THE PERMUTATION ENUMERATION OF WREATH PRODUCTS $C_k \S S_n$

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Abstract. Brenti introduced a homomorphism from the symmetric functions to polynomials in one variable with rational coefficients. His map is defined on the elementary symmetric functions. When it is applied to the homogeneous and power symmetric functions the results are generating functions for descents and excedences of permutations in the symmetric group, respectively. Beck and Remmel gave proofs of Brenti's results based on combinatorial definitions of the transition matrices between the bases. In addition, they gave an analog for the hyperoctahedral group. We extend the ideas of these proofs to obtain similar results for wreath products of an arbitrary cyclic group with the symmetric group.

Extracto. Brenti introdujo un homomorfismo de las funciones simtricas a los polinómios en una variable con coeficientes racionales. Su mapa se define en las funciones simétricas elementales. Cuando se aplica al las funciones simétricas homogéneas y las funciones simétricas de la potencia los resultados está las funciones generadoras para las pendientes y los excedences de permutaciones en el grupo simétrico, respectivamente. Beck y Remmel dieron pruebas de los resultados de Brenti basados en las definiciones combinatorias de las matrices de la transición entre las bases. Además, dieron un análogo para el grupo hyperoctahedral. Ampliamos las ideas de estas pruebas de obtener los resultados similares para los productos de la guirnalda de un grupo cíclico arbitrario con el grupo simétrico.

1. Introduction

Let Λ_k denote the space of symmetric polynomials in an infinite number of variables which are homogeneous of degree k, and let $\Lambda = \bigoplus_{k\geq 0} \Lambda_k$. Brenti [4] introduced a homomorphism $\xi : \Lambda \to \mathbf{Q}(x)$ defined on the elementary symmetric functions by

$$\xi(e_k) = \frac{(1-x)^k}{k!}.$$

He proved that the homomorphism ξ has the remarkable property that when it is applied to a homogeneous symmetric function h_k, the result is the well-known Eulerian polynomial, which is also the generating function for the number of descents of a permutation in S_k:

$$\xi(h_k) = A_k = \sum_{\sigma \in S_k} x^{des(\sigma)},$$

where $des(\sigma) = |\{i : 1 \leq i \leq n-1, \sigma(i) > \sigma(i+1)\}|$. In addition, if ξ is applied to a power symmetric function, the result is a generating function for the number of excedences over conjugacy classes:

$$\xi(p_\lambda(x)) = \sum_{\sigma \in \mathcal{C}(\lambda)} x^{e(\sigma)},$$

where $e(\sigma) = |\{i : 1 \leq i \leq n, \sigma(i) > i\}|$ and $\mathcal{C}(\lambda)$ is the conjugacy class of S_n indexed by the partition λ.

Beck and Remmel [2], [3] used combinatorial definitions of the transition matrices between bases of Λ to give combinatorial proofs of these and other related identities, including q-analogs. In addition, they used these combinatorial methods to develop an analog of Brenti's results for B_n, the hyperoctahedral group. Since B_n is of course isomorphic to $C_2 \S S_n$, it is natural to extend these ideas

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to arbitrary wreath products $C_k \S S_n$. We do this here. We begin with some necessary definitions and notation. In Section 3 we state our results. We conclude with the proof of one such result in order to give a flavor for the combinatorial interpretations used.

2. Notation and Definitions

Let $\lambda \vdash n$ denote the fact that λ is a partition of n. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$, the number of parts of λ is denoted by $l(\lambda)$ and $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)}$. If λ is written as $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where m_i is the number of parts of size i, then let $z_\lambda = 1^{m_1}2^{m_2}\cdots n^{m_n}(m_1!)(m_2!)\cdots(m_n!)$. Given k partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(k)}| = n$, we write $(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n$.

We denote the elementary, homogeneous, power, monomial, Schur, and forgotten symmetric functions by e_λ, h_λ, p_λ, m_λ, s_λ, and f_λ, respectively (see [7]).

Consider an element of $C_k \S S_n$ as a signed permutation with signs in the set $\{\epsilon, \epsilon^2, \ldots, \epsilon^k\}$ where $\epsilon = e^{\frac{2\pi i}{k}}$. An element can be written in two-line form as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \epsilon_1\sigma_1 & \epsilon_2\sigma_2 & \cdots & \epsilon_n\sigma_n \end{pmatrix},$$

in one-line form as

$$\sigma = \quad \epsilon_1\sigma_1 \quad \epsilon_2\sigma_2 \quad \cdots \quad \epsilon_n\sigma_n \quad ,$$

or in cycle form with cycles of the form

$$(\epsilon_m i_1,\ \epsilon_1 i_2, \ldots, \epsilon_{m-1}\sigma_m)$$

where i_1 is mapped to $\epsilon_i i_2$, i_2 is mapped to $\epsilon_2 i_3$ and so on. We often abuse notation by referring to $\epsilon_i\sigma_i$ as just σ_i. $\epsilon(\sigma_i)$ denotes ϵ_i and we then set $\epsilon(\sigma) = \prod_{i=1}^n \epsilon(\sigma_i)$.

We define a partial ordering Ω on the signed letters that make up elements of $C_k \S S_n$ by

$$\begin{array}{ccccccc} 1 & <_\Omega & 2 & <_\Omega & \cdots & <_\Omega n, \\ \epsilon 1 & <_\Omega & \epsilon 2 & <_\Omega & \cdots & <_\Omega \epsilon n, \\ & & & \vdots & & \\ \epsilon^{k-1}1 & <_\Omega & \epsilon^{k-1}2 & <_\Omega & \cdots & <_\Omega \epsilon^{k-1}n. \end{array}$$

Note that if $k = 2$ this is not consistent with the standard ordering for B_n. In fact we have $k-1$ other choices of orderings Ω_i for $1 \le i \le k-1$ defined by

$$\left.\begin{array}{ccccc} 1 & <_{\Omega_i} & \cdots & <_{\Omega_i} & n \\ & & \vdots & & \\ \epsilon^{i-1}1 & <_{\Omega_i} & \cdots & <_{\Omega_i} & \epsilon^{i-1}n \end{array}\right\} <_{\Omega_i} \left\{\begin{array}{ccccc} \epsilon^i n & <_{\Omega_i} & \cdots & <_{\Omega_i} & \epsilon^i 1 \\ & & \vdots & & \\ \epsilon^{k-1}n & <_{\Omega_i} & \cdots & <_{\Omega_i} & \epsilon^{k-1}1 \end{array}\right.$$

Then if $k = 2$, the standard ordering on B_n is Ω_1. We choose to use Ω because the results it renders and their proofs are more easily stated in general than those based on the other orderings. However it is important to note that there are in fact k sets of analogous results, each derived from a different ordering on these signed letters. Moreover, every ordering corresponds to different definitions of statistics on $C_k \S S_n$-elements and to different analogs of the homomorphism ξ. In addition, we use a partial ordering Γ which in equates letters that have the same base letter, i.e.

$$\epsilon^i a <_\Gamma \epsilon^j b \text{ if } a < b \text{ for all } i, j.$$

We use the partial orderings Ω and Γ to define a number of statistics on elements of $C_k \S S_n$. The number of $C_k \S S_n$-*descents* of an element σ is given by

$$des_k(\sigma) = |\{i : 1 \le i \le n-1, \sigma_i >_\Omega \sigma_{i+1}\}|.$$

The number of $C_k \S S_n$-*inversions* of an element σ is given by

$$inv_k(\sigma) = |\{(i,j) : 1 \le i < j \le n, \sigma_i >_\Gamma \sigma_j\}|.$$

If an element σ is given in cycle notation as

$$\sigma = (\sigma_{1_1}, \sigma_{1_2}, \ldots, \sigma_{1_{l_1}})(\sigma_{2_1}, \sigma_{2_2}, \ldots, \sigma_{2_{l_2}}) \cdots (\sigma_{m_1}, \sigma_{m_2}, \ldots, \sigma_{m_{l_m}}),$$

then the number of $C_k \S S_n$-*descedances* is given by

$$de_k(\sigma) = \sum_{i=1}^{m} \Big(|\{j : 1 \leq j \leq l_i - 1, \epsilon(\sigma_{i_j}) = \epsilon(\sigma_{i_{j+1}}), \sigma_{i_j} > \sigma_{i_{j+1}}\}| \\ + \chi(\sigma_{i_{l_i}} > \sigma_{i_1})\chi(\epsilon(\sigma_{i_{l_i}}) = \epsilon(\sigma_{i_1})) \Big).$$

where for a statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false.

Our final statistic depends not only on an element of $C_k \S S_n$, but also on a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}) \vdash n$. The number of λ-*descents* of an element of $C_k \S S_n$, denoted $des_{k,\lambda}$, is defined in the following way. Write $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ in one-line notation and break it into pieces of lengths $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$. Then count only the $C_k \S S_n$-descents that occur with both i and $i+1$ in the same piece. For example if $\sigma = \epsilon 7\ \epsilon 5\ 4\ 2\ 1\ \epsilon^2 8\ \epsilon^2 3\ \epsilon 6\ \in C_3 \S S_n$ and $\lambda = (1, 3, 4)$, then σ is broken into pieces $[\epsilon 7][\epsilon 5\ 4\ 2][1\ \epsilon^2 8\ \epsilon^2 3\ \epsilon 6]$. Here $des_k(\sigma) = 4$ but $des_{k,\lambda}(\sigma) = 2$ since the descents in the first and fourth positions are not counted as λ-descents because 1 and 2 do not lie in the same block, and neither do 4 and 5.

If the product of the signs in a cycle is ϵ^i, we say that it is an ϵ^i-*cycle*. The cycle structure of an element in $C_k \S S_n$ is given by $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ where $\lambda^{(i)}$ gives the lengths of the ϵ^i-cycles. Let

$$C_k \S S_n(\lambda^{(1)}, \ldots, \lambda^{(k)}) = \{\sigma \in C_k \S S_n : \text{ cycle structure of } \sigma = (\lambda^{(1)}, \ldots, \lambda^{(k)})\}.$$

In fact, $\{C_k \S S_n(\lambda^{(1)}, \ldots, \lambda^{(k)}) : (\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n\}$ is the set of conjugacy classes of $C_k \S S_n$. In [9], the author and Remmel give a characteristic map analogous to the Frobenius characteristic which sends the class functions of $C_k \S S_n$ into

$$\Lambda_{W_{k,n}}(X, Y, Z) = \bigoplus_{a+b+c=n} \Lambda_a(X) \otimes \Lambda_b(Y) \otimes \Lambda_c(Z).$$

This characteristic map is slightly different than that given by Poirier [8], which he uses to express the number of elements of $C_k \S S_n$ with a given descent set in terms of characters of $C_k \S S_n$. Set $\Lambda_{W_k} = \bigoplus_{n \geq 0} \Lambda_{k,n}$. Some of the important bases of $\Lambda_{W_{k,n}}$ are given below. With the exception of the analog of the power symmetric functions, they are given in a version of λ-ring notation which is extended to include roots of unity by requiring that $p_r(\epsilon X) = \epsilon p_r(X)$ [9]. The bases are indexed by k-tuples of partitions $(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n$. Let $X^{(1)}, \ldots, X^{(k)}$ denote k different alphabets. Then the analog of the power, Schur, elementary, homogeneous, monomial, and forgotten symmetric functions are

$$p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)}),\ \prod_{i=1}^{k} s_{\lambda^{(i)}}\left(\epsilon^{1 \cdot i} X^{(1)} + \cdots + \epsilon^{k \cdot i} X^{(k)}\right),$$

$$\prod_{i=1}^{k} e_{\lambda^{(i)}}\left(\epsilon^{1 \cdot i} X^{(1)} + \cdots + \epsilon^{k \cdot i} X^{(k)}\right),\ \prod_{i=1}^{k} h_{\lambda^{(i)}}\left(\epsilon^{1 \cdot i} X^{(1)} + \cdots + \epsilon^{k \cdot i} X^{(k)}\right),$$

$$\prod_{i=1}^{k} m_{\lambda^{(i)}}\left(\epsilon^{1 \cdot i} X^{(1)} + \cdots + \epsilon^{k \cdot i} X^{(k)}\right), \text{ and } \prod_{i=1}^{k} f_{\lambda^{(i)}}\left(\epsilon^{1 \cdot i} X^{(1)} + \cdots + \epsilon^{k \cdot i} X^{(k)}\right),$$

respectively.

The study of the permutation enumeration of $C_k \S S_n$ depends on combinatorial definitions of the transition matrices between the bases of $\Lambda_{W_{k,n}}$. Most of these transition matrices are just k-tuples of the transition matrices between the bases of Λ_n. However, the matrices between $p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})$ and the other bases are more complicated and had to be developed.

3. Results

We define a homomorphism $\xi_k : \Lambda_{W_k} \longrightarrow \mathbf{Q}[\epsilon][x]$, on $e_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})$, for each $m = 1, 2, \ldots, k$, by

$$(1)\quad \xi_k\left(e_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})\right) = \frac{1}{k^n n!}\left(\epsilon^{m\cdot 1}(\epsilon^{m\cdot 1} - \epsilon^{m\cdot 1}x)^{n-1} + \cdots + \epsilon^{m\cdot k}(\epsilon^{m\cdot k} - \epsilon^{m\cdot k}x)^{n-1}\right).$$

This is not the most efficient way of writing the definitions, but we use this form because it suggests the combinatorial proofs. We combinatorially prove the following results.

Theorem 1. *Let ξ_k be the homomorphism defined in (1). Then for $m = 1, 2, \ldots, k$, we have*

$$(2)\quad k^n n! \xi_k\left(h_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})\right) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_k(\sigma)},$$

$$k^n n! \xi_k\left(h_\lambda(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})\right) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_{k,\lambda}(\sigma)},$$

and

$$(3)\quad \frac{k^n n!}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} \xi_k\left(p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})\right) = \sum_{\sigma \in C_k \S S_n(\lambda^{(1)}, \ldots, \lambda^{(k)})} x^{de_k(\sigma)},$$

where $\lambda \vdash n$ and $(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n$.

In addition, we define a q-analog of ξ_k, $\overline{\xi}_k : \Lambda_{W_k} \longrightarrow \mathbf{Q}[\epsilon][x][q]$, on $e_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})$, for $m = 1, 2, \ldots, k$, by

$$(4)\quad \overline{\xi}_k\left(e_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})\right) = \frac{q^{\binom{n}{2}}}{k^n [n]!}\left(\epsilon^{m\cdot 1}(\epsilon^{m\cdot 1} - \epsilon^{m\cdot 1}x)^{n-1} + \cdots + \epsilon^{m\cdot k}(\epsilon^{m\cdot k} - \epsilon^{m\cdot k}x)^{n-1}\right).$$

We then prove the following.

Theorem 2. *Let $\overline{\xi}_k$ be the homomorphism defined in (4). Then for $m = 1, 2, \ldots, k$, we have*

$$k^n n! \overline{\xi}_k\left(h_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})\right) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_k(\sigma)} q^{inv_k(\sigma)},$$

$$k^n n! \xi_k\left(h_\lambda(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})\right) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_{k,\lambda}(\sigma)} q^{inv_k(\sigma)},$$

where we assume that $\lambda \vdash n$ and $(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n$.

4. An Example of a Combinatorial Interpretation in the $k = 3$ Case

In order to give a sense of the combinatorial proofs, we prove (2) in the case $k = 3$ and $m = 1$, i.e.,

$$3^n n! \xi_3(h_n(\epsilon X^{(1)} + \epsilon^2 X^{(2)} + X^{(3)})) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_3(\sigma)}.$$

But first we must define μ-brick tabloids of shape λ, objects which will be used in the proof. They are constructed in the following way. Given two partitions $\lambda, \mu \vdash n$, make bricks of lengths $\mu_1, \mu_2, \ldots, \mu_{l(\mu)}$. Place them in a Ferrers diagram of shape λ such that no two bricks overlap and all bricks lie horizontally within a row. As an example, all three $(1,1,1,2)$-brick tabloids of shape

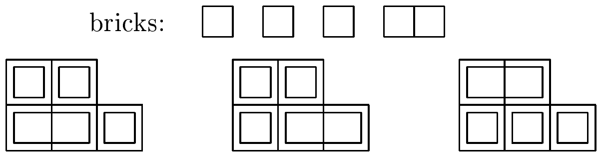

Figure 1. The (1,1,1,2)-brick tabloids of shape (2,3)

$(2,3)$ are given in Figure 1. Let $\mathcal{B}_{\mu,\lambda}$ denote the set of all μ-brick tabloids of shape λ, and set $B_{\mu,\lambda} = |\mathcal{B}_{\mu,\lambda}|$.

We begin with the following expression, which can be found in [5].

$$h_\lambda = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,\lambda} e_\mu.$$

We use the special case with $\lambda = (n)$. Multiply by $3^n n!$ and apply ξ_3 to get

$$\begin{aligned}
3^n n! \xi_3 &\left(h_n(\epsilon X^{(1)} + \epsilon^2 X^{(2)} + X^{(3)}) \right) \\
&= \sum_{\mu \vdash n} 3^n n! (-1)^{n-l(\mu)} B_{\mu,(n)} \xi_3 \left(e_\mu(\epsilon X^{(1)} + \epsilon^2 X^{(2)} + X^{(3)}) \right) \\
&= \sum_{\mu \vdash n} 3^n n! (-1)^{n-l(\mu)} B_{\mu,(n)} \prod_{i=1}^{l(\mu)} \frac{\epsilon(\epsilon - \epsilon x)^{\mu_i - 1} + \epsilon^2(\epsilon^2 - \epsilon^2 x)^{\mu_i - 1} + (1-x)^{\mu_i - 1}}{3^{\mu_i} \mu_i!} \\
&= \sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1, \dots, \mu_{l(\mu)}} \prod_{i=1}^{l(\mu)} \epsilon(\epsilon x - \epsilon)^{\mu_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\mu_i - 1} + (x-1)^{\mu_i - 1}.
\end{aligned}$$

We interpret this as a sum of signed, weighted objects o in a set of objects $\mathcal{O}$. We construct these objects in the following way. For a given partition μ, and a μ-brick tabloid of shape (n), the multinomial coefficient fills each brick with a decreasing sequence of integers such that exactly the integers $1, 2, \dots, n$ are used. Each brick is designated as *regular*, *barred*, or *double barred*. Each cell c is given a weight $w(c)$ according to the following.

- If c is in a regular brick,
$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$
This accounts for the $(x-1)^{\mu_i - 1}$ terms.
- If c is in a barred brick,
$$w(c) = \begin{cases} \epsilon, & c \text{ is at the end of a brick,} \\ -\epsilon \text{ or } \epsilon x, & \text{otherwise.} \end{cases}$$
This accounts for the $\epsilon(\epsilon x - \epsilon)^{\mu_i - 1}$ terms.
- If c is in a double barred brick,
$$w(c) = \begin{cases} \epsilon^2, & c \text{ is at the end of a brick,} \\ -\epsilon^2 \text{ or } \epsilon^2 x, & \text{otherwise.} \end{cases}$$
This accounts for the $\epsilon^2(\epsilon^2 x - \epsilon^2)^{\mu_i - 1}$ terms.

13	10	8		$\overline{\overline{7}}$	$\overline{\overline{2}}$		5	1		$\overline{12}$	$\overline{9}$	$\overline{6}$		$\overline{11}$	$\overline{4}$	$\overline{3}$
x	x	1		$\epsilon^2 x$	ϵ^2		x	1		ϵx	$-\epsilon$	ϵ		$-\epsilon$	ϵx	ϵ

FIGURE 2. An example of an object in $\mathcal{O}$.

13	10	8		$\overline{\overline{7}}$	$\overline{\overline{2}}$		5	1		$\overline{12}$	$\overline{9}$	$\overline{6}$			$\overline{11}$	$\overline{4}$	$\overline{3}$
x	x	1		$\epsilon^2 x$	ϵ^2		x	1		ϵx	$-\epsilon$	ϵ			$-\epsilon$	ϵx	ϵ

$\updownarrow$

13	10	8		$\overline{\overline{7}}$	$\overline{\overline{2}}$		5	1		$\overline{12}$	$\overline{9}$		$\overline{6}$		$\overline{11}$	$\overline{4}$	$\overline{3}$
x	x	1		$\epsilon^2 x$	ϵ^2		x	1		ϵx	ϵ		ϵ		$-\epsilon$	ϵx	ϵ

FIGURE 3. An example of the involution on $\mathcal{O}$.

13	8	2		$\overline{\overline{10}}$	$\overline{\overline{7}}$		5	1		$\overline{12}$	$\overline{9}$	$\overline{6}$		$\overline{11}$	$\overline{4}$	$\overline{3}$
x	x	1		$\epsilon^2 x$	ϵ^2		x	1		ϵx	ϵx	ϵ		ϵx	ϵx	ϵ

FIGURE 4. An example of a fixed point of the involution.

Define the weight of an object o by $w(o) = \prod_{c \in o} w(c)$. We can then write

$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1, \ldots, \mu_{l(\mu)}} \prod_{i=1}^{l(\mu)} \left(\epsilon(\epsilon x - \epsilon)^{\mu_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\mu_i - 1} + (x-1)^{\mu_i - 1} \right) = \sum_{o \in \mathcal{O}} w(o).$$

An example of these objects is given in Figure 2.

We perform a sign-changing, weight-preserving involution on the objects in $\mathcal{O}$. Traverse the tabloid from left to right until one of the following occurs, then perform the appropriate operation.

- If a cell c has weight -1, $-\epsilon$, or $-\epsilon^2$, split the brick after c and change the weight of c from -1, $-\epsilon$, or $-\epsilon^2$ to $+1$, $+\epsilon$, or $+\epsilon^2$.
- If there is a decrease between the integer filling of the last cell c of a brick and that of the first cell of the next brick and both bricks are of the same type, join the two bricks and change the weight of c from $+1$, $+\epsilon$, or $+\epsilon^2$ to -1, $-\epsilon$, or $-\epsilon^2$.

An example of the involution is given in Figure 3. The fixed points then are μ-brick tabloids of shape (n) filled with the integers $1, 2, \ldots, n$ such that the integer fillings decrease within bricks and increase between consecutive bricks of the same type. Each brick is designated as regular, barred, or double barred. Each cell is weighted as follows. In a regular brick, the cell at the end has weight 1 while the other cells have weight x. In a barred brick, the cell at the end has weight ϵ while the other cells have weight ϵx. In a double barred brick, the cell at the end has weight ϵ^2 while the other cells have weight $\epsilon^2 x$. An example of a fixed point of the involution is given in Figure 4.

Interpret the integer filling of a fixed point as an element of $C_3 \S S_n$ with fillings of cells in regular, barred, and double barred bricks corresponding to letters signed with 1, ϵ, and ϵ^2, respectively. Note that there is exactly one fixed point that corresponds to each element of $C_3 \S S_n$. Then each $C_3 \S S_n$-descent has a weight of x while no other transitions have an x-weight. In addition, each cell in a barred brick contributes ϵ and each cell of a double barred brick contributes ϵ^2. This contribution of signs corresponds to the sign of the element, $\epsilon(\sigma)$. This gives $\sum_{o \in \mathcal{O}} w(o) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_3(\sigma)}$, and thus the result.

The proofs of the expressions in (2) are essentially the same for all values of m, with $m = k$ giving an unsigned version of the proof. It is important to note that the definition of ξ_k was developed to prove not only these identities, but also (3). The proof of (3) is much too intricate to include here, but it relies heavily on the signs included in (1).

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