COMBINATORIAL AND GEOMETRIC PROPERTIES OF GRAPH VARIETIES

JEREMY L. MARTIN

ABSTRACT. A *configuration variety* is an algebraic variety whose points correspond to tuples of vector spaces with certain intersection conditions. Classical examples include the Grassmannian, flag, and Schubert varieties, all of which have been studied extensively by combinatorial means. Given a graph G, we construct and study a configuration variety $\mathcal{V}(G)$, the *graph variety*, whose points correspond to "pictures" of G in a projective plane. The structure of $\mathcal{V}(G)$ depends heavily on the theory of *combinatorial rigidity* of graphs [2]. In addition, a number of patterns emerge when we consider the initial ideal of $\mathcal{V}(G)$ as the Stanley-Reisner ring [1], [5] of a certain simplicial complex.

RÉSUMÉ. Un *variété de configuration* est une variété algébrique dont les points correspondent aux multiples des espaces vectoriel à certains conditions d'intersection. Des exemples classiques incluent les variétés de Grassmann, les variétés des drapeaux, et les variétés de Schubert, qui ont été étudiées intensivement par des moyens combinatoires. Donné un graphe G, nous construisons et étudions une variété de configuration $\mathcal{V}(G)$, le *variété du graphe*, dont les points correspondent des "pictures ' de G dans un plan projectif. La structure de $\mathcal{V}(G)$ dépend fortement de la théorie de *rigidité combinatoire* des graphes [2]. En outre, beaucoup de phénomènes émergent quand nous considérons l'idéal initial de $\mathcal{V}(G)$ être l'anneau Stanley-Reisner [1], [5] d'un certain complexe simplicial.

1. INTRODUCTION

A *configuration variety* is an algebraic variety whose points correspond to tuples of vector spaces with certain intersection conditions. The geometry of configuration varieties is closely connected to combinatorics; classical examples include the Grassmannian, flag, and Schubert varieties. Our research here focuses on a class of configuration varieties, related to graphs, which appear to have a rich combinatorial structure.

We begin in Section 2 by constructing varieties $\mathcal{X}(G)$ and $\mathcal{V}(G)$ whose points correspond to "pictures" of a graph G built out of lines and points in a projective plane. Briefly, $\mathcal{X}(G)$ is the set of all pictures of G, while $\mathcal{V}(G)$, the *graph variety*, is the most important irreducible component of $\mathcal{X}(G)$. The structure of the ideal determining $\mathcal{V}(G)$ depends to a large extent on the theory of *combinatorial rigidity*. We give a brief summary of rigidity theory in Section 3 (a complete treatment may be found in [2]) and, in Section 4 study how it helps relate combinatorial properties of G to algebraic properties of $\mathcal{V}(G)$. In the final section, we describe a number of patterns that appear when we consider the initial ideal of $\mathcal{V}(G)$ as the Stanley-Reisner ring of a certain simplicial complex. This view leads to strong conjectures about the geometry of $\mathcal{V}(G)$ and also offers a promi sing route to their proofs.

All algebra and algebraic geometry is assumed to be over an algebraically closed field k of characteristic $\neq 2$. All graphs are undirected, without loops or multiple edges. A graph with vertex set V and edge set E will be written $G = (V, E)$. The symbol $[n]$ denotes the set $\{1, \ldots, n\}$. K_n denotes the complete graph with vertex set $[n]$.

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2. Basic Definitions

Let $G = (V, E)$ be an undirected graph without loops or multiple edges, with $|V| = n$, $|E| = r$. Our object is to study G by defining an associated variety of "pictures," in which the vertices and edges of G are thought of as points and lines in a plane, respectively, and assigned coordinates.

We associate with each vertex $v \in V$ a copy P_i of the projective plane $\mathbb{P}^2_k$, and with each edge $e = \{v, w\} \in E$ a copy $L_e = L_{vw}$ of its dual $(\mathbb{P}^2_k)^*$. (While $\mathbb{P}^2_k$ and its dual are geometrically isomorphic, we wish to emphasize that an element of L_e is to be considered as a line in the projective plane.) Define

$$\mathbb{P}^2_k(G) = \prod_{v \in V} P_v \times \prod_{e \in E} L_e.$$

Definition 1. A *picture* of G is a point

$$(\mathbf{p}, \mathbf{l}) = (p_1, \ldots, p_n, l_1, \ldots, l_r) \in \mathbb{P}^2_k(G)$$

such that $p_v \in l_e$ whenever $v \in e$. The set of all pictures of G is called the *incidence variety* of G and denoted $\mathcal{X}(G)$.

Note that points and lines in different factors are allowed to coincide. Thus, for instance, four copies of the origin and six copies of the x-axis constitute a picture of K_4. This brings us to the definition of celloids.

Definition 2. Let $\mathcal{S}$ be a set partition of V and $\sim_{\mathcal{S}}$ the associated equivalence relation. The corresponding *celloid* $\mathcal{X}_{\mathcal{S}}(G)$ is then the set of pictures $(\mathbf{p}, \mathbf{l}) \in \mathcal{X}(G)$ such that $p_v = p_w$ iff $v \sim w$. (The terminology is intended to suggest a comparison with the decomposition of the flag variety into Schubert cells.)

Of particular importance is the *discrete partition* $\mathcal{S}_0$ of V into n singleton sets. We call $\mathcal{X}_{\mathcal{S}_0}(G)$ the *generic celloid*, and its elements *generic pictures*. One might say loosely that a picture is generic if a graph theorist would be willing to draw it. We also distinguish the *indiscrete partition* $\mathcal{S}_\infty$, in which there is only one equivalence class, namely V itself. For ease of notation, we put $\mathcal{X}_0(G) = \mathcal{X}_{\mathcal{S}_0}(G)$ and $\mathcal{X}_\infty(G) = \mathcal{X}_{\mathcal{S}_\infty}(G)$.

Since $\mathcal{X}_0(G)$ is defined by inequalities, it is not a closed subset of $\mathcal{X}(G)$. This motivates the following:

Definition 3. The *graph variety* $\mathcal{V}(G)$ of G is the closure of $\mathcal{X}_0(G)$ in $\mathcal{X}(G)$. We note that the coordinates of the lines of any generic picture are determined uniquely by those of its points. Therefore $\mathcal{X}_0(G)$ is isomorphic to a dense open subset of $(\mathbb{P}^2_k)^n$, and $\mathcal{V}(G)$ is irreducible.

The celloids are quasiprojective subvarieties of $\mathcal{X}(G)$, and their dimensions can be computed easily by considering local degrees of freedom. Let $\mathcal{S} = \{S_1, \ldots, S_h\}$. First, each S_i contributes 2 to $\dim \mathcal{X}_{\mathcal{S}}(G)$. Now consider some $e = \{v, w\} \in E$. If $v \not\sim_{\mathcal{S}} w$, then the coordinates of e are uniquely determined by those of v and w for any picture in $\mathcal{X}_{\mathcal{S}}(G)$. On the other hand, if $v \sim_{\mathcal{S}} w$, then e can be any line through $p_i = p_j$, which adds one degree of freedom. Therefore

$$\dim \mathcal{X}_{\mathcal{S}}(G) \;=\; 2r + |\{\{v, w\} \in E \;\mid\; v \sim_{\mathcal{S}} w\}|\,. \tag{1}$$

In particular, $\dim \mathcal{X}_0(G) = 2n$ and $\dim \mathcal{X}_\infty(G) = 2 + r$.

Example 1. $\mathcal{X}(K_2)$ is the blowup of $\mathbb{P}^2_k \times \mathbb{P}^2_k$ along the diagonal $D = \{(p_1, p_2) \mid p_1 = p_2\}$. The blowup map

$$\alpha : \mathcal{X}(K_2) \to \mathbb{P}^2_k \times \mathbb{P}^2_k$$

is just the projection on the vertex coordinates. The generic celloid $\mathcal{X}_0(K_2)$ has dimension 4 and is dense in $\mathcal{X}(K_2)$; thus $\mathcal{V}(K_2) = \mathcal{X}(K_2)$. The indiscrete celloid has dimension 3 and is the exceptional divisor $\alpha^{-1}(D)$ of the blowup.

This example suggests that it might be possible to construct $\mathcal{X}(G)$ by starting with $\mathcal{X}((V, \emptyset)) = (\mathbb{P}^2_k)^n$ and then iteratively blowing up the loci corresponding to the coincidence of the endpoints of each edge. If this were true, then $\mathcal{X}(G)$ would always be irreducible, and $\mathcal{X}_{gen}(G)$ would be a dense subset. However, this is not always the case.

Example 2. Let $G = K_4$. By the formula (1), the indiscrete and discrete celloids each have dimension 8. In particular, neither of these celloids can lie in the closure of the other. Therefore $\mathcal{V}(K_4) \subsetneq \mathcal{X}(K_4)$. Indeed, $\mathcal{V}(K_4)$ and $\mathcal{X}_\infty(K_4)$ are exactly the irreducible components of $\mathcal{X}(K_4)$.

Taken together, these examples suggest that the degeneracy of the indiscrete pictures in $\mathcal{X}(K_4)$ is somehow "worse" than those of $\mathcal{X}(K_2)$. The following questions immediately present themselves: when does $\mathcal{V}(G) = \mathcal{X}(G)$, and when equality does not hold, what additional equations define $\mathcal{V}(G)$ as a subvariety of $\mathcal{X}(G)$? To solve these and other problems, we will need some results from the theory of combinatorial rigidity.

3. Combinatorial Rigidity Theory

In this section, we describe those elements of rigidity theory which are relevant to our investigation of graph varieties. For a detailed account, we refer the reader to [2].

3.1. The Rigidity Matroid.

Fix a graph $G = (V, E)$ with $V = [n]$, so that K_n is the complete graph on V. Imagine building a physical model of a generic picture of G, where the constituent points and lines are represented by "ball joints" and "straight rods" respectively. The rods are considered to be fixed in length, but are free to rotate about the joints.

Intuitively, a graph G is *length-rigid*, or simply *rigid*, if the physical model of any picture in $\mathcal{X}_0(G)$ will "hold its shape." More precisely, G is rigid if the distance between any two points of a picture is determined by the lengths of the edges in E. (In [2], this property is called "generic rigidity", as distinguished from several other types of rigidity, which we will not need to consider here.)

For instance, the "quadrilateral" graph $Q = ([4], \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\})$ is not rigid; there are an infinite number of incongruent rhombuses with the same side length. However, adding the edge $\{\{2, 3\}\}$ produces a rigid graph.

Definition 4. The *rigidity closure* $\bar{E}$ of an edge set $E \subset E(K_n)$ is the set of all edges $e \in E(K_n)$ whose lengths are determined by the lengths of the edges in E.

It can be shown that $E \mapsto \bar{E}$ is a matroid closure operator. Indeed, the induced matroid $\mathcal{M}_n$ on $E(K_n)$ is the *2-dimensional generic rigidity matroid*, the maximal matroid satisfying certain axioms [2, p. 96]. Thus the rigid graphs are precisely the ones whose edges are a spanning set of $\mathcal{M}_n$. The bases of $\mathcal{M}_n$ are characterized as follows:

Theorem 1. *(Laman, 1973) Let $E \subset E(K_n)$. E is a basis of $\mathcal{M}_n$ if and only if*

(1) $|E| = 2n - 3$, *and*
(2) $|F| \leq 2|V(F)| - 3$ *for any* $F \subset E$.

Informally, condition (2) says that the edges of a rigidity basis must be "spread out" enough to hold all vertices in place.

3.2. **Rigidity Cycles.** Of particular importance to our research are the *cycles*of $\mathcal{M}_n$, or minimal dependent subsets. A graph $G = (V, E)$ is called a *rigidity pseudocycle* if E can be partitioned into two disjoint spanning trees on V. A rigidity cycle is then a minimal element in the set of rigidity pseudocycles.

Example 3. The *n-wheel* W_n is the graph with vertex set $[n+1]$ and edges

$$\{\{1, n+1\}, \{2, n+1\}, \ldots \{n, n+1\}\} \cup \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n, 1\}\}. \tag{2}$$

W_n is a rigidity cycle for $n \geq 3$. In fact, $W_3 \cong K_4$ and W_4 are the only rigidity cycles on 4 and 5 vertices, respectively. The 5-wheel is one of four isomorphism classes on 6 vertices other than W_5.

4. Application of Rigidity to Graph Varieties

Fix a graph $G = (V, E)$, with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_r\}$.

4.1. **Affine Coordinates.** Let $A \subset \mathbb{P}^2_k$ be an affine plane, and fix an xy-coordinate system on A. Instead of attempting to work with the obvious multiprojective coordinates on the incidence variety, we restrict our attention to an open subset $\mathcal{X}_a(G)$ consisting of those pictures $(\mathbf{p}, \mathbf{l}) \in \mathcal{X}(G)$ such that each point p_i lies in A and no line l_j is parallel to the y-axis. Nothing is lost by this restriction, since $\mathcal{X}(G)$ is covered by finitely many such sets. Now a picture consists of the following data: for each $v_i \in V$, a point $(x_i, y_i) \in A$, and for each $\{v_i, v_j\} \in E$, a line with slope m_{ij} and y-intercept b_{ij}, subject to the equations

$$y_i = m_{ij} x_i + b_{ij} \qquad \text{and} \qquad y_j = m_{ij} x_j + b_{ij}. \tag{3}$$

One immediate advantage of the affine coordinates is that we can eliminate the variables b_{ij} by simply subtracting the two equations (3). This yields

$$(y_i - y_j) - m_{ij}(x_i - x_j) = 0. \tag{4}$$

We may now regard our "affine incidence variety" $\mathcal{X}_a(G)$ as the locus of points in affine $(2n+r)$-space cut out by the ideal of the ring $k[\mathbf{x}, \mathbf{y}, \mathbf{m}]$ generated by the equations (4).

4.2. **The Slope-Rigidity Matroid.** Since our coordinates on graph varieties measure the slope of edges rather than their length, we are led to an alternate definition of rigidity. Instead of fixing the lengths of edges and letting their slopes vary, we fix the slopes and let the lengths vary. We then repeat the construction of $\mathcal{M}_n$ by defining a closure operator $E \mapsto \tilde{E}$. Thus, a graph is *slope-rigid* if its edge set spans the resulting matroid $\mathcal{M}'_n$.

Theorem 2. (1) $\mathcal{V}(G) = \mathcal{X}(G)$ *if and only if* E *is an independent set of the matroid* $\mathcal{M}_n$.
(2) E *is length-rigid if and only if it is slope-rigid. That is,* $\mathcal{M}_n = \mathcal{M}'_n$.

Sketch of proof.

(1) Since $\mathcal{X}(G)$ is defined by r equations (4) among $2n+r$ variables, each of its irreducible components must have dimension at least $2n$. $\mathcal{V}(G)$ has exactly this dimension. If every celloid other than $\mathcal{X}_0(G)$ has dimension strictly less than $2n$, then $\mathcal{V}(G)$ must be the only irreducible component. On the other hand, any celloid of dimension $\geq 2n$ cannot be contained in $\mathcal{V}(G)$. Using the celloid dimension formula (1), it can be shown that $\mathcal{X}(G)$ contains a nongeneric celloid of dimension $\geq 2n$ if and only if it is rigidity-dependent (we omit the details).

(2) Since $\mathcal{M}'_n$ satisfies the requirements of an *abstract rigidity matroid* [2, p. 87], it is immediate that its independent sets are also independent sets of $\mathcal{M}_n$. The first part of the theorem implies the reverse.

4.3. **Equations Defining $\mathcal{V}(G)$.** By Theorem 2, the minimal graphs for which $\mathcal{V}(G) \subsetneq \mathcal{X}(G)$ are the rigidity cycles.

Let $G = (V, E)$ be a rigidity cycle with $|V| = n$ (so that $|E| = 2n-2$), and $T \subset E$ a spanning tree. Fix some orientation on E. For each $e = \{v, w\} \in E \setminus T$, let $p(T, e)$ be the unique path from v to w consisting of edges in T; this can be considered as a formal linear combination

$$e \equiv \sum_{f \in T} c_{e,f} f$$

with $c_{e,f} \in \{0, -1, 1\}$ for all e, f.

The $n-1$ equations (4) may be written simultaneously as a matrix equation $MX = 0$, where M be the $(n-1) \times (n-1)$ square matrix with rows indexed by T and columns indexed by $E \setminus T$, whose (e, f) entry is

$$c_{e,f}(m_e - m_f)$$

and X is a column vector with an entry $x_v - x_w$ for each $\{v, w\} \in T$.

Example 4. Let $G = K_4$, $T = \{(1 \to 2), (1 \to 3), (1 \to 4)\}$. The T-circuits are

$$\begin{aligned} e = (2 \to 3):&\quad p(T,e) = \{e, -(1 \to 2), (1 \to 3)\}, \\ e = (2 \to 4):&\quad p(T,e) = \{e, -(1 \to 2), (1 \to 4)\}, \\ e = (3 \to 4):&\quad p(T,e) = \{e, -(1 \to 3), (1 \to 4)\}. \end{aligned}$$

The matrix M is therefore

$$\begin{bmatrix} m_{12} - m_{23} & m_{23} - m_{13} & 0 \\ m_{24} - m_{12} & 0 & m_{14} - m_{24} \\ 0 & m_{13} - m_{34} & m_{34} - m_{14} \end{bmatrix}.$$

Indeed, the three equations (4) for K_4 can be written in the form

$$M \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \\ x_1 - x_4 \end{bmatrix}.$$

Theorem 3. *Let $G = ([n], E)$ be a rigidity cycle.*

(1) *The polynomial* $\det M$ *is homogeneous of degree $n-1$, square-free, and independent of the choice of spanning tree T. Thus we are justified in defining*

$$\Delta(G) = \det M$$

up to sign.

(2) *The graph variety $\mathcal{V}(G)$ is the projective hypersurface consisting of the zero set of the ideal $(\Delta(G))$.*

(3) *For any spanning tree* $S \subset E$, *let* $m_S = \prod_{e \in S} m_e$. *Then*

$$\Delta(G) = \sum_S \pm m_S,$$

where S *ranges over all "special spanning trees" of* E, *i.e., spanning trees whose complements are also spanning trees. Moreover, the product of the coefficients of* m_S *and* $m_{E \setminus S}$ *is* $(-1)^{n-1}$.

Sketch of proof.

(1) The first two claims are obvious; the third claim follows because changing the choice of T corresponds to replacing MX to $(MB)(B^{-1}X)$, where B is a "change-of-basis matrix" between the two spanning trees.
(2) Any $P \in \mathcal{X}(G)$ satisfies the equation $MX = 0$; if P is generic, then the vector X has nonzero entries and so $\det M = 0$.
(3) One shows combinatorially that the monomial m_T appears exactly once in the determinant, and, for any $S \subset E$, that all copies of m_S cancel if S is not a special spanning tree.

When $G = W_n$ (2), the matrix M is particularly nice. If we take

$$T = \{\{1, n+1\}, \{2, n+1\}, \ldots \{n, n+1\}\},$$

then M has exactly two nonzero entries in each row, on the main diagonal and the main super-diagonal. Thus expanding $\det M$ produces $2 \cdot 2^n$ monomials. It is not hard to see that the only cancellation involves two copies each of m_T and $m_{E \setminus T}$, yielding the following curious result (which one can verify easily for $W_3 = K_4$):

Corollary 1. W_n *has* $2^{n+1} - 4$ *special spanning trees.*

The construction of $\Delta(G)$ goes through in the case where G is a rigidity pseudocycle. All of the previous facts hold, and in addition:

Theorem 4. *Let* G *be a rigidity pseudocycle. The polynomial* $\Delta(G)$ *is irreducible if and only if* G *is a rigidity cycle.*

Sketch of proof. For the $\Longrightarrow$ direction, it is not hard to see that if $G' \subsetneq G$ is also a pseudocycle, then $\Delta(G') \mid \Delta(G)$. The $\Longleftarrow$ direction is more difficult. Suppose that G is a rigidity cycle and that $\Delta(G) = f_1 f_2$. Since $\Delta(G)$ is squarefree, no edge of G can occur in both f_1 and g_2. For $i = 1, 2$, let V_i be the set of $v \in V$ such that each monomial of f_i contains at least one edge with v as an endpoint. It can be shown that $V = V_1 \cup V_2$ and that $V_1 \cap V_2 = \emptyset$. Next, one shows that for any $e \in E$ and any $v \in e$, there exists a special spanning tree of G such that v is a leaf of T via the edge e. Finally, combining this information, one shows that no edge has one endpoint in each V_i. But then G is disconnected. The only way to resolve the contradiction is if one of the V_i is empty, which means that the factorization was trivial to begin with.

Theorem 5. *Let* $G = (V, E)$ *be any graph and* $\mathrm{Cyc}(G)$ *the set of rigidity cycles which are subgraphs of* G. *Then* $\mathcal{V}(G)$ *is set-theoretically determined by the ideal* $J_G = (\Delta(C) \mid C \in \mathrm{Cyc}(G))$. *That is,* $I(\mathcal{V}(G)) = \sqrt{J_G}$.

Sketch of proof. It is clear that the functions in J_G all vanish on $\mathcal{V}(G)$. On the other hand, given a picture P in the vanishing locus of J_G, it is possible to "perturb" it slightly to produce a generic picture, showing that $P \in \overline{\mathcal{X}_0(G)} = \mathcal{V}(G)$.

5. The Associated Simplicial Complex

Let $G = ([n], E)$ be a graph, and $\mathrm{Cyc}(G)$ the set of rigidity cycles which are subgraphs of G. Let R be the polynomial ring $k[m_e]$ in the slope variables for $e \in E$, and $J \subset R$ the ideal generated by $\{\Delta(C) \mid C \in \mathrm{Cyc}(G)\}$. Fix a term ordering $<$ on R. For instance, we may order the edge variables by

$$m[1,2] < m[1,3] < m[1,4] < \cdots < m[1,n] < m[2,3] < \cdots < m[n-1,n],$$

then use the induced pure lexicographic ordering on monomials of R.

Let J' be the ideal generated by $\{\mathrm{in}(\Delta(C)) \mid C \in \mathrm{Cyc}(G)\}$, and let $B(G)$ be the set of square-free monomials not in J'. One can regard $B(G)$ as a simplicial complex on E, in which a monomial $m_{e_1} \ldots m_{e_s} \in B(G)$ corresponds to the face $\{e_1, \ldots, e_s\}$. Thus R/J' is the *Stanley-Reisner ring* [1, ch. 5] of the $B(G)$.

While the isomorphism type of $B(G)$ depends on the choice of term ordering, a substantial amount of combinatorial data, including the f-vector and h-vector, remains invariant. Our experimental study of these complexes has uncovered a number of surprising patterns, leading to the following conjectures.

Conjecture 1. *For any graph G, $\mathrm{in}(J)$ is generated by $\{\mathrm{in}(\Delta(C)) \mid C \in \mathrm{Cyc}(G)\}$. Equivalently, the $\Delta(C)$ form a Gröbner basis for J.*

The remaining conjectures concern the simplicial complexes constructed from the complete graph K_n. Throughout, let $F(K_n)$ denote the set of *facets*, or maximal elements, of $B(K_n)$.

Conjecture 2. *Each facet is a basis of the matroid $\mathcal{M}_n$. In particular, each facet contains $2n-3$ edges, and $B(K_n)$ is pure of dimension $2n-4$.*

Conjecture 3. $|F(K_n)| = (2n-5)!! = (2n-5)(2n-7)\ldots(3)(1)$.

Conjecture 4. *The h-vector of $B(K_n)$ is $(h_0, h_1, \ldots, h_{n-2})$, where h_j is the number of matchings on $[2n-4]$ with j long pairs, i.e., pairs not of the form $\{i, i+1\}$ (see* [3]*).*

Conjecture 5. *$B(K_n)$ is shellable. In particular, R/J' and $\mathcal{V}(K_n)$ are Cohen-Macaulay* [1, p. 217].

Conjectures 2, 3 and 4 have been verified computationally for $n \leq 7$, and Conjecture 5 for $n \leq 6$. Conjecture 2 is particularly fundamental, since $(2n-5)!!$ is the cardinality of $M(2n-4)$, the set of matchings (partitions into 2-element sets) on $[2n-4]$. A bijection between $F(K_n)$ and $M(2n-4)$ describable in combinatorial terms would be an extremely useful tool in establishing the other conjectures. We have some experimental indications as to how such a bijection may be constructed.

We expect to prove these conjectures soon for K_n, and hope then to extend the results to arbitrary graphs. Our plan for future research is to study how the patterns observed in $B(K_n)$ reappear in the simplicial complexes associated with other graphs, and to explore further connections between the combinatorics of G and the structures of $\mathcal{V}(G)$ and $B(G)$.

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E-mail address: jmartin@euclid.ucsd.edu

Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA