HARUSPICY AND NON-D-FINITE FUNCTIONS

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Abstract. Guttmann and Enting [1] proposed the examination of the anisotropic generating function as a test of the solvability of models of bond animals. In this work we describe a technique for examining some properties of anisotropic generating functions. Using this technique we are able to prove that the anisotropic generating functions of self-avoiding polygons and directed bond animals are not D-finite functions.

Résumé. Guttmann et Enting [1] ont suggéré que l'examen de la série génératrice anisotrope de familles d'animaux de liens pouvait servir de test de solvabilité de ces modèles. Nous décrivons ici une technique permettant d'étudier les propriétés de ces séries anisotropes. Nous démontrons en particulier que les séries anisotropes des polygones auto-évitants et des animaux dirigés de liens ne sont pas holonomes.

1. Introduction

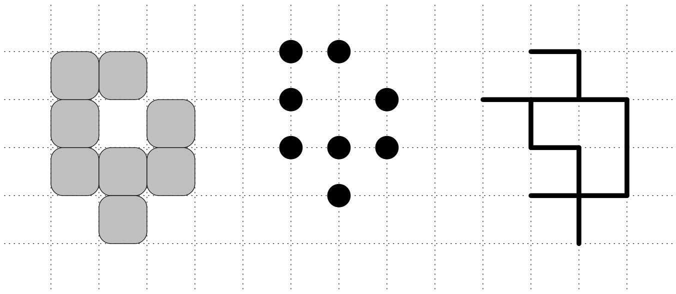

Figure 1. Three basic types of lattice objects (from left to right): polyominoes, site-animals and bond-animals. Polyominoes and site-animals are equivalent under lattice duality; the polyomino and site-animal depicted here are equivalent to one another.

The enumeration of lattice animals is arguably one of the most famous problems in combinatorics, and is of considerable importance in the study of lattice models in statistical physics and theoretical chemistry. Considering the intensive study that these models have been subjected to over their 40+ year history, it is perhaps a little surprising that the number of rigorous results is very small, and the number of models that have been solved exactly (either implicitly or explicitly) smaller still.

There are three basic types of lattice "objects"; polyominoes, site animals and bond animals. A polyomino is a finite connected set of lattice faces (or cells), a site animal is a connected set of lattice vertices (or sites) and a bond animal is a connected set of lattice edges (or bonds); all are defined up to translation (if two animals are equal under some translation, then we consider them to be the same animal). By replacing each cell of a polyomino with a site at its centre, one obtains an equivalent site-animal on the dual lattice and *vice-versa* (see figure 1). In this paper we will only study bond animals on the square lattice. We will write "*animal*" to mean "*square lattice bond animal*".

The fundamental question in the study of animals is

Question 1. *How many animals are there that contain n bonds?*

It is not difficult to compute the answer to this question for small n, either with a pencil and paper, or with a computer and a C compiler. In principle one can do this for any n, but it quickly becomes obvious that the number of animals is growing rapidly with n and so the time it takes to answer the question is also growing rapidly with n. If we denote the number of animals (of a given type) with n bonds by c_n, then one can show using a concatenation argument [2] that the limit

$$\lim_{n\to\infty} (c_n)^{1/n} = \mu$$

exists and so the number of animals grows exponentially with n (*i.e.* $c_n \sim A\mu^n$). So although brute force methods[1] always work, they are not very satisfactory, and so are not normally considered to to be a solution to question 1.

More mathematically appealing are solutions in the form of (non-trivial) expressions for c_n, a recurrence for c_n that can be computed quickly, or (as we will concentrate on here) an expression for the generating function, $\sum c_n x^n$. Let us consider two families of bond animals; self-avoiding polygons and staircase polygons.

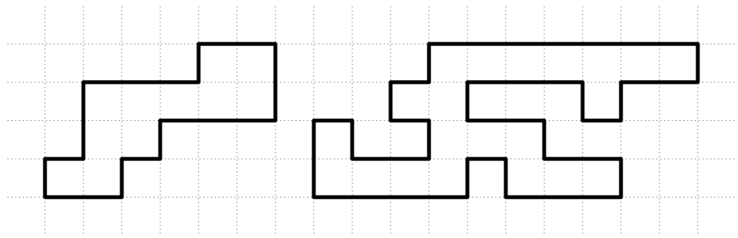

FIGURE 2. A staircase polygon (left) and a self-avoiding polygon (right).

Example 1 (Staircase polygons)**.** An animal is a *staircase polygon* if each row and column of the animal contains exactly 2 bonds, and further the lower and upper edges are directed paths (taking only north and east steps).

In the case of staircase polygons we are able to answer question 1 in a number of several different (equivalent) forms:

- The coefficient: $c_n = \frac{1}{n+1}\binom{2n}{n}$,
- The generating function: $\sum_{n\geq 1} c_n x^n = \frac{1-2x-\sqrt{1-4x}}{2x}$,
- c_n satisfies $(n+1)c_n - (4n-2)c_{n-1} = 0$, with $c_0 = 0$ and $c_1 = 1$.

Exact solutions (such as these) are only known for those models with severe topological restrictions: spiral walks (see [3] for example), three choice polygons [4], a number of families of column-convex polygons (see [5] for example), and three-dimensional convex polygons [6]. For other problems we have to use brute force, or other algorithms that are still exponential in time.

Example 2 (Self-avoiding polygons)**.** An animal is a *self-avoiding polygon* (SAP) if it is the embedding of a simple closed loop into the square lattice (*i.e.* the vertices of the underlying graph are all of degree 2). Despite a great deal of effort over many years, no (non-trivial) expression for the number, c_n, of SAPs with $2n$ bonds is known. Nor do we have an non-trivial expression for the corresponding generating function, nor do we know a non-trivial recurrence satisfied by c_n.

There are a number of other interesting animal models (general bond-animals, bond trees, directed bond animals and self-avoiding walks, ...) that have proved to be equally difficult to count. To date, the best way of computing c_n for these models is the "*finite lattice method*" which is an algorithm that requires exponential time and space, but is still exponentially faster than brute-force methods (see for example [7, 8, 9]).

The history of animal enumeration on the square (triangular, hexagonal, cubic...) lattice suggests that attempts to find c_n (either in the form of an expression, a generating function or a recurrence) are very likely to be frustrated. So rather than starting down an almost certainly doomed path, in this work we try to determine some of the properties of the answer to question 1 and so get some idea of what the answer might be, what tools we may use to find it, and why it is so difficult to find. To do this we examine

[1] *i.e.* methods of the form: • input n, • list out every animal with n bonds, • output the length of the list.

anisotropic generating functions; in particular one observes marked difference in the singularity structure of the coefficients of anisotropic generating functions for solved and unsolved models [1].

In this paper we prove that this is indeed the case for self-avoiding polygons and directed bond animals. This, in turn, proves that the anisotropic generating functions of these models (in contrast to most solved models) are excluded from what is perhaps the most common class of functions in mathematics and physics, *differentiably-finite* functions.

2. Anisotropic generating functions

The *isotropic* generating function of a family of animals enumerates the animals according to the total number of bonds. The *anisotropic* generating function, on the other hand, distinguishes between horizontal and vertical bonds. For a given bond animal B, we denote the number of horizontal (resp. vertical) bonds it contains by $|B|_{\Leftrightarrow}$ (resp. $|B|_{\Updownarrow}$). Very generally speaking, it is not too difficult to extend isotropic techniques (be they exact solutions or numerical expansions) to the anisotropic case, so one can obtain the anisotropic generating function without having to do too much more work.

Let $\mathcal{G}$ be a set of bond animals on the square lattice. Let us enumerate the elements of $\mathcal{G}$ according to the number of horizontal and vertical bonds and form the *anisotropic* generating function:

$$gf(\mathcal{G}) = \sum_{Q \in \mathcal{G}} x^{|Q|_{\Leftrightarrow}} y^{|Q|_{\Updownarrow}} = \sum_{n,m=0}^{\infty} c_{n,m}\, x^n y^m,$$

where $c_{n,m}$ is the number of elements of $\mathcal{G}$ containing exactly n horizontal bonds and m vertical bonds[2]. Let us start by rewriting this generating function as a power series in y with coefficients that are power series in x. Writing $\mathcal{G}_n = \{Q \in \mathcal{G}_n : |Q|_{\Updownarrow} = n\}$ we have:

$$gf(\mathcal{G})(x,y) = \sum_{n=1} y^n \sum_{Q \in \mathcal{G}_n} x^{|Q|_{\Leftrightarrow}} = \sum_{n=1} H_n(x) y^n.$$

The coefficient of y^n in the above generating function, $H_n(x)$, is a power series in x that enumerates all bond animals in $\mathcal{G}$ containing n vertical bonds (the set $\mathcal{G}_n$), according to the number of horizontal bonds.

In some sense, the anisotropic generating function is a more manageable object than the isotropic. Splitting the set of animals $\mathcal{G}$, into separate simpler subsets, $\mathcal{G}_n$, gives us smaller pieces, each of which is easier to study than the whole. If one seeks to compute (or even just understand) the *isotropic* generating function then one must somehow examine *all* possible topologies or configurations[3] that can occur in $\mathcal{G}$ — it is perhaps for this reason that the only families of bond animals that have been solved are those with severe topological restrictions (such as column-convex polygons). On the other hand, if we examine the generating function of $\mathcal{G}_n$, then the number of different "*topologies*" that can occur is always finite. For example, consider self-avoiding polygons with $2n$ vertical bonds. If $n = 1$ all configurations are rectangles. If $n = 2$, then all configurations are vertically *and* horizontally convex, while if $n = 3$ all configurations are vertically *or* horizontally convex. The anisotropy allows one to study the effect that these configurations have on the generating function in a more controlled manner.

Similarly, instead of trying to study the properties of the whole (possibly unknown) generating function, the anisotropy breaks the generating function into separate simpler pieces, $H_n(x)$, that can be calculated exactly (for small n). By studying the properties of these coefficients, particularly their singularities, we can obtain some idea of the properties of the generating function as a whole.

2.1. Some examples.

Let us consider two examples of these anisotropic generating functions; staircase polygons and self-avoiding polygons. However, before we examine the examples, we need to define the *cyclotomic polynomials* which appear throughout the following work[4].

[2] If the animal is a polygon, then the numbers of vertical and horizontal bonds are always even numbers, and so rather introducing extraneous factors of 2 we will enumerate families of polygons according to their horizontal and vertical *half-perimeters* (being exactly half the number of horizontal and vertical bonds). Rather than defining different notation for polygons we will simply take $|B|_{\Leftrightarrow}$ and $|B|_{\Updownarrow}$ to mean the horizontal and vertical half-perimeters (respectively) of any polygon B.

[3] We deliberately use imprecise words here — we will be more precise below.

[4] We will show that the generating functions, $H_n(x)$, can be expressed in terms of generating functions of the form $\frac{x^k}{1-x^k}$, and so the factors of the denominators are *cyclotomic polynomials*.

Definition 1. The *cyclotomic polynomials*, $\Psi_k(x)$, are the factors of the polynomials $(1-x^n)$. In particular, $(1-x^n) = \prod_{k|n} \Psi_k(x)$.

The first few cyclotomic polynomials are underlined below.

$$\begin{array}{lll}
\Psi_1: & (1-x) = & \underline{\boldsymbol{(1-x)}} \\
\Psi_2: & (1-x^2) = & (1-x)\ \underline{\boldsymbol{(1+x)}} \\
\Psi_3: & (1-x^3) = & (1-x)\ \underline{\boldsymbol{(1+x+x^2)}} \\
\Psi_4: & (1-x^4) = & (1-x)(1+x)\ \underline{\boldsymbol{(1+x^2)}} \\
\Psi_5: & (1-x^5) = & (1-x)\ \underline{\boldsymbol{(1+x+x^2+x^3+x^4)}} \\
\Psi_6: & (1-x^6) = & (1-x)(1+x)(1+x+x^2)\ \underline{\boldsymbol{(1-x+x^2)}}
\end{array}$$

We call $\Psi_k(x)$ the kth cyclotomic polynomial, and say that it its *order* is k. For any given integers $\{\alpha_i\}_{i\geq 1}$, the we have the following factorisation

$$\prod_{n=1}(1-x^n)^{\alpha_n} = \prod_{n=1}\prod_{k=n/d}\Psi_k(x)^{\alpha_n} = \prod_{k=1}\Psi_k(x)^{\sum_{d=1}\alpha_{dk}}$$

Example 3 (Staircase polygons)**.** The anisotropic generating function of staircase polygons[5] is known in closed form [10]:

$$P(x,y) = \frac{1}{2}\left(1-x-y-\sqrt{(1-x-y)^2-4xy}\right)$$

Expanding $P(x,y)$ as a power series in y gives:

$$\begin{aligned}
P(x,y) \quad = \quad & \frac{x}{1-x}y + \frac{x}{(1-x)^3}y^2 + \frac{x(1+x)}{(1-x)^5}y^3 \\
& + \frac{x(1+3x+x^2)}{(1-x)^7}y^4 + \frac{x(1+6x+6x^2+x^3)}{(1-x)^9}y^5 + \cdots
\end{aligned}$$

The coefficients, $H_n(x)$, have the following properties:

- $H_n(x)$ is a rational function of x,
- the degree of the numerator of $H_n(x)$ is $(n-1)$, for all $n \geq 2$.
- the denominator of $H_n(x)$ is $(1-x)^{2n-1}$.
- the coefficients of the numerators are positive, symmetric and unimodal.

Many other polygons models for which a closed form solution is known (such as the subsets of column-convex polygons) have similar properties (though the symmetry of numerator coefficients is lost in some cases) [11].

Example 4 (Self-avoiding polygons)**.** The generating function of self-avoiding polygons, $P(x,y)$, remains elusive, and so we must use either brute-force or the finite-lattice method to compute its coefficients. Expanding $P(x,y)$ as a power series in y, one finds [12] that the coefficients (which have been computed up to order $n=14$), $H_n(x)$, have the following properties:

- $H_n(x)$ is a rational function of x,
- the degree of the numerator of $H_n(x)$ is equal to the degree of its denominator.
- the coefficients of the numerators are positive and unimodal, but not symmetric.

[5]Since this is a family of polygons the generating function enumerates anisotropic *half-perimeter*; the coefficient of $x^m y^n$ is the number of staircase polygons with $2m$ horizontal bonds and $2n$ vertical bonds.

- if we write the denominator of $H_n(x)$ as $D_n(x)$, then the first ten are:

$$\begin{aligned}
D_1(x) &= (1-x) \\
D_2(x) &= (1-x)^3 \\
D_3(x) &= (1-x)^5 \\
D_4(x) &= (1-x)^7 \\
D_5(x) &= (1-x)^9(1+x)^2 \\
D_6(x) &= (1-x)^{11}(1+x)^4 \\
D_7(x) &= (1-x)^{13}(1+x)^6(1+x+x^2) \\
D_8(x) &= (1-x)^{15}(1+x)^8(1+x+x^2)^3 \\
D_9(x) &= (1-x)^{17}(1+x)^{10}(1+x+x^2)^5 \\
D_{10}(x) &= (1-x)^{19}(1+x)^{12}(1+x+x^2)^7(1+x^2)
\end{aligned}$$

  These suggests that a new cyclotomic factor enters every third coefficient, and that it enters with exponent 1, and increases by 2 in each subsequent coefficient (with the exception of $(1+x)$ which has exponent 1 less than this pattern predicts).

The denominator structure of self-avoiding polygons is starkly different to that of the previous (solvable) example. The denominators of staircase polygons contain only a single cyclotomic factor, $(1-x)$, and so $H_n(x)$ contains only a finite number of poles as $n \to \infty$. On the other hand, if one extrapolates from the observed pattern of self-avoiding polygon denominators, then any cyclotomic factor will appear eventually, and so $H_n(x)$ has a dense set of poles on $|x| = 1$ as $n \to \infty$.

2.2. **A test of solvability.** If we examine the coefficients, $H_n(x)$ of almost all *solved* models (excepting spiral-walks) then, like staircase polygons, one finds that they are rational functions whose denominator is a product of cyclotomic polynomials. In particular we note that only a finite number of different cyclotomic factors appear in these functions and so they have only a finite number of singularities (as $n \to \infty$).

Then if we examine the coefficients, $H_n(x)$ of many unsolved families of bond animals (including self-avoiding walks, general animals, directed bond animals and bond trees [12]), then we also find that they are rational functions whose denominators are products of cyclotomic polynomials. However, unlike the solved models, we find that the denominators contain many different cyclotomic polynomials — a new factor seems to appear every few coefficients (just as was the case for SAPs). If this pattern persists then the singularities of the coefficients form a dense set on $|x| = 1$.

The observation of this demarcation between the properties of solved and unsolved families of bond animals led Guttmann and Enting [1] to propose this as a (numerical) test of the *solvability* of a model[6]. In particular, if a function, $P(x, y)$, has coefficients whose poles form a dense set on $|x| = 1$, then the function is not a member of the most common functions of mathematics and physics — *differentiably-finite* functions.

2.3. **Differentiably-finite functions.** The solutions of ordinary differential equations with polynomial coefficients are arguably the most important class of formal power series in mathematics and physics. Such power series are called *differentiably-finite* power series [13, 14]. The solutions to many combinatorial and physical models belong to this class. They are very easy objects to work with since knowledge of the differential equation satisfied by the power series (even if it cannot be solved in closed form), is sufficient to compute their coefficients very efficiently (in polynomial time) and also to determine their asymptotic behaviour. There also exist a number of numerical methods that conjecture D-finite solutions based on the first few terms of the series expansion of a generating function (such as Differential Approximants and the Maple packages *GFUN* and *MGFUN* [15, 16])

[6]Similar patterns have also been observed in the thermodynamic functions of the Ising and Potts models [1]. Such functions can be interpreted as enumerating families of graphs on the square lattice with complicated weights (the weights can be negative and the graphs can be disconnected). We have not yet been able to extend the techniques described in this paper to these problems.

Definition 2. A power series $f(t) \in \mathbb{C}[[t]]$ is *differentiably finite* (or D-finite) if there exist $k+1$ polynomials, $P_0(t), P_1(t), \dots, P_k(t)$ such that

$$P_0(t)f(t) + P_1(t)f'(t) + \cdots + P_k(t)f^{(k)}(t) = 0. \tag{1}$$

Similarly a power series $f(x_1 \dots, x_k) \in \mathbb{C}[[x_1 \dots, x_k]]$ is *differentiably finite* if there exists k differential equations (one for each variable) of the form:

$$\left(A_{in_i}(x) \left(\frac{\partial}{\partial x_i}\right)^{n_i} + A_{in_i-1}(x) \left(\frac{\partial}{\partial x_i}\right)^{n_i-1} + \cdots + A_{i0} \right) f = 0, \tag{2}$$

with $A_{ij} \in \mathbb{C}[x_1 \dots, x_k]$.

It can be shown that any rational or algebraic function is D-finite [17].

The classical theory of linear differential equations implies that a D-finite formal power series in $\mathbb{C}[[x]]$, satisfying a linear ordinary differential equation with polynomial coefficients, has a finite number of singularities. This is a very simple test of "D-finiteness"; if the function has an infinite number of singularities, then it cannot be D-finite. For example, a function such as $f(t) = \frac{1}{\cos(t)}$ is not D-finite. The two variable case is a little more complicated:

Theorem 1 (Bousquet-Mélou [18]). *Let $f(x,y) = \sum_{n \geq 0} y^n H_n(x)$ be a D-finite series in y with rational coefficients, $H_n(x) \in \mathbb{C}(x)$. For $n \geq 0$ let S_n be the set of poles of $H_n(x)$, and let $S = \bigcup_n S_n$. Then S has only a finite number of accumulation points.*

From this we see that the singularity structure of the coefficients of solved models is compatible (it does not imply) with their being D-finite functions. Whereas the observed denominator patterns of many unsolved families of bond animals imply the singularities of the coefficients are dense on $|x| = 1$, have an infinite number of accumulation points, and consequently the anisotropic generating functions are not D-finite. The remainder of this paper is devoted to proving that this is indeed the case for self-avoiding polygons and directed bond animals.

3. Haruspicy

Definition 3 (From [19]). **haruspicy:** /hə'rsʌspɪsi/ Divination by inspection of the entrails of sacrifices.

Question 2. Why "haruspicy?"

Answer. Because we will examine the properties of anisotropic generating functions by examining the configuration of bonds within some specially prepared bond animals.

3.1. Columns, squashing and a partial order.

Definition 4. We will define a *column* of an animal to be the horizontal bonds within a single horizontal lattice spacing of an animal. See figure 3. If the column contains k horizontal bonds we say it is a k*-column*[7]. The number of k-columns in an animal, A, is denoted by $\gamma_k(A)$.

The animal "squashing" procedure is based on the idea of reducing the width of the animal by removing a column if it is the same as its neighbour.

Definition 5. We say that a column is a *duplicate column* if the column immediately on its left (without loss of generality) is identical (see figure 4) and there are no vertical bonds between them. One can squash or reduce animals by *deletion* of duplicate columns by slicing the animal on either side of the duplicate column, removing the column and recombining the animal, as illustrated in figure 4). By reversing the column deletion process we define *duplication* of a column. An animal is a *minimal-animal* if it contains no duplicate columns.

Let us take an animal, B, and delete one (or more) of its columns to obtain another animal, A. This process of column-deletion induces a partial order on the set of animals; since A is obtained by deleting some of the columns of B we write, $A \preceq B$. See figure 5 for example.

Definition 6. For any two animals $P, Q \in \mathcal{G}_n$, we define the binary relation $\preceq_c$ by:

[7] If we are considering a set of polygons then a column with $2k$ horizontal bonds is a k-column.

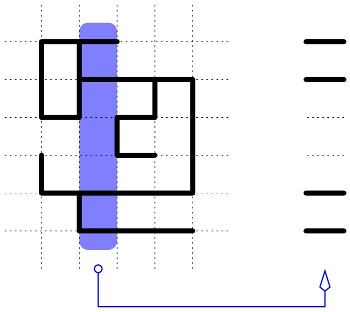

FIGURE 3. A 4-column of an animal.

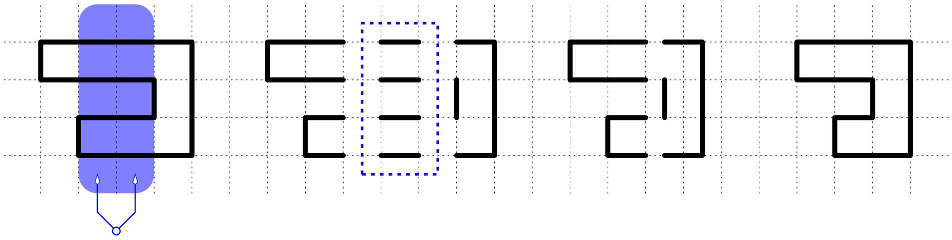

FIGURE 4. Animal surgery — the process of column deletion. The two indicated columns are identical. Slice either side of the duplicate and separate the polygon into three pieces. The middle piece, being the duplicate, is removed and the remainder of the polygon is recombined. Reversing the steps leads to column duplication.

- $P \preceq_c Q$ if $P = Q$ or P can be obtained from Q by a sequence of column deletions.

It is not hard to show that this binary relation is a partial order on the set of animals.

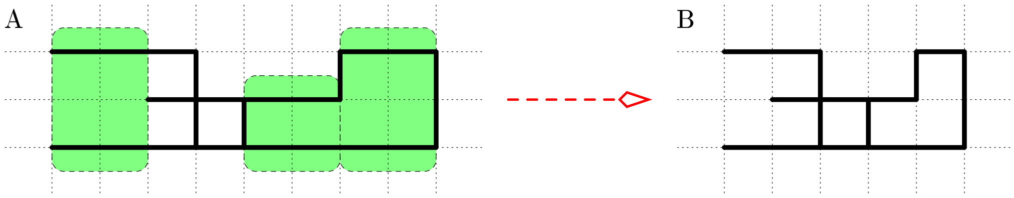

FIGURE 5. The duplicate columns in the animal on the left are highlighted. Removing these duplicate columns from the animal reduces it to the animal on the right (which is column-minimal). Hence we write $B \preceq_c A$.

It can be shown that the set of bond animals with n vertical bonds, $\mathcal{G}_n$, contains only a finite number of minimal animals. Further, since each animal reduces to a unique minimal animal we have a natural equivalence relation; two animals are equivalent if they reduce to the same minimal animal. We can then partition $\mathcal{G}_n$ into a finite number of equivalence classes, each containing a single minimal animal. Let us denote the horizontal bond generating function of the equivalence class with column-minimal animal P, by $G_c(P)$. The generating function is given by:

$$G_c(P) \quad = \quad \prod_k \left(\frac{x^k}{1 - x^k} \right)^{\gamma_k(P)} \tag{3}$$

One has to be a little careful to ensure that the animals are closed under column deletion and duplication; most animals of interest do satisfy this technical requirement.

Since $\mathcal{G}_n$ is split into these equivalence classes, the generating function $H_n(x)$ can be written as the sum of the generating function of each equivalence class. Further, since there are only a finite number of these equivalence classes, the sum is finite and we have the following theorem:

Theorem 2. *If $P(x,y) = \sum_{n\geq 0} H_n(x)y^n$ is the anisotropic generating function of some set of animals, $\mathcal{G}$, that is closed under column deletion and duplication then*

- *$H_n(x)$ is a rational function,*
- *the degree of the numerator of $H_n(x)$ is at most equal to the degree of its denominator, and*
- *the denominator of $H_n(x)$ is a product of cyclotomic polynomials.*
- *if the denominator of $H_n(x)$ contains $\Psi_k(x)$ then there exists an minimal-animal that contains a K-column, for some $K \in \mathbb{N}$ divisible by k.*

Note that this theorem does not imply that if one can construct a minimal-animal with n vertical bonds and a k-column, that there will be a factor of $\Psi_k(x)$ in the denominator of $H_n(x)$, since there can be common factors in the numerator and denominator that will cancel. In general to show that there are no cancellations is very difficult, however we have been able to do this in the case of the first appearance of $\Psi_k(x)$.

3.2. **The first occurrence of $\Psi_k(x)$.** Consider the column-minimal polygons in figure 6. From this example it is quite obvious that there is no minimal polygon with a k-column and less than $4k-2$ vertical bonds. By theorem 2 we know that the factor $\Psi_k(x)$ cannot occur before $H_{2k-1}(x)$.

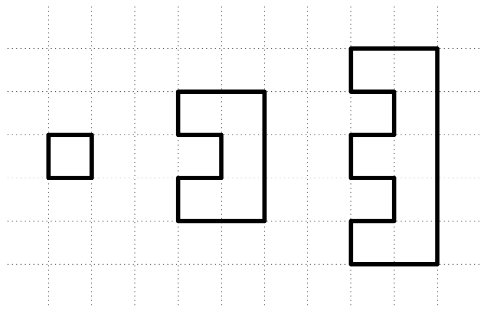

FIGURE 6. Constructing a polygon with a k column.

This example also implies that there are minimal polygons with k-columns and $4k-2$ vertical bonds, which contribute factors of $\Psi_k(x)$ to the denominator of $H_{2k-1}(x)$. The numerical results show that $\Psi_k(x)$ does not appear until $H_{3k-2}(x)$, and so the contributions from these k-columns must cancel with the numerators. We can overcome this problem by considering "*section-deletion*" instead of column-deletion[8].

One can show that with $6k-4$ vertical bonds it is possible to construct a minimal polygon with a k-section, but not with any less. These minimal polygons contribute a factor of $\Psi_k(x)$ to the denominator of $H_{3k-2}(x)$ (in agreement with the data above). We can now split the function $H_{3k-2}(x)$ into two parts; $f_k(x)$ which is the contribution from minimal animals that contain a k-section, and $g_k(x)$ which is the remainder.

$$H_{3k-2}(x) = f_k(x) + g_k(x). \tag{4}$$

Since none of the minimal animals contributing to $g_k(x)$ contain K-sections with $K \geq k$, the denominator of $g_k(x)$ is a product of cyclotomic polynomials of order less than k. Consequently if the denominator factor $\Psi_k(x)$ cancels with the numerator of $H_{3k-2}(x)$ then it must also cancel in $f_k(x)$. We are able to prove that this does not occur by computing $f_k(x)$.

In a number of cases we have been able to determine which animals are counted by $f_k(x)$. For SAPs it is those polygons with 2 vertical bonds in their odd-numbered rows, and 4 vertical bonds in their even-numbered rows, so that there are a total of $6k-4$ vertical bonds. For directed bond animals, general animals and bond trees, it is those animals with exactly 2 vertical bonds in each row, and $2k-2$ vertical bonds in total.

[8]There is insufficient space here to describe section-deletion; it is a very similar process to column-deletions and all of the above results still hold with the word "column" is replaced by "section".

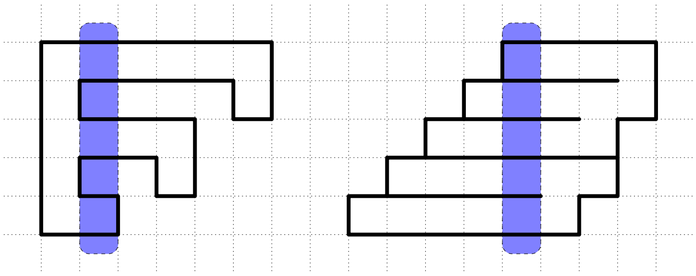

FIGURE 7. Examples of the minimal-animals that contribute to the first occurrence of $\Psi_k(x)$ for SAPs (left) and directed bond animals (right). In each case the k-section has been highlighted.

Using the Temperley method (see [5] for example) we are able to find a functional recurrence satisfied by the $f_k(x)$ for SAPs and directed bond-animals; we have not yet done so for general animals and bond trees, since the topologies that can occur are more complicated. Analysis of this recurrence gives the following result:

Theorem 3. *For self-avoiding polygons, the factor of $\Psi_k(x)$ appears in the denominator of $H_{3k-2}(x)$ without cancellations (except for $k = 2$).*

For directed bond-animals, the factor of $\Psi_k(x)$ appears in the denominator of $H_{2k-2}(x)$ without cancellations.

Consequently the singularities of the coefficients of the anisotropic generating functions for these two families of animals form a dense set on $|x| = 1$, and so the *anisotropic* generating functions cannot be D-finite.

Consider the function:

$$P(x, y) = \sum_{n\geq 1} \frac{y^n}{(1 - x^n)(1 - x^{n+1})}.$$

Setting $x = y$ gives $P(x, x) = \frac{x}{(1-x)^2}$, and so this is an example of a two-variable function that is not D-finite, but when $x = y$ reduces to D-finite function. So though we know the anisotropic generating functions are not D-finite, this does not give us information about the *isotropic* generating functions. In every solved case, however, we find that "anisotropising" the generating function does not change the (analytic) nature of the generating function. Since we know so little about the nature of the generating functions for unsolved models, we cannot rigorously examine the effect of anisotropisation. There do exist non-rigorous Renormalisation Group arguments that imply that anisotropisation should not affect the analytic nature of the generating function [20].

4. Summary of results

We have introduced a method for the examination of the anisotropic generating function of general families of animals, $P(x, y)$. The coefficient of y^n in this function is found to be a rational function whose denominator is a product of cyclotomic polynomials. For many families of animals we are able to determine which cyclotomic factors can occur in these coefficients, but we are not able to show that there are no cancellations.

In the cases of directed bond animals and self-avoiding polygons we are able to show that cancellations do not occur for specific cyclotomic factors. This, in turn, proves that the singularities of the coefficients of y in the anisotropic generating function form an infinite set that becomes dense on $|x| = 1$. And so directed bond animals and self-avoiding polygons, enumerated anisotropically, are not solvable in terms of D-finite functions. We hope to prove similar results for bond animals and lattice trees.

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