A GENERALIZATION OF THE RANDOM ASSIGNMENT PROBLEM

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ABSTRACT. We give a conjecture for the expected value of the optimal k-assignment in an $m \times n$-matrix, where the entries are all exp(1)-distributed random variables or zeros. This conjecture generalizes a conjecture of Coppersmith and Sorkin. We prove this conjecture for the case that there is a zero-cost $k-1$-assignment. Assuming our conjecture, we determine some limits, as $k = m = n \to \infty$, of the expected cost of an optimal n-assignment in an n by n matrix with zeros in a given region. If we take the region outside a quarter-circle inscribed in the square matrix, this limit is thus conjectured to be $\pi^2/24$.

1. INTRODUCTION

Suppose we are given an m by n array of nonnegative real numbers. A *k-assignment* is a set of k entries, no two of which are in the same row or the same column (we "assign" rows to columns, or vice versa). The *cost* of the assignment is the sum of the entries. A k-assignment is *optimal* if its cost is minimal among all k-assignments.

We let $F_k(m,n)$ denote the expected cost of the optimal k-assignment in an m by n array of independent exponentially distributed random variables with mean 1.

D. Aldous [A92, A00] has recently shown that of $\lim_{n\to\infty} F_n(n,n) = \pi^2/6$. Moreover, Parisi [P98] has conjectured that $F_n(n,n) = 1 + 1/4 + 1/9 + \cdots + 1/n^2$. Computer generated verifications of this conjecture have been made for $n \leq 7$, see [LW].

In [CS99], a more precise conjecture was formulated.

Conjecture 1.1.

$$F_k(m,n) = \sum_{\substack{i+j<k \\ i,j\geq 0}} \frac{1}{(m-i)(n-j)}.$$

It is also shown in [CS99] that this conjecture is in accordance with the conjecture of Parisi if $k = m = n$. We have verified Conjecture 1.1 for $k \leq 6$, (and arbitrary m and n). The cases $k = 3, 4$ are treated in [AS99]. The verification for the case $k = 5$ can be found in [LW]. For $k = 6$, about 10,000 cases were needed.

1.1. **The Main Conjecture.** As is shown in [CS99], [AS99], and in Section 2, the calculation of $F_k(m,n)$ can in some (but not all, see [LW]) cases be done recursively by reducing the problem to several assignment problems for matrices which have zeros in certain positions.

Let Z be a finite set of matrix-positions, that is,

$$Z = \{(i_1, j_1), \ldots, (i_r, j_r)\},$$

for some positive integers $i_1, j_1, \ldots, i_r, j_r$. Suppose $m \geq \max(i_1, \ldots, i_r)$ and $n \geq \max(j_1, \ldots, j_r)$. We define $F_{k,Z}(m,n)$ to be the expected cost of the optimal k-assignment in an m by n matrix with zeros in the positions belonging to Z, and independent exponential random variables with mean 1 in all other positions.

We will consider sets of rows and columns in the $m \times n$-matrix. A set α of rows and columns is said to *cover* Z if every entry in Z is on either a row or on a column in α. A covering with s rows and columns will be called an s-covering. $k-1$-coverings will be of particular importance. By a *partial $k-1$-covering* of Z, we mean a set of rows and columns which is a subset of a $k-1$-covering of Z. Let $p_{i,j}$ be the probability that a set of i rows and j columns, chosen from uniform distribution on all such sets, is a partial $k-1$-covering.

We now present a conjecture for the values of $F_{k,Z}(m,n)$.

Conjecture 1.2 (Main Conjecture)**.**

$$F_{k,Z}(m,n) = \sum_{\substack{i+j<k \\ i,j \geq 0}} \frac{p_{i,j}}{(m-i)(n-j)} \tag{1}$$

Example 1.3. When Z is empty, Conjecture 1.2 clearly specializes to Conjecture 1.1.

Example 1.4. When $k=2$ and Z has just one element, then $p_{1,0} = 1/m$, $p_{0,1} = 1/n$ and $p_{0,0} = 1$, so according to Conjecture 1.2 we get

$$F_{2,Z}(m,n) = \frac{1}{mn} + \frac{1/m}{(m-1)n} + \frac{1/n}{m(n-1)} = \frac{-1}{mn} + \frac{1}{(m-1)n} + \frac{1}{m(n-1)},$$

which is also the correct value.

In fact, we prove in Theorem 3.7 that Conjecture 1.2 holds under the assumption that Z contains a $k-1$-assignment. This is our main theorem.

Let P be the poset of all intersections of $k-1$-coverings of Z, ordered by reverse inclusion, and with an artificial minimal element $\hat{0}$ corresponding to the empty intersection. We prove in Section 3 that the following is an equivalent form of the Conjecture 1.2:

$$F_{k,Z}(m,n) = \sum_{\alpha \in P \setminus \hat{0}} \frac{-\mu_P(\alpha)}{(m-i_\alpha)(n-j_\alpha)}, \tag{2}$$

where $\mu_P(\alpha)$ is the Möbius function of the interval $(\hat{0}, \alpha)$ in P and i_α, j_α are the number of rows and columns in α respectively.

In Section 2 we describe our tools to recursively compute $F_{k,Z}(m,n)$. In Section 4 we prove that some asymptotic results follows from Conjecture 1.2.

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2. Computation of $F_{k,Z}(m,n)$

We say that a set α of rows and columns is an *optimal covering* of a set Z, if α covers Z, and α has minimal cardinality among all coverings of Z.

If A is a (random) m by n matrix, and u is an assignment, then we let $\Psi_u(A)$ be the cost of u, that is,

$$\Psi_u(A) = \sum_{(i,j) \in u} A_{i,j}.$$

If k is an integer not greater than m or n, we let $\Psi_k(A)$ be the cost of the optimal k-assignment in A.

The following lemma is proved in Section 2.1.

Lemma 2.1. *If α is an optimal covering of a subset of Z, then there is an optimal k-assignment which intersects every row and every column of α.*

Our calculations of $F_{k,Z}(m,n)$ are based on the following theorem.

Theorem 2.2. *Let $k \leq m, n$ be positive integers. Let A be an m by n matrix, with zeros in a certain set Z of positions, and (possibly random) positive values outside Z. Let α be an optimal covering of Z. Let ϵ be a positive real (possibly random) number, not greater than any of the entries not covered by α. Let A' be the matrix obtained from A by subtracting ϵ from every entry not covered by α, and adding ϵ to every doubly covered entry of A. Then*

$$\Psi_k(A) = \epsilon \cdot (k - |\alpha|) + \Psi_k(A').$$

Proof. Let u be an optimal k-assignment of A' intersecting every row and column of α. Such an optimal assignment exists by Lemma 2.1. Note that subtracting ϵ from every entry not covered by α, and adding ϵ to every doubly covered entry is the same thing as first subtracting ϵ from all entries not covered by the rows of α, and then adding ϵ to all the entries covered by the columns of α. Let i and j be the number of rows and columns in α, respectively. Then

$$\begin{aligned}(3)\quad \Psi_u(A) = \Psi_u(A') + \epsilon \cdot (k-i) - \epsilon \cdot j = \\ = \epsilon \cdot (k - |\alpha|) + \Psi_u(A') = \epsilon \cdot (k - |\alpha|) + \Psi_k(A').\end{aligned}$$

For every k-assignment v we have

$$\Psi_v(A) \geq \epsilon \cdot (k - |\alpha|) + \Psi_v(A') \geq \epsilon \cdot (k - |\alpha|) + \Psi_k(A').$$

Hence u is an optimal k-assignment also in A, and $\Psi_k(A)$ is given by (3). ☐

In the typical use of the theorem, ϵ is the minimum of all the non-covered entries, which gives a new zero when we subtract ϵ. The special case of this theorem where there are no doubly covered entries or all such entries are known not to be in the optimal k-assignment, was proved and used in [CS99] and [AS99] to prove Conjecture 1.1 for $k \leq 4$.

2.1. **Proof of Lemma 2.1.** We begin by citing a famous theorem of D. König:

Theorem 2.3. *If a set of matrix positions cannot be covered by $k-1$ rows and columns, then it contains a k-assignment.* ☐

Lemma 2.4. *Let $k \leq m, n$ be positive integers. Let A be an m by n matrix, with zeros in a certain set Z of positions, and (possibly random) positive values outside Z. Suppose that Z does not contain a $k+1$-assignment. Let α be an optimal covering of Z. Then every row and every column of α contains an element of every optimal k-assignment of A.*

Proof. Suppose that α contains rows $1, \dots, r$ and no other rows. Since α is optimal it follows by König's theorem that there is an r-assignment v in Z containing no element from the columns in α. Suppose (for a contradiction) that there is an optimal k-assignment u which does not use row 1.

We now construct a sequence of matrix positions (all with zeros) as follows: Let v_0 be the element of v which is in the first row. Suppose that we have defined $v_0, \dots, v_h$ and $u_1, \dots, u_h$. Then if there is an element of $u \cap Z$ in the same column as v_h, let this element be u_{h+1}, and let v_{h+1} be the element of v which is in the same row as u_{h+1}. Since u does not contain any element from the first row, the sequences $v_0, v_1, v_2, \dots$ and $u_1, u_2, \dots$ cannot contain any repetitions of the same element. Hence the sequence must end with an element v_h such that there is no element of $u \cap Z$ in the same column.

We now consider two cases. Suppose first that no element in the column of v_h belongs to u. Since Z does not contain a $k+1$-assignment, the cardinality of α is at most k. Each row and column of α covers at most one element of u, and row 1 does not cover any element of u. Consequently there is an element (i,j) of u which is not covered by α, hence does not belong to Z. Then $u \setminus \{u_1, \dots, u_{h-1}, (i,j)\} \cup \{v_0, \dots, v_h\}$ is a k-assignment of smaller cost than u, a contradiction. If on the other hand there is an element (i',j') of u in the first column, then $u \setminus \{u_1,\dots,u_{h-1},(i',j')\} \cup \{v_0,\dots,v_h\}$ is a k-assignment of smaller cost than u. This contradiction proves the lemma. □

Now Lemma 2.1 follows by a continuity argument. Let the values in the positions of Z not covered by α be ϵ, and let ϵ tend to zero. Since there are only finitely many k-assignments, there has to be an assignment which is optimal for all sufficiently small ϵ, which by Lemma 2.4 has to intersect every row and column of α. This assignment is also optimal for $\epsilon = 0$.

Note that it is not true that every optimal k-assignment has to intersect the rows and columns of an optimal covering of a subset of Z.

2.2. **Superfluous matrix elements.** It can be of great help to know that a certain entry of the matrix cannot occur in any optimal assignment. Sufficient conditions to draw this conclusion were given in Lemma 26 of [CS99] and Lemma 9 of [AS99]. The following lemma gives a necessary and sufficient condition. The condition here is in fact equivalent to the condition given in [AS99].

Theorem 2.5. *Let $k \le m,n$ be positive integers. Let $A = (a_{i,j})$ be an m by n matrix. Suppose that a set Z of entries are zero, and the remaining entries are random variables that can take arbitrary positive values. Suppose that $(i,j) \notin Z$. If every set of $k-1$ rows and columns covering Z also covers (i,j), then (i,j) cannot belong to an optimal k-assignment. Conversely, if there is a $k-1$-covering of Z which does not cover (i,j), then it is possible to assign positive values to the matrix entries not in Z in such a way that (i,j) belongs to a unique optimal k-assignment.*

Proof. By the theorem of König, a set contains a k-assignment if and only if it cannot be covered by fewer than k rows and columns. The minimal cost of a k-assignment is of course the same as the minimal cost of a set containing a k-assignment. In trying to minimize the cost of such a set, it suffices to consider sets containing Z, since the entries in Z do not increase the cost.

Suppose that (i,j) is covered whenever Z is covered with fewer than k rows and columns. Then from any set containing Z, which cannot be covered with fewer than k rows and columns, we can delete (i,j) and obtain another such set of smaller cost. This shows that (i,j) can never belong to an optimal k-assignment.

Suppose on the other hand that we can cover Z with a set α of $k-1$ rows and columns, without covering (i,j). Then we assign the value 1 to $a_{i,j}$, and to the entries not in Z which are covered by α, and values larger than k to all other entries not in Z. Since m and n are not smaller than k, the set of entries which are covered by α cannot be covered in any other way by $k-1$ rows and columns. Hence the set of entries which are ≤ 1 cannot be covered with $k-1$ rows and columns. By König's theorem there is a k-assignment of cost at most k. However, since every k-assignment must use at least one entry not covered by α, every k-assignment of cost at most k must use $a_{i,j}$. Hence $a_{i,j}$ belongs to the unique optimal k-assignment. □

In this situation, the fact that $F_{k,Z}(m,n) = F_{k,Z\cup\{(i,j)\}}(m,n)$ can sometimes simplify computation. It is possible that several elements have the property of (i,j) in the above theorem. Indeed, as the following theorem shows, it is sometimes possible to dispose of an entire row or column. If r is a row, we let $Z\backslash r$ denote the set where all positions of Z in row r has been removed and similarly for a column.

Theorem 2.6. *Suppose that a row r is used in every covering of Z by $k-1$ rows and columns (in particular, this is the case if at least k elements in r belong to Z). Let $Z' = Z \setminus r$. Then*

$$F_{k,Z}(m,n) = F_{k-1,Z'}(m-1,n). \tag{4}$$

Similarly, if a column c belongs to every covering of Z with fewer than k rows and columns, and $Z'' = Z \setminus c$, then $F_{k,Z}(m,n) = F_{k,Z''}(m,n-1)$.

Proof. If r is used in every covering of Z by $k-1$ rows and columns, then by Theorem 2.5, no non-zero element in row r, no matter how small, is used in any optimal k-assignment. Hence we do not change the cost of the optimal k-assignment if we replace all non-zero entries in row r by zeros. Let A' be the matrix obtained from A by deleting the row r. Every k-assignment of A must of course contain (at least) a $k-1$-assignment of A'. It is not a problem if Z contains an entry in row m, since we can always permute the rows in Z so row m becomes empty instead of row r.

Conversely, if row r consists of only zeros, then every $k-1$-assignment in A' can be extended to a k-assignment in A of the same cost. Hence $\Psi_k(A) = \Psi_{k-1}(A')$. Passing to expected values, we obtain (4).

The corresponding statement for columns is of course equivalent. □

3. Proof of the Main Theorem

In this section we give a proof of Conjecture 1.2 under the assumption that Z contains a $k-1$-assignment.

Let k, m, n and Z be as usual. Let $B(m,n)$ be the Boolean algebra of all sets of rows and columns in A, ordered by reverse inclusion. We define a function $f = f_{k,Z}$ on $B(m,n)$ by

$$f(\alpha) = \begin{cases} 1, \text{ if } \alpha \text{ is a partial } k-1\text{-covering} \\ 0, \text{ otherwise} \end{cases}$$

We define $g = g_{k,Z}$ by

$$\sum_{\beta \leq \alpha} g(\beta) = f(\alpha).$$

Lemma 3.1. *Let $P = P_{k,Z}(m,n)$ be the sub-poset of $B(m,n)$ consisting of all intersections of $k-1$-coverings, together with an artificial minimal element. Then*

$$g(\alpha) = \begin{cases} -\mu_P(\alpha), \text{ if } \alpha \in P \\ 0, \text{ if } \alpha \notin P \end{cases}$$

Proof. Suppose that α is minimal contradicting the statement. Then $\alpha \notin P$ and $g(\alpha) \neq 0$. Let α' be the intersection of all atoms of P which are smaller than α, that is, which are supersets of α. Then $\alpha' \in P$, and for every $\beta \in P$, if $\beta \leq \alpha$, then $\beta \leq \alpha'$. Hence $\sum_{\beta<\alpha} g(\beta) = 0$, and consequently, $g(\alpha) = 0$. This is a contradiction. □

For $\alpha \in B(m,n)$, we let i_α and j_α be the number of rows and columns in α, respectively.

As we will prove, a reformulation of our main conjecture is the following:

Conjecture 3.2.

$$F_{k,Z}(m,n) = \sum_{\alpha \in B(m,n)} \frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}. \tag{5}$$

Example 3.3. Let $k=3$ and $Z = \{(1,1)\}$. We will compute $F_{3,Z}(m,n)$ using Conjecture 3.2 and Lemma 3.1. The $k-1$-coverings of Z are all sets of two rows and columns including row one or column one. The poset $P = P_{3,Z}(m,n)$ and the Möbius function is shown in Figure 1. The sets with an index i, e.g. $r_1 r_i$, represents all $m-1$ sets with $2 \leq i \leq m$ and similarly the sets with an index j represents all $n-1$ sets with $2 \leq j \leq n$. We get

$$\begin{aligned} F_{3,Z}(m,n) = {} & (m-1)\frac{1}{(m-2)n} + (n-1)\frac{1}{(m-1)(n-1)} + \frac{1}{(m-1)(n-1)} + \\ & + (m-1)\frac{1}{(m-1)(n-1)} + (n-1)\frac{1}{m(n-2)} + \frac{-m-n+2}{(m-1)n} + \\ & + (m-1)\frac{-1}{(m-1)n} + (n-1)\frac{-1}{m(n-1)} + \frac{-m-n+2}{m(n-1)} + \frac{m+n-2}{mn} = \\ & = \frac{1}{(m-2)n} + \frac{1}{m(n-2)} + \frac{1}{(m-1)(n-1)} + \frac{1}{m(n-1)} + \frac{1}{(m-1)n} + \frac{-2}{mn} \end{aligned} \tag{6}$$

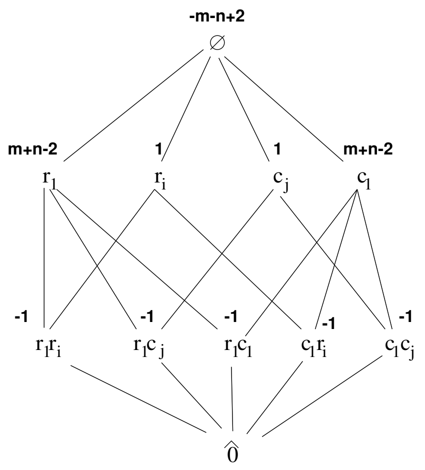

FIGURE 1. The poset $P_{3,Z}(m,n)$ in Example 3.3. The Möbius function is written in boldface.

We now state a lemma, which will also be useful later. We omit the proof.

Lemma 3.4. *If m and r are positive integers, then*

$$\sum_{i=0}^{r} \frac{(-1)^i \binom{r}{i}}{m-i} = \frac{(-1)^r}{m\binom{m-1}{r}}.$$

Proposition 3.5. *Conjecture 1.2 $\Leftrightarrow$ Conjecture 3.2.*

Proof. Let C be the set of partial $k-1$-coverings. For $\beta \in C$, we define

$$f_\beta(\alpha) = \begin{cases} 1, \text{ if } \alpha = \beta \\ 0, \text{ otherwise} \end{cases}$$

Obviously, $f = \sum_{\beta \in C} f_\beta$. Moreover,

$$g = \sum_{\beta \in C} g_\beta,$$

where g_β is defined by

$$\sum_{\gamma \le \alpha} g_\beta(\gamma) = f_\beta(\alpha).$$

We have $g_\beta(\alpha) = \mu(\beta,\alpha)$, the Möbius function of the interval (β,α) in the Boolean algebra over the set of rows and columns (ordered by reverse inclusion). This interval is itself a Boolean algebra. Hence

$$g_\beta(\alpha) = \begin{cases} (-1)^{|\beta|-|\alpha|}, \text{ if } \alpha \subseteq \beta \\ 0, \text{ otherwise} \end{cases}$$

For the right hand side of (5), we have

$$\sum_{\alpha\in B(m,n)}\frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}=\sum_{\beta\in C}\sum_{\alpha\geq\beta}\frac{(-1)^{|\beta|-|\alpha|}}{(m-i_\alpha)(n-j_\alpha)}= \tag{7}$$

$$=\sum_{\beta\in C}\left(\sum_{i=0}^{i_\beta}(-1)^i\frac{\binom{i_\beta}{i}}{m-i}\right)\left(\sum_{j=0}^{j_\beta}(-1)^j\frac{\binom{j_\beta}{j}}{n-j}\right)=$$

$$=[\text{by Lemma 3.4}]=\sum_{\beta\in C}\frac{1}{m\binom{m-1}{i_\beta}n\binom{n-1}{j_\beta}}=$$

$$=\sum_{\beta\in C}\frac{1}{\binom{m}{i_\beta}\binom{n}{j_\beta}(m-i_\beta)(n-j_\beta)}.$$

The factor

$$\frac{1}{\binom{m}{i_\beta}\binom{n}{j_\beta}}$$

can be interpreted as the probability of β under the uniform probability measure on all sets of i_β rows and j_β columns. Hence (7) equals (1). □

We need the following lemma.

Lemma 3.6. *Suppose Z contains a $k-1$-assignment u. Let $Z'=Z\cup\{(i,j)\}$, where (i,j) lies in a column not used by u. Then*

$$g_{k,Z'}(\alpha)=\begin{cases}g_{k,Z}(\alpha), \text{ if } \alpha \text{ contains row } i\\ 0, \text{ otherwise}\end{cases}$$

Proof. It follows from the assumptions that in order to cover Z' with $k-1$ rows and columns, one must use row i. This implies that $P_{k,Z'}(m,n)$ consists of precisely those elements of $P_{k,Z}(m,n)$ which contain row i. Since this means that $P_{k,Z'}(m,n)$ is an order-ideal in $P_{k,Z}(m,n)$, the Möbius function on $P_{k,Z'}(m,n)$ is the restriction of the Möbius function on $P_{k,Z}(m,n)$ to $P_{k,Z'}(m,n)$. Now the statement follows from Lemma 3.1. □

We now prove the formula of Conjecture 3.2 under the condition that Z contains a $k-1$-assignment.

Theorem 3.7. *If Z contains a $k-1$-assignment, then*

$$F_{k,Z}(m,n)=\sum_{\alpha\in B(m,n)}\frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}. \tag{8}$$

Proof. If Z contains a k-assignment, the statement is true. We will therefore assume that Z does not contain a k-assignment. We will use induction on k, and for a specific value of k, we will also use induction on the number of elements *not* belonging to Z. Hence we will assume that the statement has been proved for $F_{k,Z'}$, for every superset Z' of Z.

Notice that if there is a row r which belongs to every $k-1$-covering of Z, then by Theorem 2.6,

$$F_{k,Z}(m,n)=F_{k-1,Z\setminus r}(m-1,n).$$

By induction on k, this is

$$\sum_{\alpha\in P_{k-1,Z\setminus r}(m-1,n)\setminus\hat{0}}\frac{g(\alpha)}{(m-1-i_\alpha)(n-j_\alpha)}.$$

But $P_{k-1,Z\setminus r}(m-1,n)$ is isomorphic to $P_{k,Z}(m,n)$ via the isomorphism $\alpha\mapsto\alpha\cup\{r\}$, and $i_{\alpha\cup\{r\}}=i_\alpha+1$. Hence $F_{k-1,Z\setminus r}(m-1,n)$ equals (8).

We can assume that the $k-1$-assignment in Z is contained in the first $k-1$ rows and the first $k-1$ columns. Actually, we can assume that all elements of Z are contained in this $k-1$ by $k-1$ square, since if there is an element of Z in another column, say, then the row containing this element must be used in every $k-1$-covering of Z, and we can use induction on k.

We now subtract the minimum of the elements in columns $k,\dots,n$, and sum over the m possibilities for the row r containing the new zero, using induction:

$$F_{k,Z}(m,n)=\frac{1}{m(n-k+1)}+\frac{1}{m}\sum_{r}\sum_{\substack{\alpha\,\in\,P_{k,Z}(m,n)\setminus\hat{0}\\ r\,\in\,\alpha}}\frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}.$$

Now we change the order of summation, and use the fact that the number of rows r for which $r\in\alpha$ is i_α.

$$(9)\quad F_{k,Z}(m,n)=\frac{1}{m(n-k+1)}+\frac{1}{m}\sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}}\frac{i_\alpha g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}=$$

$$=\frac{1}{m(n-k+1)}+\sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}}\frac{g(\alpha)}{(m-i_\alpha)(n-j_\alpha)}-\sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}}\frac{g(\alpha)}{m(n-j_\alpha)}.$$

We have to show that the first term and the last sum cancel, that is, that

$$\sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}}\frac{g(\alpha)}{(n-j_\alpha)}=\frac{1}{n-k+1}.$$

When α is the set of the first $k-1$ columns, we get a term which equals the right hand side. Hence we are going to show that everything else cancels. It clearly suffices to show that if J is a proper subset of the set of the first $k-1$ columns, then

$$\sum_{\substack{\alpha\,\in\,P_{k,Z}(m,n)\\ column-set(\alpha)\,=\,J}} g(\alpha)=0.$$

If J is nonempty, then the induced sub-poset of $P_{k,Z}(m,n)$ consisting of those elements that contain J is isomorphic to $P_{k-|J|,Z\setminus J}(m,n-|J|)$ via the isomorphism $\alpha\mapsto\alpha\setminus J$. Hence the claim follows by induction on k.

If on the other hand $J=\emptyset$, we have

$$\sum_{\substack{\alpha\,\in\,P_{k,Z}(m,n)\\ column-set(\alpha)\,=\,\emptyset}} g(\alpha)=\sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}} g(\alpha)-\sum_{\substack{\alpha\,\in\,P_{k,Z}(m,n)\,\setminus\,\hat{0}\\ column-set(\alpha)\,\neq\,\emptyset}} g(\alpha).$$

By what we already know, the second sum equals 1, since everything cancels except the term where α is the set of the first $k-1$ columns. If we let α_0 be the intersection of all coverings of Z with $k-1$ rows and columns, then

$$\sum_{\alpha\in P_{k,Z}(m,n)\setminus\hat{0}} g(\alpha)=\sum_{\alpha\le\alpha_0} g(\alpha)=1,$$

by the definition of g. Hence

$$\sum_{\substack{\alpha\,\in\,P_{k,Z}(m,n)\\ column-set(\alpha)\,=\,\emptyset}} g(\alpha)=1-1=0,$$

which proves the theorem. □

As a special case Theorem 3.7 enables us to compute the expected cost for completing a zero cost $k-1$-assignment. This question was considered in [CS99] and answered in [CS98].

Corollary 3.8. Let $Z_k = \{(1,1),(2,2),\dots,(k-1,k-1)\}$. Then

$$F_{k,Z_k}(m,n) = \sum_{i+j<k} \binom{k-1}{i,j,k-1-i-j} \frac{(-1)^{k-1-i-j}}{(m-i)(n-j)}.$$

In particular,

$$F_{k,Z_k}(k,n) = \frac{1}{k}\sum_{j=0}^{k-1} \frac{1}{n-j},$$

and

$$F_{k,Z_k}(k,k) = \frac{1}{k}\left(1+1/2+\cdots+1/k\right) \sim \frac{\log k}{k}.$$

Proof. Omitted in this extended abstract. □

4. Asymptotic consequences

In this section, we show that certain limits can be computed from Conjecture 1.2. We assume throughout this section that Conjecture 1.2 holds.

Let K be a subset of the unit square $[0,1]\times[0,1]$. Let n be a positive integer. Later, we will let n tend to infinity. We divide the unit square into squares $s_{i,j}$ of side $1/n$, such that

$$s_{i,j} = \left[\frac{i-1}{n},\frac{i}{n}\right] \times \left[\frac{j-1}{n},\frac{j}{n}\right].$$

Let Z_n be the set of all (i,j) such that $s_{i,j}$ intersects K. Under certain circumstances, we can compute the limit of $F_n = F_{n,Z_n}(n,n)$, as $n\to\infty$. For a fixed n we have

$$F_n = \sum_{i,j} \frac{p_{i,j}}{(n-i)(n-j)},$$

where $p_{i,j}$ is the probability that a set of i rows and j columns is a partial $k-1$-covering.

By the substitutions $x=i/n$, $y=j/n$, $p_n(x,y) = p_{\lfloor xn\rfloor,\lfloor yn\rfloor}$, we can rewrite this as

$$\int_0^1\int_0^1 \frac{p_n(x,y)}{(1-x)(1-y)}\,dx\,dy.$$

If S is a subset of $\mathbb{R}^2$, we let $M_{a,b}(S)$ be the infimum of $a\lambda(S_1)+b\lambda(S_2)$ (Lebesgue-measure), taken over all sets S_1, S_2 of real numbers such that $S\subseteq (S_1\times\mathbb{R})\cup(\mathbb{R}\times S_2)$.

Under some reasonable assumptions on K, we can determine the limit of $p_n(x,y)$ as $n\to\infty$. We say that a covering of Z_n is *minimal*, if no proper subset is a covering of Z_n. It turns out that if K is compact, and for every ϵ, the number of minimal coverings of Z_n grows slower than $(1+\epsilon)^n$, as $n\to\infty$, then as $n\to\infty$,

$$p_n(x,y) \to \begin{cases} 1, \text{ if } M_{1-x,1-y}(K) < 1-x-y,\\ 0, \text{ if } M_{1-x,1-y}(K) > 1-x-y.\end{cases}$$

Corollary 4.1. If K is compact, and the number of minimal coverings of Z_n grows slower than $(1+\epsilon)^n$ for every ϵ, as $n\to\infty$, then

$$\lim_{n\to\infty} F_n = \int_D \frac{dx\,dy}{(1-x)(1-y)},$$

where D is the region in which $M_{1-x,1-y}(K) < 1-x-y$.

4.1. **An example: Zeros outside a quarter of a circle.** Let K be the region $x^2 + y^2 \geq 1$, $0 \leq x, y \leq 1$. To find $M_{a,b}(K)$, for given a and b, we have to find the point where the slope of the curve $x^2 + y^2 = 1$ is $-a/b$. The part of the curve which is above this point should be covered with a horizontal strip, and the part which is to the right of this point should be covered with a vertical strip. This point is

$$\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

Hence

$$M_{a,b}(K) = a\left(1 - \frac{a}{\sqrt{a^2+b^2}}\right) + b\left(1 - \frac{b}{\sqrt{a^2+b^2}}\right) = a + b - \sqrt{a^2+b^2}.$$

The number $T(n)$ of minimal coverings of Z_n is at most $n+1$, since any minimal covering consists of, for some i, $0 \leq i \leq n$, columns $i, \dots, n$, and all rows that contain some element of Z_n not covered by the columns. The region D is given by the inequality $M(K_{1-x,1-y}) < 1 - x - y$, which becomes

$$(1-x)^2 + (1-y)^2 > 1.$$

In other words, D is the part of the unit square which is outside the circle of radius 1 centered in the point $(1, 1)$. By Corollary 4.1, we have

$$\lim_{n\to\infty} F_n = \int_D \frac{dx\,dy}{(1-x)(1-y)} = \frac{\pi^2}{24}.$$

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