p = p_1p_2\cdots p_n$ be an n-permutation. We say that $i \in [n-1]$ is a* descent *of p if $p_i > p_{i+1}$. The set of all descents of p is called the* descent set *of p and is denoted* $\mathrm{Des}(p)$.

Now we are in a position to define the poset P_n^A of 132-avoiding permutations we want to study.

Definition 2. *Let p and q be two 132-avoiding n-permutations. We say that $p < q$ in P_n^A if* $\mathrm{Des}(p) \subset \mathrm{Des}(q)$.

Clearly, P_n^A is a poset as inclusion is transitive. The Hasse diagram of P_4^A is shown in Figure 1.

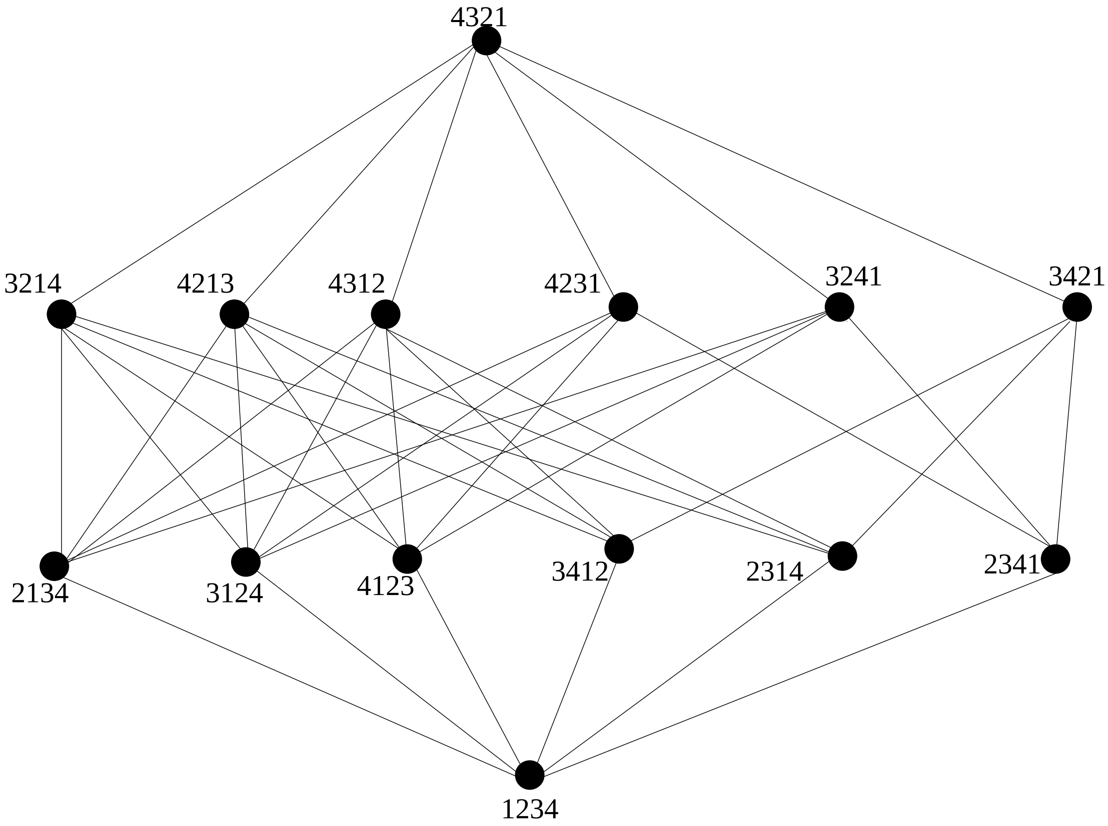

FIGURE 1. The Hasse diagram of P_4^A

Observation 1. *In a 132-avoiding permutation, i is a descent if and only if p_{i+1} is smaller than every entry on its left. Such an element is called a* left-to-right minimum. *So $p < q$ in P_n^A if and only if the set of positions of left-to-right minima in p is a proper subset of the set of positions of left-to-right minima in q.*

Proposition 1. *The bijection f has the following property: Let $i \geq 1$. Then $i \in \mathrm{Des}(f(\pi))$ if and only if $i+1$ is the smallest element of its block in $\pi \in NC_n^A$.*

2.2. **Properties of P_n^A.** Proposition 1 implies that P_n^A is isomorphic to the order on noncrossing partitions in which $\pi < \pi'$ if the set of minima of the blocks of π' is contained in the set of minima of the blocks of π. This yields the first result of this section.

Theorem 1. *The lattice of noncrossing partitions NC_n^A is a subposet of P_n^A.*

Proof. We show that our bijection f is an order-reversing map $NC_n^A \to P_n^A$. The conclusion then follows from the self-duality of the lattice of noncrossing partitions. Suppose $\pi < \tau$ in NC_n^A. This means π is a finer partition than τ, so every element which is the minimum of its block in τ is also the minimum of its block in π. By Proposition 1 this implies $\mathrm{Des}(f(\tau)) \subset \mathrm{Des}(f(\pi))$, so $f(\pi) > f(\tau)$ in P_n^A. $\square$

Clearly, P_n^A is a ranked poset (with rank function $\mathrm{rk}_{P_n^A}(p) = \#\mathrm{Des}(p)$), and we have $\mathrm{rk}_{NC_n^A}(\pi) = n - 1 - \mathrm{rk}_{P_n^A}(f(\pi))$.

Corollary 1. *The poset P_n^A is rank-symmetric, rank-unimodal and strongly Sperner, and its rank generating function is equal to that of NC_n^A.*

We now turn to showing that P_n^A is self-dual. For $S \subseteq [n-1]$, let $Perm_n(S)$ denote the number of 132-avoiding n-permutations with descent set S.

Lemma 1. *Let S be any subset of $[n-1]$ and let $\alpha(S)$ denote its "reverse complement," that is, $i \in \alpha(S) \Longleftrightarrow n-i \notin S$. Then $Perm_n(S) = Perm_n(\alpha(S))$.*

Proof. We use induction on n. For $n = 1, 2, 3$ the statement is true. Now suppose we know it for all positive integers smaller than n. Denote by t the smallest element of S, and let p be a 132-avoiding n-permutation whose descent set is S.

(1) Suppose that $t > 1$. Then we have $p_1 < p_2 < \cdots < p_t$ and, because p avoids the pattern 132, the values of $p_1, p_2, \ldots, p_t$ are *consecutive* integers. So, for given values of p_1 and t, we have only one choice for $p_2, p_3, \ldots, p_t$. This implies

$$Perm_n(S) = Perm_{n-(t-1)}(S - (t-1)), \tag{1}$$

where $S - (t-1)$ is the set obtained from S by subtracting $t-1$ from each of its elements.

On the other hand, we have $n-t+1, n-t+2, \cdots, n-1 \in \alpha(S)$, meaning that in any permutation q counted by $Perm_n(\alpha(S))$ the chain of inequalities $q_{n-t+1} > q_{n-t+2} > \cdots > q_n$ holds. To avoid forming a 132-pattern in q, we must have $(q_{n-t+2}, \ldots, q_n) = (t-1, t-2, \ldots 1)$. Therefore,

$$Perm_n(\alpha(S)) = Perm_{n-(t-1)}(\alpha(S)|n-(t-1)) \tag{2}$$

where $\alpha(S)|n-(t-1)$ denotes the set obtained from $\alpha(S)$ by removing its last $t-1$ elements. Clearly, $Perm_{n-(t-1)}(S-(t-1)) = Perm_{n-(t-1)}(\alpha(S)|n-(t-1))$ by the induction hypothesis, so equations (1) and (2) imply $Perm_n(S) = Perm_n(\alpha(S))$.

(2) If $t = 1$, but $S \neq [n-1]$, then let u be the smallest index which is *not* in S. Then again, to avoid forming a 132-pattern, the value of p_u must be the smallest positive integer a which is larger than p_{u-1} and is not equal to any p_i for $i \leq u-1$. So again, we have only one choice for p_u. On the other hand, the largest index in $\alpha(S)$ will be $n-u$. Therefore, in permutations q counted by $Perm_n(\alpha(S))$, we must have $q_{n-u} = 1$ as Observation 1 implies that q_{n-u} must be the rightmost left-to-right minimum in such permutations, and that is always the entry 1.

In order to use this information to reduce our permutations in size, we define $S' \subset [n-2]$ as follows: $i \in S'$ if and only if either $i < u$ and then, by the definition of u, $i \in S$, or $i > u$ and $i+1 \in S$. In other words, we decrease elements larger than u by 1; intuitively, we remove u from $[n-1]$, and translate the interval on its right one notch to the left. If we now take $\alpha(S')$, that will consist of entries j so that $j < n-u$ and $(n-1)-(j-1) = n-j \notin S$. So in other words, we simply remove $n-u$ from $[n-1]$ (there has been nothing on the right of $n-u$ in $\alpha(S)$ to translate). Note that the size of $\alpha(S)$ decreases with this operation as $n-u \in \alpha(S)$. As we have seen in the previous paragraph, we had only one choice for p_u and p_{n-u}, so removing them this way does not change the number of permutations with a given descent set. Thus we have $Perm_n(S) = Perm_{n-1}(S')$, and also $Perm_n(\alpha(S)) = Perm_{n-1}(\alpha(S'))$. By induction hypothesis, the right hand sides of thes two equations agree, and therefore the left hand sides must agree, too.

Example 2. If $n = 8$ and $S = \{1, 6\}$, and so $\alpha(S) = \{1, 3, 4, 5, 6\}$, then $u = 2$, $n-u = 6$, and indeed, $S' = \{1, 5\}$ and $\alpha(S') = \{1, 3, 4, 5\}$.

(3) Finally, if $S = [n-1]$, then the statement is trivially true as $Perm_n(S) = Perm_n(\alpha(S)) = 1$.

So we have seen that $Perm_n(S) = Perm_n(\alpha(S))$ in all cases. ◻

It is now easy to verify that the reverse complementation of the descent set can be used to construct an anti-automorphism of P_n^A.

Theorem 2. *The poset P_n^A is self-dual.*

2.3. **A poset derived from excedences.** It is shown in [10] that the joint distribution of the excedence and Denert statistics on 321-avoiding permutations agrees with the joint distribution of the block and rb statistics on noncrossing partitions. This suggests the definition of the poset Q_n^A consisting of the 321-avoiding n-permutations ordered by containment of the set of excedences, and invites the question of how Q_n^A compares with the poset P_n^A.

A permutation σ has an *excedence* at i if $\sigma(i) > i$. For example, the *excedence set* of $\sigma = 32514$ is $\mathrm{Exc}(\sigma) = \{1, 3\}$. Let $\mathrm{exc}(\sigma)$ denote the number of excedences of σ. Following [10], there is a bijection θ from NC_n^A to 321-avoiding n-permutations such that $\mathrm{exc}(\theta(\pi)) = \mathrm{bk}(\pi) - 1$. Namely, if the set of minima of the blocks of $\pi \in NC_n^A$, omitting the block containing 1, is $\{f_2 < \cdots < f_k\}$ and the set of maxima of the blocks, again, omitting the block containing 1, is $\{l_2 < \cdots < l_k\}$, then let $\theta(\pi)$ be the permutation whose value at $f_i - 1$ is l_i for $i = 2, 3, \ldots, k$, and whose other values constitute an increasing subsequence in the remaining positions. For instance, if $\pi = \{1, 5, 7\}\{2\}\{3, 4\}\{6\}\{8, 10\}\{9\} \in NC_{10}^A$, then we have $(f_2, \ldots, f_6) = (2, 3, 6, 8, 9)$ and $(l_2, \ldots, l_6) = (2, 4, 6, 9, 10)$, and we obtain $\theta(\pi) = 2\ 4\ 1\ 3\ 6\ 5\ 9\ 10\ 7\ 8$.

Recall from [10] that the set of excedences of $\theta(\pi)$ is precisely $\{f_2 - 1, f_3 - 1, \ldots, f_k - 1\}$. Similarly to the case of descents discussed for 132-avoiding permutations, a covering relation $\pi < \pi'$ in NC_n^A corresponds to the deletion of an excedence: $\mathrm{Exc}(\theta(\pi')) = \mathrm{Exc}(\theta(\pi)) - \{i\}$, for a suitable $i \in \mathrm{Exc}(\theta(\pi))$. Hence, taking advantage of the self-duality of NC_n^A, one can establish directly that the poset Q_n^A enjoys the same properties as P_n^A: There is an embedding of NC_n^A into the poset Q_n^A of 321-avoiding n-permutations ordered by containment of the set of excedences; the embedding is rank-preserving and Q_n^A is a strongly Sperner poset.

The fact that the posets P_n^A and Q_n^A have strongly similar properties is not accidental.

Proposition 2. *The posets P_n^A and Q_n^A are isomorphic.*

Proof. For each $S \subseteq [n-1]$, let $E_n^{321}(S)$ be the set of 321-avoiding n-permutations with excedence set $S \subseteq [n-1]$.

Let also $D_n^{132}(\alpha(S))$ be the set of 132-avoiding n-permutations with descent set equal to $\alpha(S)$, the reverse-complement of S. Thus, in the notation of the previous subsection, the cardinality of $D_n^{132}(\alpha(S))$ is $Perm_n(\alpha(S))$.

We construct a bijection $s\colon E_n^{321}(S) \to D_n^{132}(\alpha(S))$ (illustrated by example 3). If $p \in E_n^{321}(S)$, then, as seen earlier in the definition of θ, the entries p_j with $j \notin S$ form an increasing subsequence. This, and the definition of excedence imply that p_j is a *right-to-left minimum* (that is, smaller than all entries on its right) if and only if $j \notin \mathrm{Exc}(p) = S$.

Now let $p' = p_n p_{n-1} \cdots p_1$ be the reverse of p. Then p' is a 123-avoiding permutation having a left-to-right minimum at position $i \leq n$ exactly if $n + 1 - i \notin S$.

There is exactly one 132-avoiding permutation p'' which has this same set of left-to-right minima at these same positions [7]. Namely, p'' is obtained by keeping the left-to-right minima of p' fixed, and successively placing in the remaining positions, from left to right, the smallest available element which does not alter the left-to-right minima. We set $s(p) = p''$. Observation 1 then tells us that $i \in \mathrm{Des}(p'')$ if and only if $n - i \notin S$, in other words, when $i \in \alpha(S)$, and so p'' belongs indeed to $D_n^{132}(\alpha(S))$.

It is easy to see that s is invertible. Clearly, p' can be recovered from p'' as the only 123-avoiding permutation with the same values and positions of its left-to-right minima as p''. (All entries which are not left-to-right minima are to be written in decreasing order). Then p can be recovered as the reverse of p'.

The bijections $s\colon E_n^{321}(S) \to D_n^{132}(\alpha(S))$ for all the choices of $S \subseteq [n-1]$ produce an order-reversing bijection from Q_n^A to P_n^A. But P_n^A is self-dual, so the proof is complete. □

Example 3. Take $p = 2\ 4\ 1\ 6\ 3\ 5\ 9\ 10\ 7\ 8 \in E_{10}^{321}(S)$ for $S = \{1,2,4,7,8\}$. Then its reversal $p' = 8\ 7\ 10\ 9\ 5\ 3\ 6\ 1\ 4\ 2$ has left-to-right minima 8, 7, 5, 3, 1 in positions 1, 2, 5, 6, 8. We obtain $s(p) = p'' = 8\ 7\ 9\ 10\ 5\ 3\ 4\ 1\ 2\ 6$, a permutation in $D_{10}^{132}(\{1,4,5,7\})$.

3. The type-B case

3.1. **The type-B noncrossing partitions.** The hyperplane arrangement of the root system of type B_n consists of the hyperplanes with equations $x_i = \pm x_j$ for $1 \le i < j \le n$ and the coordinate hyperplanes $x_i = 0$, for $1 \le i \le n$. The subspaces of $\mathbf{R}^n$ arising as intersections of hyperplanes from among these can be encoded by partitions of $\{1,2,\ldots,n,\overline{1},\overline{2},\ldots,\overline{n}\}$ satisfying the following properties: i) if $\{a_1,\ldots,a_k\}$ is a block, then $\{\overline{a_1},\ldots,\overline{a_k}\}$ is also a block, where the bar operation is an involution; and ii) there is at most one block, called *the zero-block*, which is invariant under the bar operation. The collection of such partitions are the *type-B partitions of* $[n]$. If $1,2,\ldots,n,\overline{1},\overline{2},\ldots,\overline{n}$ are placed around a circle, clockwise in this order, and if cyclically successive elements of the same block are joined by chords drawn inside the circle, then, following [6], the class of *type-B noncrossing partitions*, denoted NC_n^B, is the class of type-B partitions of $[n]$ which admit a circular diagram with no crossing chords. Alternatively, a type-B partition is noncrossing if there are no four elements a,b,c,d in clockwise order around the circle, so that a,c lie in one block and b,d lie in another block of the partition. The total number of type-B noncrossing partitions of $[n]$ is $\binom{2n}{n}$ (see [6]). As in the case of type A, the refinement order on type-B partitions yields a geometric lattice (in fact, isomorphic to a Dowling lattice with an order-2 group), and the noncrossing partitions constitute a sub-meet-semilattice as well as a lattice in its own right. As a poset under the refinement order, NC_n^B is ranked, with $\mathrm{rk}(\pi) = n - \#($ of pairs of non-zero blocks $)$. For example, $\pi = \{1,\overline{3},\overline{5}\},\{\overline{1},3,5\},\{4\},\{\overline{4}\},\{2,\overline{2}\}$ is an element of NC_5^B having 2 pairs of non-zero blocks and its rank is equal to 3. The rank-sizes in NC_n^B are given by $\left(\binom{n}{k}^2\right)_{0\le k\le n}$ (see [6]).

The numerous properties of NC_n^A which also hold for NC_n^B (as shown in [6]), establish the latter as a natural B-analogue. In particular, NC_n^B is a self-dual, rank-unimodal, strongly Sperner poset, analogously to the properties of NC_n^A of concern in Section 2. We now turn to a type-B counterpart of the restricted permutations.

3.2. **A class of pattern-avoiding signed permutations.** We will view the elements of the hyperoctahedral group B_n as signed permutations written as words of the form $b = b_1b_2\ldots b_n$ in which each of the symbols $1,2,\ldots,n$ appears, and may or may not be barred. Thus, the cardinality of B_n is $n!2^n$. To find a B-analogue of the poset P_n^A, we need a subset of B_n whose cardinality is $\#NC_n^B = \binom{2n}{n}$, which is characterized via pattern-avoidance, and over which the distribution of the descent statistic agrees with the distribution across ranks of the type-B noncrossing partitions of $[n]$. Such a class of signed permutations is $B_n(12,\overline{2}\,\overline{1})$ which appears in [9].

Consider the elements of B_n which avoid simultaneously the patterns 21 and $\overline{2}\,\overline{1}$. That is, the set of elements $b = b_1b_2\cdots b_n \in B_n$ such that there are no indices $1 \le i < j \le n$ for which i) either both b_i, b_j are barred, or neither is barred, and ii) $|b_i| > |b_j|$ (the absolute value of a symbol means $|a| = a$ if a is not barred, and $|a| = \overline{a}$ if a is barred; effectively, the absolute value removes the bar from a barred symbol). The following is immediate: a $(21,\overline{2}\,\overline{1})$-avoiding permutation in B_n is a shuffle of an increasingly ordered subset L of $[n]$ whose elements we then bar, with its increasingly ordered complement in $[n]$. For example, $b = \overline{2}13\overline{5}4\overline{6}7$ is one of $\binom{7}{3}$ elements of $B_7(21,\overline{2}\,\overline{1})$ associated with the subset $L = \{2,5,6\} \subseteq [7]$. Obviously, summing over the choices of L of cardinality ranging from zero to n and over the shuffles, it follows that

$$\#B_n(21,\overline{2}\,\overline{1}) = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} = \#NC_n^B, \tag{3}$$

as desired.

Furthermore, the distribution of descents over $B_n(21, \overline{2}\,\overline{1})$ is as desired. We say that $b = b_1 b_2 \dots b_n \in B_n$ has a *descent* at i, for $1 \le i \le n-1$, if $b_i > b_{i+1}$ with respect to the total ordering $1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}$, and that it has a descent at n if b_n is barred. As usual, the *descent set* of b, denoted $\mathrm{Des}(b)$, is the set of all $i \in [n]$ such that b has a descent at i. For example, for $b = \overline{2}1\overline{3}\overline{5}47\overline{6}$ we have $\mathrm{Des}(b) = \{1, 3, 4, 7\}$. It is then transparent that if $b \in B_n(21, \overline{2}\,\overline{1})$, then its descent set is precisely the set of positions occupied by barred symbols. In conclusion,

Observation 2. *For an element b of the hyperoctahedral group B_n, let $L(b)$ denote the set of symbols which are barred in b, and* $\mathrm{Des}(b)$ *denote the descent set of b. Then the map $b \mapsto (L(b), \mathrm{Des}(b))$ gives a bijection between the class of restricted signed permutations $B_n(21, \overline{2}\,\overline{1})$ and ordered pairs of subsets of $[n]$ of equal cardinality.*

3.3. **The poset P_n^B.** As the B-analogue of the poset of 132-avoiding permutations P_n^A of the preceding section, we consider the poset P_n^B consisting of the $(21, \overline{2}\,\overline{1})$-avoiding elements of the hyperoctahedral group B_n, with the order relation given by $b < b'$ if and only if $\mathrm{Des}(b) \subset \mathrm{Des}(b')$.

Based on the preceding discussion and an encoding of type-B noncrossing partitions appearing in [6], one readily obtains the properties of P_n^B which parallel those of P_n^A.

Theorem 3. *The poset P_n^B of $(12, \overline{2}\,\overline{1})$-avoiding elements of the hyperoctahedral group B_n, ordered by containment of the descent set, is an extension of the refinement order on the type-B noncrossing partition lattice NC_n^B. The poset P_n^B has the same rank-generating-function as NC_n^B, therefore it is rank-symmetric and rank-unimodal, and it is a self-dual and strongly Sperner poset.*

Proof. It is immediate from its definition and Observation 2 that P_n^B is a ranked poset (namely, $\mathrm{rk}(b) = \#\mathrm{Des}(b)$) and has rank-sizes given by $\left(\binom{n}{k}^2\right)_{0 \le k \le n}$, equal to the rank-sizes in NC_n^B. Also, P_n^B is a self-dual poset: clearly, if b' is the $(21, \overline{2}\,\overline{1})$-avoiding signed permutation which corresponds to the pair $([n] - L(b), [n] - \mathrm{Des}(b))$, then the mapping $b \leftrightarrow b'$ is an order-reversing involution on P_n^B.

Toward checking that there is an order-preserving bijection from NC_n^B to P_n^B, we first recall a fact from [6]: every partition $\pi \in NC_n^B$ can be encoded by a pair $(L(\pi), R(\pi))$ of subsets of $[n]$ whose cardinality is the number of pairs of non-zero blocks of π. Informally, these sets consist of the Left and Right delimiters of non-zero blocks when the elements are read in clockwise order (in the circular diagram of π). More precisely, if $n = 0$ or if π has only a zero-block, we set $L = R = \emptyset$. Otherwise, $\pi \in NC_n^B$ has some non-zero block consisting of cyclically consecutive elements in its diagram. If such a block consists of $j_1, j_2, \dots, j_k$ in clockwise order, then $|j_1|$ belongs to $L(\pi)$ and $|j_k|$ belongs to $R(\pi)$. By deleting this block and its image under barring, a type-B noncrossing partition of $[n-k]$ is obtained and the construction of the sets $L(\pi)$ and $R(\pi)$ is completed by repeating this process as long as non-zero blocks arise. For instance, if $\pi = \{1, \overline{3}, \overline{5}\}, \{\overline{1}, 3, 5\}, \{4\}, \{\overline{4}\}, \{2, \overline{2}\}$, then $L(\pi) = \{3, 4\}$ and $R(\pi) = \{1, 4\}$.

Now suppose that $\pi < \pi'$ in NC_n^B, and that this is a covering relation (i.e., $\mathrm{rk}(\pi') = \mathrm{rk}(\pi) + 1$). Then there exist $l \in L(\pi)$ and $r \in R(\pi)$ such that $L(\pi') = L(\pi) - \{l\}$ and $R(\pi') = R(\pi) - \{r\}$, as a result of the merging of blocks entailed by the covering relation. Thus it is clear that if $\pi \in NC_n^B$ is mapped to the signed permutation $b \in P_n^B$ with the property that $(L(b), \mathrm{Des}(b)) = (L(\pi), R(\pi))$, then one obtains an order-reversing embedding of NC_n^B into P_n^B. Combining this with the self-duality of P_n^B we obtain the desired embedding of NC_n^B into P_n^B.

Finally, the strong Sperner property of P_n^B follows as in type A, from the strong Sperner property of NC_n^B (see [6]) and the rank-preserving embedding of NC_n^B into P_n^B. ☐

3.4. **A poset based on type-B excedences.** As in the type-A case, there is a self-dual poset of $\#NC_n^B$ restricted signed permutations ordered by containment of the set of excedences. In fact, there is more than one definition of the excedence statistic in the literature, in the case of the hyperoctahedral group. We mention two possibilities [13].

Given a signed permutation b, let k be the number of symbols which are *not* barred in b. We associate to b an $(n+1)$-permutation $\sigma(b)$ by setting $\sigma(b)_{n+1} = k+1$ and, for $1 \le i \le n$, letting $\sigma(b)_i = j$ if b_i is the jth smallest among the symbols $b_1, b_2, \ldots, b_n, n+1$ with respect to the linear ordering $1 < 2 < \cdots < n < n+1 < \overline{1} < \overline{2} < \cdots < \overline{n}$. For example, if $b = 1\ \overline{3}\ 2\ 4\ 5\ \overline{6}\ \overline{8}\ 7$, then $\sigma(b) = 1\ 7\ 2\ 3\ 4\ 8\ 9\ 5\ 6$. Now, the excedence set of b is defined to be that of $\sigma(b)$. It turns out that for $b \in B_n(21, \overline{2}\ \overline{1})$ this definition makes the excedence set coincide with the descent set for each b. Therefore, this leads to the poset P_n^B again.

An alternative definition for excedences of "indexed permutations" appears in [13]. Specialized to the hyperoctahedral group it is the following.

Definition 3. *If $b \in B_n$, its* excedence set *is the union of the sets $S(b)$ and $F(b)$, where $S(b)$ is the set of excedences computed in the symmetric group for the permutation $|b_1||b_2|\ldots|b_n|$ obtained by removing all bars from the symbols in b, and $F(b) = \{i\ :\ b_i = \overline{i}\}$, the set of barred fixed points of b.*

Thus, for $b = 1\ \overline{3}\ 2\ 4\ 5\ \overline{6}\ \overline{8}\ 7$ we obtain excedences at $\{2,6,7\}$ by either of the two definitions. But $b = \overline{1}3\overline{2}$ has excedences at $\{1,3\}$ by the first definition (based on $\sigma(b) = 3\ 1\ 4\ 2$), and $\{1,2\}$ if the second definition is adopted.

Proposition 3. *Let Q_n^B denote the poset of $(21, \overline{2}\ \overline{1})$-avoiding signed permutations in B_n, ordered by containment of their excedence set. The poset Q_n^B is self-dual.*

By [13], the rank generating function of Q_n^B is equal to that of P_n^B. Therefore it is natural to ask whether the posets P_n^B and Q_n^B are isomorphic, just as their type-A counterparts are (Proposition 2). The answer in this case is negative. Indeed, if $n = 3$ it is straightforward to verify that all atoms of P_3^B are covered by six elements, while the atom $\overline{1}\ 2\ 3$ of Q_3^B is covered by seven elements (namely, $\overline{1}\ \overline{2}\ 3, \overline{1}\ 2\ \overline{3}, \overline{1}\ 3\ \overline{2}, 2\ 3\ \overline{1}, \overline{2}\ \overline{3}\ 1, 2\ \overline{1}\ \overline{3}$, and $\overline{2}\ 1\ \overline{3}$).

4. Remarks and questions for further investigation

(1) *Is NC_n^B a subposet of Q_n^B?* We do not know whether the lattice of type-B noncrossing partitions can be embedded in the poset Q_n^B of $(21, \overline{2}\ \overline{1})$-avoiding signed permutations ordered by their excedence set of definition 3.

(2) *Self-duality of NC_n^A and NC_n^B extending to self-duality for P_n^A and P_n^B.*

We have seen that *each* of the posets NC_n^A and P_n^A is self-dual and that NC_n^A is a subposet of P_n^A. The same is true for the pair NC_n^B, P_n^B. Both for type A and for type B one can exhibit an order-reversing involution on the larger poset which restricts to an order-reversing involution on the smaller one.

We first construct such an involution g for NC_n^A which will be similar, though not identical, to the involution defined in [8].

Write the elements $1, 2, \cdots, n$ clockwise around a circle, and write elements $1', 2', \cdots, n'$ interlaced in counterclockwise order, so that $1'$ is between 1 and n, $2'$ is between n and $n-1$, and so on, i' is between $n+2-i$ and $n+1-i$. For $\pi \in NC_n^A$, join by chords – as usual – cyclically successive (unprimed) elements belonging to the same block of π. Then define $g(\pi)$ to be the coarsest noncrossing partition on the elements $1', 2', \cdots, n'$ so that the chords joining primed elements of the same block do not intersect the chords of π. See Figure 2 for an example.

The map g is certainly a bijection, and it is order-reversing in NC_n^A since merging two blocks of π subdivides a block of $g(\pi)$. We claim that g is also order-reversing on P_n^A. To see this, observe that for any $i > 1$, the element i is the smallest in its block in π if and only if the element $(n+2-i)'$ is *not* the smallest in its block in $g(\pi)$. Indeed, the definition of g implies that exactly one of i and $(n+2-i)'$ can be connected to smaller elements by a chord. Therefore, g takes the set of block-minima (not equal to 1) of π into its reverse complement in $[n+2]$, so g is indeed order-reversing on P_n^A.

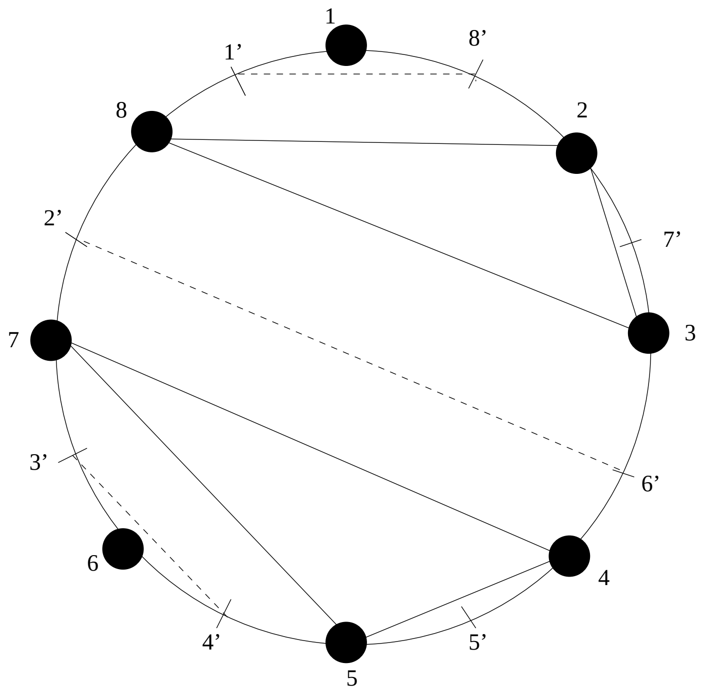

FIGURE 2. The partition $\pi = (\{1\}, \{2, 3, 8\}, \{4, 5, 7\}, \{6\})$ and its image $g(\pi)$

In the type-B case, one can obtain an analogous bijection h in a similar way: take the circular (clockwise) representation of π, then write the elements $1', 2', \cdots, n', \bar{1}', \bar{2}', \cdots, \bar{n}'$ so that the primed numbers interlace the unprimed, placing $1'$ between 1 and $\bar{n}$ and continuing counterclockwise. For $\pi \in NC_n^B$, define $h(\pi)$ as above, that is, as the unique coarsest partition on the primed set whose chords do not cross those of π.

Added in Proof At the end of our joint work with Rodica, we asked what other combinatorial objects admitted a natural partial order which is self-dual and possibly, had other nice properties. A natural candidate was the class of *two-stack sortable permutations* [14]. It is known [3] that there are as many of them with k descents as with $n - 1 - k$ descents. However, the poset obtained by the descent ordering is not self-dual, even for $n = 4$, so another ordering, and another duality map was needed. Present author has recently found [1] such a duality map for not just 2-stack sortable, but in general, for t-stack sortable permutations. The obtained posets are not only rank-symmetric, but also rank-unimodal. So Rodica's research interests are very much alive...

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