COMBINATORIAL ADDITION FORMULAS

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ABSTRACT. We derive addition formulas at the combinatorial level, that is equations of the form $F(X_1 + X_2 + \cdots + X_k) = \Phi_F(X_1, X_2, \ldots, X_k)$, where $F = F(X)$ is a given combinatorial species and Φ_F is a species on k sorts of singletons $X_1, X_2, \ldots, X_k$, depending on F.

RÉSUMÉ. Nous obtenons des formules d'addition combinatoires, c'est-à-dire, des équations de la forme $F(X_1+X_2+\cdots+X_k) = \Phi_F(X_1, X_2, \ldots, X_k)$, où $F = F(X)$ est une espèce de structures donnée et Φ_F est une espèce, dépendant de F, sur k sortes de singletons $X_1, X_2, \ldots, X_k$.

1. INTRODUCTION

Addition formulas are well known in classical analysis. They involve trigonometric, elliptic and other special functions and algebraic identities. Examples include

$$e^{x+y} = e^x \cdot e^y, \quad \sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)\,,$$
$$(x + y + z)^n = \sum_{i+j+k=n} \tfrac{n!}{i!j!k!} x^i y^j z^k\,.$$

The purpose of this paper is to study addition formulas at the combinatorial level. In this context, the functions are replaced by combinatorial species and the variables are replaced by various *sorts* of *singletons* representing various sorts of elements (labels) in the underlying sets of the structures.

Informally, a combinatorial species is a class, G, of finite labelled structures which is closed under relabeling along bijections. When the labels are of different sorts, $X_1, \ldots, X_k$, the relabellings must be made along bijections that preserve each sort X_i of elements. We write $G = G(X_1, \ldots, X_k)$ to express the fact that G is a species of structures on the sorts $X_1, \ldots, X_k$. The theory of species was founded by Joyal in [5] and a comprehensive English presentation of the theory can be found in [3].

Definition 1.1. Let $F = F(X)$ be a combinatorial species of one sort. A *combinatorial addition formula* for the species F is a combinatorial equation of the form

$$F(X_1 + X_2 + \cdots + X_k) = \Phi_F(X_1, X_2, \ldots, X_k) \tag{1.1}$$

where Φ_F is a species, depending on F, on k sorts of singletons $X_1, X_2, \ldots, X_k$, $k \geq 2$. △

Note that the species Φ_F in (1.1) is necessarily *symmetric* in $X_1, X_2, \ldots, X_k$, that is: for every permutation $\sigma \in S_k$ we have a corresponding combinatorial equation

$$\Phi_F(X_1, X_2, \ldots, X_k) = \Phi_F(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(k)}). \tag{1.2}$$

One of the simplest instances of a combinatorial addition formula is the «exponential» formula satisfied by the species, E, of sets

$$E(X_1 + X_2 + \cdots + X_k) = E(X_1) \cdot E(X_2) \cdots E(X_k). \tag{1.3}$$

Indeed, any set U made up of elements of various sorts $X_1, \ldots, X_k$ can be canonically identified with a k-uple, $(U_1, U_2, \ldots, U_k)$, of sets where

$$U_i = \{u \in U \mid u \text{ is of sort } X_i\}, \quad i = 1, \ldots, k. \tag{1.4}$$

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Another example, for $k=2$, is the addition formula satisfied by the species, C, of oriented cycles [3], namely

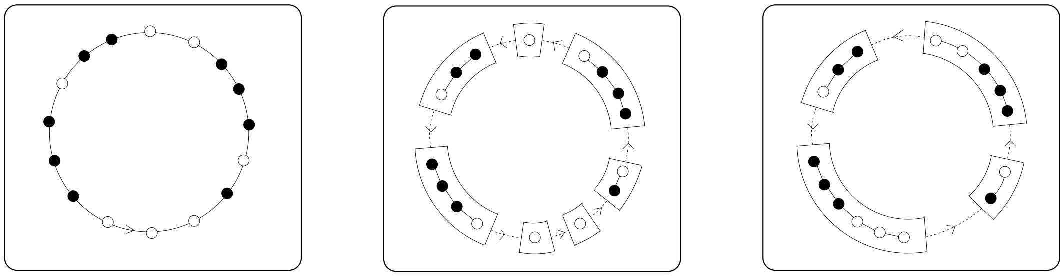
$$C(X+Y)=C(X)+C(L(X)Y), \tag{1.5}$$

where L is the species of (possibly empty) lists. This formula expresses the fact that if a $C(X+Y)$-structure has at least one element of sort Y, then it can be identified, in a natural manner, to an oriented cycle formed of disjoint lists of the form $x_1x_2\cdots x_\nu y$, where the x_i are distinct elements of sort X and y is an element of sort Y(see Figure 1b).

There exists, however, a more «symmetrical» version of (1.5), that is

$$C(X+Y)=C(X)+C(Y)+C(L_+(X)\cdot L_+(Y)), \tag{1.6}$$

where L_+ is the species of non empty lists (see Figure1c).



a) b) c)

FIGURE 1. $C(X+Y)$

Since any species can be written as a sum of irreducible ones (molecular species) and since the transformation

$$F(X)\longmapsto F(X_1+X_2+\cdots+X_k) \tag{1.7}$$

is obviously linear in F, we focus our attention, in the present paper, on addition formulas of the form

$$M(X_1+X_2+\cdots+X_k)=\sum_N c_{M,N}N(X_1,X_2,\ldots,X_k) \tag{1.8}$$

where $M=M(X)$ is a molecular species on the sort $X, N=N(X_1,\ldots,X_k)$ runs through molecular species on the sorts $X_1,X_2,\ldots,X_k$ and the coefficients $c_{M,N}$ are nonnegative integers. Finite molecular decompositions such as (1.8) can be considered as generalized multinomial formulas since they reduce to

$$(X_1+\cdots+X_k)^n=\sum_{n_1+\cdots+n_k=n}\frac{n!}{n_1!\cdots n_k!}X_1^{n_1}\cdots X_k^{n_k}, \tag{1.9}$$

in the case $M(X)=X^n$, the species of n-lists (lists of length n).

In Section 2, we make use of suitable weighted versions of Burnside's lemma, in the context of species, to develop some general addition formulas of type (1.8) when M is written in the canonical form $M(X)=X^n/H$, H being a subgroup of the symmetric group $S_n(H\leq S_n)$.

In Section 3, we analyze specific instances of addition formulas for special molecular species $M(X)$: the species X^n, of n-lists; $\mathrm{cha}_n(X)$, of n-chains; $E_n(X)$, of n-sets; $E_n^{\pm}(X)$, of oriented n-sets; $C_n(X)$, of oriented n-cycles, and $P_n(X)$, of n-gons (unoriented n-cycles).

Addition formulas can be used in various combinatorial settings, including:

- the computation of cycle index series and asymmetry index series [9] of species;

- the study of general substitution of species via Yeh's formulas [13] (see also [7])

$$F(m_1X_1+\cdots+m_kX_k)=F(X_1+\cdots+X_k)\times E(X_1)^{m_1}\cdots E(X_k)^{m_k} \tag{1.10}$$

  where "×" denotes the cartesian product of species and

$$F(X-Y)=F(X+kY)\big|_{k:=-1}. \tag{1.11}$$

- the analysis of generalized binomial coefficients $\binom{M}{N}_k$ for molecular species, introduced by the authors in [1] and [2], defined by the equations

$$M(X+k)=M(X+kY)\big|_{Y:=1}=\sum_N \binom{M}{N}_k N(X). \tag{1.12}$$

2. General addition formulas

In order to develop general addition formulas of type (1.8), we need to recall first some basic notions from the theory of group actions in the context of sets weighted in a commutative ring $\mathbb{A}$. A finite *weighted set* is an ordered pair $(\mathbb{W},w)$ where $\mathbb{W}$ is a finite set and $w:\mathbb{W}\longrightarrow\mathbb{A}$ is an arbitrary function, called a *weight function.* The *total weight* of $\mathbb{W}$ is denoted by,

$$|\mathbb{W}|_w=\sum_{s\in\mathbb{W}} w(s). \tag{2.1}$$

Let G be a finite group. A group action

$$G\times\mathbb{W}\longrightarrow\mathbb{W},\quad (g,s)\longmapsto g\cdot s, \tag{2.2}$$

is said to be *weight preserving* if $w(g\cdot s)=w(s)$, for every $g\in G$ and $s\in\mathbb{W}$. The *G-orbit* of an element $s\in\mathbb{W}$ is defined by $Gs=\{g\cdot s\mid g\in G\}$. The set of all G-orbits is denoted $\mathbb{W}/G$. It is a weighted set with the weight function $w^*:\mathbb{W}/G\longrightarrow A,\quad w^*(Gs)=w(s)$.

Lemma 2.1 (Burnside Lemma). *For any weight-preserving finite group action* (2.2), *the total weight of the G-orbits is given by*

$$|\mathbb{W}/G|_{w^*}=\frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|_w, \tag{2.3}$$

where

$$\mathrm{Fix}(g)=\{s\in\mathbb{W}\mid g\cdot s=s\}. \tag{2.4}$$

□

Now, our first general addition theorem for molecular species involves the Young subgroups $S_{n_1,n_2,\ldots,n_k}\simeq S_{n_1}\times S_{n_2}\times\ldots\times S_{n_k}$ of the symmetric group S_n which are defined by

$$S_{n_1,n_2,\ldots,n_k}=\{\sigma\in S_n\mid \sigma(\bar n_i)=\bar n_i,\ i=1,\ldots,k\}, \tag{2.5}$$

where $n=n_1+n_2+\cdots+n_k$ and

$$\bar n_1=[n_1],\ \bar n_2=[n_1+n_2]\setminus[n_1],\ \bar n_3=[n_1+n_2+n_3]\setminus[n_1+n_2],\ \ldots \tag{2.6}$$

where $[n]$ denotes the set $\{1,2,\ldots,n\}$. More generally, for $H\le S_n$, we write

$$H_{n_1,n_2,\ldots,n_k}=H\cap S_{n_1,n_2,\ldots,n_k}. \tag{2.7}$$

Theorem 2.1 (Generalized multinomial expansion). *Let $M=M(X)$ be a molecular species of degree n, then*

$$M(X_1+\cdots+X_k)=\sum_{n_1+\cdots+n_k=n}\sum_{s\in M[n]}\frac{|\mathrm{Aut}(s)_{n_1,\ldots,n_k}|}{n_1!\cdots n_k!}\frac{X_1^{n_1}\cdots X_k^{n_k}}{\mathrm{Aut}(s)_{n_1,\ldots,n_k}} \tag{2.8}$$

where

$$M[n] = \text{the set of all } M\text{-structures on } [n], \tag{2.9}$$

$$\mathrm{Aut}(s) = \{\sigma \in S_n \mid \sigma \text{ is automorphism of } s\} \leq S_n. \tag{2.10}$$

Proof

Any $M(X_1 + X_2 + \cdots + X_k)$-structure can be expressed in the form $(s, \bar{n}_1, \bar{n}_2, \ldots, \bar{n}_k)$, up to isomorphism, where $s \in M[n], n_1 + n_2 + \cdots + n_k = n$, the elements of $\bar{n}_i$ being interpreted as elements of sort X_i. The result then follows by applying, for each $(n_1, n_2, \ldots, n_k)$, Burnside Lemma with $G = S_{n_1,n_2,\ldots,n_k}, \mathbb{W} = \{(s, \bar{n}_1, \bar{n}_2, \ldots, \bar{n}_k) \mid s \in M[n]\} \simeq M[n], \mathbb{A} = \mathbb{Z}||X_1, X_2, \ldots, X_k||$, the ring of virtual species [6], [7], [8], [13], [3],

$$w(s, \bar{n}_1, \ldots, \bar{n}_k) = X_1^{n_1} X_2^{n_2} \ldots X_k^{n_k} / \mathrm{Aut}(s)_{n_1,\ldots,n_k}, \tag{2.11}$$

and the action of $\sigma \in S_{n_1,\ldots,n_k}$ on $\mathbb{W}$ is by transport of structures (relabelling). Note that the weight function w is compatible with this action. □

Here is a variant of the generalized multinomial expansion formula which uses the standard form X^n/H for the molecular species M.

Lemma 2.2. *Let $M = M(X) = X^n/H$ be a molecular species with $H \leq S_n$. Then*

$$M(X_1 + \cdots + X_k) = \sum_{U_1 + \cdots + U_k = [n]} \frac{|H_{U_1,\ldots,U_k}|}{|H|} \frac{X_1^{U_1} \cdots X_k^{U_k}}{H_{U_1,\ldots,U_k}} \tag{2.12}$$

where the sum runs over all k-uples $(U_1, U_2, \ldots, U_k)$ of disjoint subsets $U_i \subseteq [n]$, respecting $U_1 + U_2 + \cdots + U_k = [n]$, $H_{U_1,U_2,\ldots,U_k} = \{h \in H \mid h(U_i) = U_i, i = 1, \ldots k\}$ and $X_1^{U_1} X_2^{U_2} \cdots X_k^{U_k} / H_{U_1,U_2,\ldots,U_k}$ is the molecular species whose set of structures, on $[V_1, V_2, \ldots, V_k]$, is defined to be the set

$$\{\lambda H_{U_1,U_2,\ldots,U_k} \mid \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),\ \lambda_i : U_i \xrightarrow{\sim} V_i\}. \tag{2.13}$$

Proof

Let s_0 be a M-structure on $[n]$, having H as its stabilizer. Then any $M(X_1 + X_2 + \cdots + X_k)$-structure can be expressed, up to isomorphism, in the form $(s_0, U_1, U_2, \ldots, U_k)$ where $U_1 + U_2 + \cdots + U_k = [n]$. The result then follows by applying Burnside's lemma with

$$G = H, \quad \mathbb{W} = \{(s_0, U_1, U_2, \ldots, U_k) \mid U_1 + U_2 + \cdots + U_k = [n]\},$$

$$\mathbb{A} = \mathbb{Z}||X_1, \ldots, X_k||, \quad w(s_0, U_1, \ldots, U_k) = X_1^{U_1} \cdots X_k^{U_k} / H_{U_1,\ldots,U_k},$$

the action being given by

$$h \cdot (s_0, U_1, U_2, \ldots, U_k) = (s_0, h(U_1), h(U_2), \ldots, h(U_k)). \tag{2.14}$$

□

Our last general addition formula involves words on an ordered alphabet

$$\mathfrak{X} = \{x_1 < x_2 < \ldots < x_k\} \tag{2.15}$$

having one letter x_i associated with each sort X_i. Note that any subgroup $H \leq S_n$ acts on the set $\mathfrak{X}^n$ of words $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ of length n on $\mathfrak{X}$ by

$$h \cdot (\alpha_1 \alpha_2 \ldots \alpha_n) = \alpha_{h^{-1}(1)} \alpha_{h^{-1}(2)} \ldots \alpha_{h^{-1}(n)}. \tag{2.16}$$

An orbit of this action is called a *H-class* of words of length n on $\mathfrak{X}$.

Theorem 2.2. *Let $M = M(X) = X^n/H$ be a molecular species with $H \leq S_n$. Then*

$$M(X_1 + \cdots + X_k) = \sum_{n_1 + \cdots + n_k = n} \sum_{\alpha \in \Omega_{n_1,\ldots,n_k}} \frac{X_1^{n_1} \cdots X_k^{n_k}}{\sigma_\alpha (\mathrm{Aut}_H \alpha) \sigma_\alpha^{-1}} \tag{2.17}$$

where $\Omega_{n_1,n_2,\ldots,n_k}$ is a set of representatives (for example, lexicographically smallest words) of the H-classes of words on $\mathfrak{X} = \{x_1, x_2, \ldots, x_k\}$ having exactly n_i occurences of letter x_i for $i = 1, 2, \ldots, k$, and where for each word α, σ_α is any permutation of S_n such that $\sigma_\alpha(U_i) = \bar{n}_i$, where

$$U_i = \{j \mid 1 \leq j \leq n,\ \alpha_j = x_i\},\ i = 1, \ldots, k, \tag{2.18}$$

and

$$\mathrm{Aut}_H\alpha = \{h \in H \mid h \cdot \alpha = \alpha\}. \tag{2.19}$$

Proof

This follows from the proof of Lemma 2.2 using the canonical bijection $\alpha = \alpha_1\alpha_2 \ldots \alpha_n \longleftrightarrow (U_1, U_2, \ldots, U_k)$ where each U_i is given by (2.18). Taking into account that $\mathrm{Aut}_H\alpha = H_{U_1,U_2,\ldots,U_k}$ and that, in the ring of species,

$$X_1^{U_1} \cdots X_k^{U_k}/H_{U_1,\ldots,U_k} = X_1^{n_1} \ldots X_k^{n_k}/\sigma_\alpha(\mathrm{Aut}_H\alpha)\sigma_\alpha^{-1}, \tag{2.20}$$

formula (2.18) turns out to be simply the total weight $|\mathfrak{X}^n/H|_w$ of the set $\mathfrak{X}^n/H$ of H-classes of words on $\mathfrak{X}$ of lenght n. □

3. Specific addition formulas

3.1. **Lists, chains, sets and oriented sets. 3.1.1 Lists.** In the case of the species $M = M(X) = X^n$ of n-lists (here $H = \{\mathrm{id}_n\}$, the trivial one-element subgroup of S_n), it is easily checked that our general addition formulas reduce to the classical multinomial formula (1.9) mentioned in the Introduction.

3.1.2 Chains. Since chains can be seen as lists up to "reversal", the molecular species $M = M(X) = \mathrm{cha}_n(X)$ of n-chains is characterized by the combinatorial equation

$$\mathrm{cha}_n(X) = X^n/\mathbb{Z}_2, \tag{3.1}$$

where $\mathbb{Z}_2$ is interpreted as the subgroup $H = \langle\tau\rangle \leq S_n$ generated by the "global" transposition $\tau = (1, n)(2, n-1)(3, n-2) \ldots$. In this case, addition formula (2.17), takes the following form:

Theorem 3.1. *We have*

$$\begin{aligned} \mathrm{cha}_n(X_1 + \cdots + X_k) \quad &= \sum_{n_1+\cdots+n_k=n} \mathrm{pal}_{n_1,\ldots,n_k} \frac{X_1^{n_1} \cdots X_k^{n_k}}{\mathbb{Z}_2} \\ &\qquad + \frac{1}{2}\left(\frac{n!}{n_1! \cdots n_k!} - \mathrm{pal}_{n_1,\ldots,n_k}\right) X_1^{n_1} \cdots X_k^{n_k} \end{aligned} \tag{3.2}$$

where $\mathrm{pal}_{n_1,\ldots,n_k}$ is the number of palindromic words on $\mathfrak{X} = \{x_1, \ldots, x_k\}$ having n_i occurences of letter x_i for $i = 1, \ldots, k$ and the action of $\mathbb{Z}_2$ on $X_1^{n_1} \ldots X_k^{n_k}$-structures corresponds to the simulaneous reversal of the induced n_i-lists on $\bar{n}_i$.

The following explicit expressions are easily checked:

$$\mathrm{pal}_{n_1,\ldots,n_k} = \begin{cases} \dfrac{(\frac{n}{2})!}{(\frac{n_1}{2})! \cdots (\frac{n_k}{2})!}, & \text{if every } n_i \text{ is even;} \\[2ex] \dfrac{(\frac{n-1}{2})!}{(\frac{n_1}{2})! \cdots (\frac{n_j-1}{2})! \cdots (\frac{n_k}{2})!}, & \text{if only } n_j \text{ is odd;} \\[2ex] 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

3.1.3 Sets. Since sets are unordered lists, the molecular species $M = M(X) = E_n(X)$ of n-sets can be written in the form

$$E_n(X) = X^n/S_n. \tag{3.4}$$

There is only one E_n-structure on $[n]$ and addition formula (2.8) takes the form

$$E_n(X_1 + \cdots + X_k) = \sum_{n_1+\cdots+n_k=n} E_{n_1}(X_1)\cdots E_{n_k}(X_k), \tag{3.5}$$

since $X_1^{n_1}\cdots X_k^{n_k}/S_{n_1,\ldots,n_k} = (X_1^{n_1}/S_{n_1})\cdots(X_k^{n_k}/S_{n_k})$.

3.1.4 Oriented sets. By definition [3], an *oriented set* is a list up to an *even* permutation of its elements. This kind of structures has been used by Pólya in [11]. The molecular species $M = M(X) = E_n^{\pm}(X)$ of oriented n-sets is characterized by

$$E_n^{\pm}(X) = X^n/A_n, \tag{3.6}$$

where A_n is the alternating subgroup of S_n. For $n \geq 2$, there are exactly 2 oriented sets on $[n]$ each with stabilizer A_n (since A_n is normal in S_n). This time, addition formula (2.8) can be written in the following form.

Theorem 3.2. *We have*

$$E_n^{\pm}(X_1 + \cdots + X_k) = \sum_{n_1+\cdots+n_k=n} \frac{E_{n_1}^{\pm}(X_1)\cdots E_{n_k}^{\pm}(X_k)}{\equiv},$$

where two $E_{n_1}^{\pm}(X_1)\cdots E_{n_k}^{\pm}(X_k)$-structures are congruent ($\equiv$) if one can be obtained from the other by an even number of local non-trivial orientation reversals.

3.2. **Cycles and polygons. 3.2.1 Cycles.** The molecular species $M = M(X) = C_n(X)$ of (oriented) n-cycles is characterized by

$$C_n(X) = X^n/H, \quad H = \langle\rho\rangle \leq S_n, \tag{3.7}$$

where ρ is the n-cycle $(1, 2, \cdots, n)$. Addition formula (2.16), which uses words on $\mathfrak{X}$, is more suitable in this situation.

Theorem 3.3. [10] *We have*

$$C_n(X_1 + \cdots + X_k) = \sum_{d/n} \sum_{\psi_1+\cdots+\psi_k=d} \lambda_\psi C_{n/d}\left(X_1^{\psi_1}\cdots X_k^{\psi_k}\right) \tag{3.8}$$

where for $\psi = (\psi_1, \ldots, \psi_k), \lambda_\psi$ is the number of Lyndon words on $\mathfrak{X} = \{x_1 < \ldots < x_k\}$ having distribution ψ, i.e. ψ_ν occurences of the letter $x_\nu, \nu = 1, \ldots, k$.

Recall that a *Lyndon word* is a primitive (not a power of another word) lexicographically minimal word in its cyclical conjugacy class. Making use of Moebius inversion, it can be shown that

$$\lambda_\psi = \frac{1}{\psi_1 + \cdots + \psi_k} \sum_{\delta|\gcd(\psi_1,\cdots,\psi_k)} \mu(\delta) \binom{\frac{\psi_1+\cdots+\psi_k}{\delta}}{\frac{\psi_1}{\delta}, \cdots, \frac{\psi_k}{\delta}} \tag{3.9}$$

3.2.2 Polygons. By definition, a *polygon* is an unoriented cycle. The molecular species $M = M(X) = P_n(X)$ of n-gons (polygons on n vertices) is characterized by

$$P_n(X) = X^n/D_n, \tag{3.10}$$

where D_n is the dihedral $2n$-element subgroup of S_n with generators $\rho = (1, 2, \cdots, n)$ and $\tau = (1, n)(2, n-1)(3, n-2)\cdots$.

In order to state our addition formula for $P_n(X)$, we need some preliminary definitions and notations. Given the ordered alphabet $\mathfrak{X} = \{x_1 < x_2 < \cdots < x_k\}$, and a word $\mu = a_1a_2\cdots a_n$ on $\mathfrak{X}$:

- μ is *primitive* if it is not a power of another word on $\mathfrak{X}$;
- μ is a *palindrome* if $\mu = \bar{\mu}$ where $\bar{\mu} = a_na_{n-1}\cdots a_1$ (that is, $\bar{\mu}$ is μ read "backwards");
- μ is a *dexterpalindrome* if $\mu = x\nu$ where $x \in \mathfrak{X}$ and $\nu \in \mathfrak{X}^*$ satisfies $\nu = \bar{\nu}$; we say that the letter x is a *pivot* of μ.

Given a word $\mu \in \mathfrak{X}^n$, the C_n-orbit of μ is denoted $[\mu]$ and the D_n-orbit of μ is denoted $[[\mu]]$. It is clear that $[[\mu]] = [\mu] \cup [\bar{\mu}]$. A D_n-class of words (on $\mathfrak{X}$) is called a *dihedral class* (on $\mathfrak{X}$) (i.e., an element of $\mathfrak{X}^n/D_n$). Let $c = [[\mu]]$ be a dihedral class and $\psi \in \mathbb{N}^k, \psi = (\psi_1, \ldots, \psi_k)$,

- c is *primitive* if $c = [[\mu]]$ with μ primitive;
- c is *skew* if $c = [\mu] \cup [\bar{\mu}]$ with $[\mu] \neq [\bar{\mu}]$;
- c is *palindromic* if $\exists \mu, \mu$ palindrome such that $c = [\mu]$;
- c is *dexterpalindromic* if $\exists \mu, \mu$ dexterpalindrome such that $c = [\mu]$;
- c has *distribution* ψ if $c = [[\mu]]$ where μ has distribution ψ, that is, ψ_i is the number of occurences of x_i in μ.

The numbers σ_ψ, π_ψ, δ_ψ, and $\delta_\psi(i)$ are then defined as follows:

(3.11) σ_ψ is the number of primitive skew dihedral classes having distribution ψ,

(3.12) π_ψ is the number of primitive palindromic dihedral classes having distribution ψ,

(3.13) δ_ψ is the number of primitive dexterpalindromic dihedral classes having distribution ψ,

(3.14) $\delta_\psi(i)$ is the number of primitive dexterpalindromic dihedral classes having distribution ψ, for which x_i is a pivot.

We also need two families of quotient species, denoted by

$$C_m(X_1^{\psi_1} \ldots X_k^{\psi_k})/_{i_0}\mathbb{Z}_2 \text{ and } C_m(X_1^{\psi_1} \ldots X_k^{\psi_k})/\!\!/_{i_0}\mathbb{Z}_2, \tag{3.15}$$

where $0 \leq i_0 \leq k$, $\psi_{i_0} > 0$, according to actions "$\cdot_{i_0}$" and "$\bullet_{i_0}$" of $\mathbb{Z}_2 = \{1, \tau\}$ on the species $C_m(X_1^{\psi_1} \cdots X_k^{\psi_k})$ defined as follows: Given any $s = [\mu_1, \mu_2, \ldots, \mu_m] \in C_m(X_1^{\psi_1} \cdots X_k^{\psi_k})$, where $\mu_j = \mu_{1,j}\mu_{2,j} \cdots \mu_{k,j}$, $\mu_{i,j} \in X_i^{\psi_i}$, $1 \leq i \leq k$, $1 \leq j \leq m$, we define $\tau \cdot_{i_0} s$ by

$$\tau \cdot_{i_0} s = [\mu'_m, \mu'_{m-1}, \ldots, \mu'_1], \tag{3.16}$$

with $\mu'_j = \bar{\mu}_{1,j} \ldots \bar{\mu}_{i_0,j+1 (\mathrm{mod}\, m)} \ldots \bar{\mu}_{k,j}$, and if $\mu_{i_0,j} = a_j \nu_{i_0,j}$, with $a_j \in X_{i_0}$ and $\nu_{i_0,j} \in X_{i_0}^{\psi_{i_0}-1}$, we define $\tau \bullet_{i_0} s$ by

$$\tau \bullet_{i_0} s = [\mu''_m, \mu''_{m-1}, \ldots, \mu''_1 \tag{3.17}$$

with $\mu''_j = \bar{\mu}_{i,j} \cdots a_{j+1 (\mathrm{mod}\, m)} \bar{\nu}_{i_0,j} \cdots \bar{\mu}_{k,j}$.

Given a distribution $\psi \in \mathbb{N}^k$, we introduce the following notations:

$$\begin{aligned} |\psi| &= \psi_1 + \psi_2 + \cdots + \psi_k, \quad \mathrm{Supp}\, \psi = \{i \,|\psi_i \neq 0\}, \\ \mathcal{O}(\psi) &= \{i \,|\psi_i \text{ is odd}\}, \\ \chi_m(\psi) &= \chi(|\mathcal{O}(\psi)| = m) = \chi(\psi \text{ has } m \text{ odd components}), \end{aligned}$$

$m = 0, 1, 2, \ldots$, where χ denotes the truth function.

Theorem 3.4. *The species $P_n(X) = X^n/D_n$ of n-gons satisfies the addition formula*

$$P_n(X_1 + \cdots + X_k) = \sum_{d|n} \sum_{\substack{\psi \in \mathbb{N}^k \\ |\psi| = d}} \sum_{j=1}^{5} \Phi_j^\psi(X_1, \ldots, X_k) \tag{3.18}$$

where the Φ_j^ψ are species of k sorts, defined as follows:

$$\Phi_1^\psi(X_1, \ldots, X_k) = \sigma_\psi\, C_{n/d}(X_1^{\psi_1} \cdots X_k^{\psi_k}); \tag{3.19}$$

$$\Phi_2^\psi(X_1, \ldots, X_k) = \chi_1(\psi)\pi_\psi\, C_{n/d}(X_1^{\psi_1} \cdots X_k^{\psi_k})\Big/_{i_0}\mathbb{Z}_2, \tag{3.20}$$

where i_0 is the unique element of $\mathcal{O}(\psi)$;

$$\Phi_3^\psi(X_1, \ldots, X_k) = \chi_2(\psi)\delta_\psi(i_0) C_{n/d}(X_1^{\psi_1} \cdots X_k^{\psi_k})\Big/_{i_0}\mathbb{Z}_2, \tag{3.21}$$

where $\mathcal{O}(\psi) = \{i_0 < i_1\}$;

$$\Phi_4^{\psi}(X_1, \ldots, X_k) = \chi_0(\psi)\pi_\psi \; C_{n/d}(X_1^{\psi_1} \cdots X_k^{\psi_k}) \Big/_{j_0} \mathbb{Z}_2, \tag{3.22}$$

where $j_0 = \min \operatorname{Supp} \psi$;

$$\Phi_5^{\psi}(X_1, \ldots, X_k) = \chi_0(\psi) \sum_{i \in \operatorname{Supp} \psi} \delta_\psi(i) \; C_{n/d}(X_1^{\psi_1} \cdots X_k^{\psi_k}) \Big/\!\!\Big/_{i} \mathbb{Z}_2. \tag{3.23}$$

Proof
This is a consequence of Theorem 2.2. The details are omitted in this extended abstract. □

Moreover, the coefficients $\sigma_\psi, \pi_\psi, \delta_\psi$, and $\delta_\psi(i)$ can be explicitly computed using Möbius inversion as follows:

$$\pi_\psi = \frac{1}{2}(1 + \chi(|\psi| \text{ odd})) \sum_{d|\psi} \mu(d) \mathrm{pal}_{\psi/d}, \tag{3.24}$$

where $d|\psi$ means that $d|\psi_i$ for all i and $\mathrm{pal}_\psi = \mathrm{pal}_{\psi_1, \ldots, \psi_k}$, given by (3.3);

$$\delta_\psi(i) = \frac{1}{2}(1 + \chi(\psi_i \text{ odd})) \sum_{d|\psi} \mu(d) \mathrm{pal}_{\psi/d - e_i} \tag{3.25}$$

where $e_i \in \mathbb{N}^k$ is the i-th unit vector;

$$\delta_\psi = \sum_{i \in \operatorname{Supp} \psi} \delta_\psi(i), \tag{3.26}$$

and

$$\sigma_\psi = \frac{1}{2}(\lambda_\psi - \pi_\psi - \chi(|\psi| \text{ even}) \, \delta_\psi) \tag{3.27}$$

where λ_ψ is given by (3.9).

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