RIBBONS AND HOMOGENEOUS SYMMETRIC FUNCTIONS

MIKE ZABROCKI

Abstract. We demonstrate an elegant combinatorial formula for the operation which adds a column on the homogeneous symmetric functions when this function acts on the Schur basis. A q-analog of this formula adds a column on the Hall-Littlewood basis.

Résumé. Nous demontrons une formule élégante et combinatoire pour l'opérateur qui ajoute une colonne sur les fonctions symétriques homogène quand cet opérateur agit sur la base Schur. Une q-analogue de cette récurrence ajoute une colonne sur la base Hall-Littlewood.

1. Introduction

Consider the bases of the symmetric functions, each indexed by the set of partitions. To arrive at formulas for expansions of these bases in terms of the Schur functions we often consider recurrences based on the idea of adding rows or columns to the partition.

For example, to arrive at an expansion of the homogeneous symmetric function, h_λ, in terms of the Schur functions we consider the operation of successively adding rows to the partition by multiplying by h_k. The action of multiplication by h_k on the Schur basis is given by the well known Pieri formula that states

$$h_k s_\lambda = \sum_\mu s_\mu$$

where the sum is over all partitions μ such that μ differs from λ by a horizontal strip. A q-analog of this recurrence exists to build up the Hall-Littlewood symmetric functions (the Morris recurrence [5]) which is used to prove the positivity of the Kostka-Foulkes coefficients.

Similarly, to calculate the expansion of the power symmetric function, p_λ, in terms of the Schur basis and arrive at the formula for the irreducible characters of the symmetric group, one may successively add rows to the partition by multiplication by p_k and the action of this operation on the Schur basis is given by the Murnaghan-Nakayama rule (slinky rule).

Here we present the action of the operator that adds a column on the homogeneous basis when this operator acts on the Schur basis; a recurrence that is a dual Pieri rule. This operator has an elegant combinatorial description in terms of ribbons. By taking a simple q-analog of this operator, one obtains an operator that adds a column to the Hall-Littlewood symmetric functions giving a new recurrence on the Kostka-Foulkes coefficients.

It should be possible to use this recurrence to show the combinatorial interpretation of the Kostka coefficients and the Kostka-Foulkes coefficients in terms of column strict tableaux, but it remains an open problem to do so. It would also be interesting to see if this recurrence could be used to arrive at another combinatorial interpretation for these coefficients.

2. Partitions, ribbons and symmetric functions

A partition is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k > 0)$ and may be represented by a Young diagram with rows of cells aligned at their left edge such that in the i^{th} row there are λ_i cells. We use the French convention and draw these diagrams with the smallest part on top. By identifying a partition with its Young diagram we may talk about the rows, columns and cells of the partition. We will write $\lambda \vdash n$ to indicate that λ is a partition such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n = |\lambda|$. The length of the partition $\ell(\lambda)$ is the largest integer k such that $\lambda_k > 0$. The partition formed by flipping λ across the diagonal will be denoted by λ'.

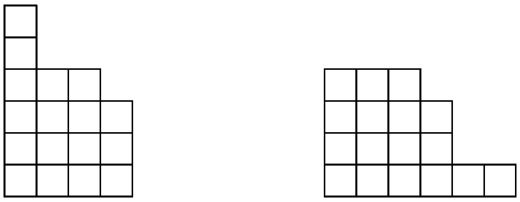


Figure 1. The Young diagrams for the partition $(4,4,4,3,1,1)$ and the conjugate partition $(6,4,4,3)$

If $\mu_i \leq \lambda_i$, then we say that $\mu \subseteq \lambda$. For such a pair of partitions define a skew partition λ/μ to be the diagram of cells that are in λ and not in μ. A ribbon in our notation will be a connected skew partition with no 2×2 subdiagrams. It is easily shown that every ribbon R is λ/λ^{rc} where $\lambda^{rc} = (\lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_{\ell(\lambda)} - 1)$ for some partition λ. We will use the symbol R to represent an arbitrary ribbon and the notation $R \models k$ to indicate that R is a ribbon of size k.

Figure 2. An example of Young diagrams for a skew partition $(6,4,4,3)/(2,1)$ and the ribbon $(6,4,4,3)/(3,3,2)$

Label the cells in a ribbon diagram of size k with the numbers $\{1,2,\ldots,k\}$ from left to right, top to bottom. This done, we let

$$D(R) = \{i \in [1, k-1] : i + 1^{st} \text{ cell of } R \text{ lies below the } i^{th} \text{ cell }\} \tag{1}$$

and refer to it as the descent set of R. The ribbons are therefore in one to one correspondence with the subsets of $\{1, 2, \ldots k-1\}$. If R is a ribbon of size k, we use $\overline{R}$ to denote the ribbon whose descent set is $\{1, 2, \ldots k-1\} - D(R)$. The diagram of $\overline{R}$ is the diagram of R flipped about the line $x = -y$.

Figure 3. The Young diagrams for the ribbon $R = (4,4,4,3)/(3,3,2)$ with descent set $\{3,5,6\}$ and the ribbon with complement descent set $\overline{R} = (4,2,1,1)/(1)$ with descent set $\{1,2,4\}$

We will consider the space of symmetric functions as the polynomials in the simple power symmetric functions, that is, $\Lambda = Q[p_1, p_2, p_3, \ldots]$. A linear basis for this space then is the set $\{p_\lambda : \lambda \vdash n, n \geq 0\}$ where $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$ since these are the monomials in this polynomial algebra. The degree of a monomial, p_λ, in Λ is given by the size of the indexing partition, $|\lambda|$. Λ is endowed with an inner product that is defined by $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$ with $\delta_{\lambda\mu} = 1$ if $\lambda = \mu$ and 0 otherwise and $z_\lambda := \prod_{i \geq 1} n_i(\lambda)! i^{n_i(\lambda)}$ with $n_i(\lambda)$ is the number of j such that $\lambda_j = i$.

This is not the classical definition of the symmetric functions since we do not refer to functions with variables that are invariant under permutations, but Λ has the same algebraic structure as the classical definition since these spaces are isomorphic under the map that sends p_k to $x_1^k + x_2^k + x_3^k + \cdots$. In this exposition, it is not necessary to refer to the variables of the symmetric function, hence we use this simplified definition.

We will also consider the homogeneous h_λ, elementary e_λ and Schur s_λ bases for the symmetric functions. We define

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}, \text{ with } h_n := \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda},$$

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}} \text{ with } e_n := \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)} p_\lambda}{z_\lambda},$$

$$\text{and } s_\lambda := det|h_{\lambda_i+i-j}|_{1 \le i,j \le \ell(\lambda)}.$$

Consider the involution ω, with the property that $\omega(s_\lambda) = s_{\lambda'}$. This operator is ubiquitous and fundamental to the theory of symmetric functions. It is an algebra homomorphism and has the property that $\omega(h_\lambda) = e_\lambda$, and $\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda$.

For any symmetric function f, we will represent by $f^\perp$ the operation which is dual to multiplication by f with respect to the inner product. That is, we define for $f, g \in \Lambda$ that

$$f^\perp g = \sum_\lambda \langle g, f s_\lambda \rangle \, s_\lambda.$$

If f is a symmetric function of degree k, then the operation of multiplication by f raises the degree of the symmetric function it is acting on by k. Similarly, the operation of $f^\perp$ lowers the degree of the symmetric function that it is acting on by k. This given, we define a Schur function indexed by a skew partition as $s_{\lambda/\mu} := s_\mu^\perp s_\lambda$. These symmetric functions may be expressed in terms of the Schur basis by using the Littlewood-Richardson rule (see for example [4] section I.9).

We will make extensive use of a family of operators due to Bernstein [9] which add either a row or a column on the Schur basis. Define $S_m = \sum_{i \ge 0} (-1)^i h_{m+i} e_i^\perp$ and $\tilde{S}_m = \sum_{i \ge 0} (-1)^i e_{m+i} h_i^\perp$. We find that $S_m s_\lambda = s_{(m,\lambda)}$ if $m \ge \lambda_1$ and $\tilde{S}_m s_\lambda = \omega s_{(m,\lambda')}$ if $m \ge \ell(\lambda)$. If $m < \lambda_1$ or $m < \ell(\lambda)$, then the operators may be 'straightened' with the relations

$$S_m S_{m+1} = \tilde{S}_m \tilde{S}_{m+1} = 0 \tag{2}$$

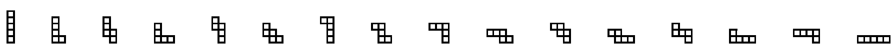
$$S_m S_n = -S_{n-1} S_{m+1} \tag{3}$$

$$\tilde{S}_m \tilde{S}_n = -\tilde{S}_{n-1} \tilde{S}_{m+1}. \tag{4}$$

Let V be an element of $Hom(\Lambda, \Lambda)$. Define the bar of V to be the element of $Hom(\Lambda, \Lambda)$ given by the formula $\overline{V} = \sum_\lambda (-1)^{|\lambda|} V(s_\lambda) s_{\lambda'}^\perp$. The bar operation is an unusual but remarkable involution on the space $Hom(\Lambda, \Lambda)$. It arises naturally in the study of operators that add rows or columns to bases of symmetric functions [6].

Proposition 1. *For $V \in Hom(\Lambda, \Lambda)$, $\overline{\overline{V}} = V$.*

Proof. $\overline{\overline{V}}(s_\gamma) = \sum_\lambda (-1)^{|\lambda|} \overline{V}(s_\lambda) s_{\gamma/\lambda'} = \sum_{\lambda,\mu} (-1)^{|\lambda|+|\mu|} V(s_\mu) s_{\lambda/\mu'} s_{\gamma/\lambda'}$ Now from [4] p. 90 we know that $\sum_\lambda (-1)^{|\lambda|} s_{\lambda'/\mu} s_{\gamma/\lambda}$ is 0 if $\mu \ne \gamma$ and if $\mu = \gamma$ then this expression is $(-1)^{|\mu|}$. Therefore $\overline{\overline{V}}(s_\gamma) = V(s_\gamma)$. $\square$



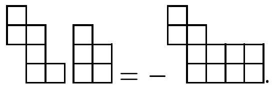
3. The ribbon rule

To each ribbon we associate an operator which has the property that it attaches a form of the ribbon to the left of the partition indexing a Schur function. For a ribbon R that is equal to λ/λ^{rc} for a partition λ we set

$$S^R := s_{\lambda^{rc}}^\perp \tilde{S}^{\lambda'_1} \tilde{S}^{\lambda'_2} \cdots \tilde{S}^{\lambda'_{\lambda_1}}. \tag{5}$$

That is, when S^R acts on a Schur function it adds a sequence of columns of form λ on the left of the partition followed by a skew operation $s_{\lambda^{rc}}^\perp$.

For example consider the ribbon $R = (3,2,2,1)/(1,1)$. The ribbon operator associated to it will be $s_{11}^{\perp}\tilde{S}_4\tilde{S}_3\tilde{S}_1$. The operator itself is very combinatorial in nature and is best calculated by using a picture and the relations given in equations 2 and 4. For this example we see that $S^R(s_{(2,2,1)})$ may be calculated by manipulating the picture



The negative sign in this operation arises from adding a column of size 1 on a column of size 3, this is then replaced with negative two columns of size 2 (from equation 4). The skew Schur function $-s_{(5,5,2,1)/(1,1)}$ can then be calculated with the Littlewood-Richardson rule to obtain $-s_{(5,5,1)} - s_{(5,4,2)} - s_{(5,4,1,1)} - s_{(4,4,2,1)}$.

We remark that every ribbon of size $k+1$ can be built up by adding a single cell to a ribbon of size k, either below or to the right of last cell. Now if R is a ribbon of size k and R^+ is a ribbon of size $k+1$ such that $D(R) = D(R^+)$ (add a cell to the right), then we have the obvious relationship between their ribbon operators $S^{R^+} = S^R\tilde{S}_1$. If R_+ is a ribbon of size $k+1$ such that $D(R_+) = D(R) \cup \{k\}$ (add a cell below), then we have the following useful relation that allows us to recursively build up any ribbon operator one cell at a time.

Proposition 2. *If $R \models k$ and $R_+ \models k+1$ such that $D(R_+) = D(R) \cup \{k\}$ then*

$$S^{R_+} = \overline{\overline{S^R}S_1}.$$

It follows from these relations that the bar of a ribbon operator is another ribbon operator (or nearly so). The bar operation is a completely algebraic construction and from the definition it is not immediately clear that this involution is even non-trivial. A priori we should not expect that the bar of a ribbon operator to be anything interesting, but what we find in the following proposition is that the bar operation permutes these operators in a very combinatorial manner.

Proposition 3. *If $R \models k$ and λ is a partition such that $\ell(\lambda) \leq k$ then $\overline{S^R}(s_\lambda) = \omega S^{\overline{R}}\omega(s_\lambda)$.*

The preceding relations also give a recursive method for defining the sum over all ribbon operators of size k.

Proposition 4. *Let $H_{1^1}^q = \tilde{S}_1$ and for $k > 1$ set $H_{1^k} = H_{1^{k-1}}\tilde{S}_1 + \overline{\overline{H_{1^{k-1}}}S_1}$, then*

$$H_{1^k} = \sum_{R \models k} S^R.$$

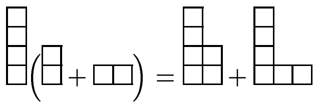
This sum of ribbon operators is important because of the commutation relation that it shares with the operation of multiplication by h_n. The following theorem shows that this operator has the remarkable property that it adds a column on the homogeneous symmetric functions. This gives us a method for computing expansions of the homogeneous symmetric functions in terms of Schur functions with a recurrence that is different from the Pieri rule.

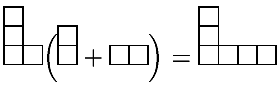
Theorem 5. *The operator $H_{1^k} = \sum_{R \models k} S^R$ has the property $H_{1^k}h_n = h_{n+1}H_{1^{k-1}}$. In particular, this implies that*

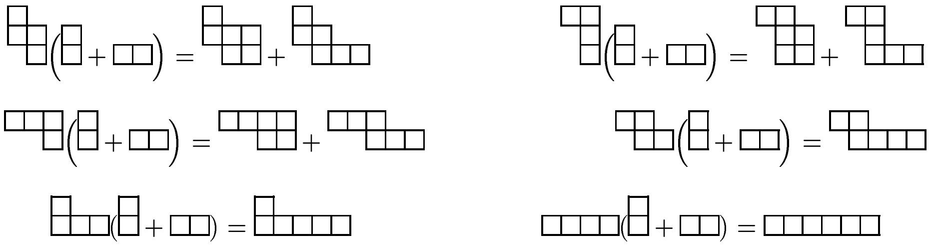
$$H_{1^{\lambda_1}}H_{1^{\lambda_2}}\cdots H_{1^{\lambda_{\ell(\lambda)}}}1 = h_{\lambda'}.$$

This combinatorial rule for computing expansions of homogeneous symmetric functions in terms of Schur functions is clearly more complicated than the Pieri rule, but it is elegant nontheless. We demonstrate here with an example.

Example 6. *To calculate the expansion of the homogeneous symmetric function $h_{(2211)}$ in terms of Schur functions we apply the 8 ribbon operators of size 4 to the homogeneous symmetric function h_{11}.*







From this calculation (applying the Littlewood-Richardson rule to reduce the Schur functions indexed by a skew shape) we see that $h_{(2211)} = s_{(2211)} + s_{(3111)} + 3s_{(411)} + 2s_{(33)} + 4s_{(321)} + 4s_{(42)} + s_{(222)} + 3s_{(51)} + s_{(6)}$.

It is clear in this example that all the terms are positive, but as was the case in the previous example of a ribbon operator, it is possible for negative terms to arise, but these subsequently cancel. Since the coefficient of s_λ in h_μ is the number of column strict tableaux of shape λ and content μ, it would be nice to have a combinatorial proof of this cancellation by giving a procedure for building the column strict tableaux of content $\mu + 1^k$ from the column strict tableaux of content μ.

4. A q-ribbon rule

This formula generalizes naturally to the q-analog of the homogeneous symmetric functions, the Hall-Littlewood symmetric functions. Define the family $H_\mu(q) := \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(q) s_\lambda$ where the coefficients $K_{\lambda\mu}(q)$ are the well known Kostka-Foulkes polynomials [4] p. 239. This family interpolates the homogeneous and Schur symmetric functions with the parameter q, since it has property that $H_\mu(1) = h_\mu$ and $H_\mu(0) = s_\mu$.

The family of operators $H^q_m := \sum_{i,j\geq 0} (-1)^i q^j h_{m+i+j} h_j^\perp e_i^\perp = \sum_{i \geq 0} q^j S_{m+i} h_i^\perp$ is due to Jing [3, 1] and have the property that $H^q_m H_\mu(q) = H_{(m,\mu)}(q)$ as long as $m \geq \mu_1$. They are a q-analog of the operation of multiplication by h_m and have the property that $H^1_m = h_m$ and $H^0_m = S_m$.

Define the major index of a ribbon to be the statistic $maj(R) = \sum_{i \in D(R)} i$. If $R \models k$, then set $comaj(R) = \binom{k}{2} - maj(R) = maj(\overline{R})$. This given, we define the operator $H^q_{1^k} := \sum_{R \models k} q^{comaj(R)} S^R$, which may be arrived at using the following recursive definition that is a q-analog of Proposition 4.

Proposition 7. *Let* $H^q_{1^1} = \tilde{S}_1$ *and for* $k > 1$ *set* $H^q_{1^k} = q^{k-1} H_{1^{k-1}} \tilde{S}_1 + \overline{\overline{H_{1^{k-1}}} S_1}$, *then*

$$H^q_{1^k} = \sum_{R \models k} q^{comaj(R)} S^R$$

This operator not only has the property that $H^q_{1^{\lambda_1}} H^q_{1^{\lambda_2}} \cdots H^q_{1^{\lambda_{\ell(\lambda)}}} 1 = H_{\lambda'}(q)$, but it also satisfies the stronger condition that it commutes in a natural way with the operator H^q_m.

Theorem 8. *The operator* $H_{1^k} = \sum_{R \models k} q^{comaj(R)} S^R$ *has the property* $H^q_{1^k} H^q_m = H^q_{m+1} H^q_{1^{k-1}}$. *In particular, this implies that*

$$H^q_{1^{\lambda_1}} H^q_{1^{\lambda_2}} \cdots H^q_{1^{\lambda_{\ell(\lambda)}}} 1 = H_{\lambda'}(q)$$

We will only mention here and refer the reader to [8, 2] that there is also a t-analog of this q-analog recurrence. An algebraic t-twisting of the operator $H^q_{1^k}$ has the property that it adds a column on the Macdonald symmetric functions. This algebraic definition can then be used to derive a combinatorial rule for calculating the Macdonald symmetric functions of size $n + k$ from the Macdonald symmetric functions of size n. It will be interesting to see if this recurrence can be used to find a combinatorial interpretation for the Macdonald q, t-Kostka coefficients.

References

[1] A. M. Garsia. Orthogonality of Milne's polynomials and raising operators, *Discrete Mathematics*, North-Holland. **99** (1992), 247-264.

[2] A. M. Garsia, M. A. Zabrocki. Polynomiality of the q, t-Kostka Revisited, `math.QA/0008199`, to appear.

[3] N. Jing. Vertex operators and Hall-Littlewood symmetric functions, *Adv. Math.* **87** (1991), 226–248.

[4] I. G. Macdonald. "Symmetric Functions and Hall Polynomials," Oxford Mathematical Monographs, Oxford UP, second edition, 1995.

[5] A. O. Morris. The characters of the group $GL(n, q)$, *Math. Zeitschr.* **81** (1963), 112-123.

[6] M. Zabrocki. Vertex operators for standard bases of the symmetric functions, *J. of Alg. Comb.*, 13, No. 1 (2000), pp. 83-101.

[7] M. Zabrocki. Ribbon operators and Hall-Littlewood symmetric functions, *Adv. in Math.*, 156, No. 1 (2000), pp. 33-43.

[8] M. Zabrocki. q-Analogs of symmetric function operators, `math.QA/0008188`, to appear in issue of Discrete Mathematics in honor of LaCIM.

[9] A. V. Zelevinsky. Representations of finite classical groups: a Hopf algebra approach, Springer Lecture Notes, **869**, 1981.

E-mail address: `zabrocki@mathstat.yorku.ca`

Mathematics and Statistics, York University, Toronto, ON M3J 1P3, Canada