RANDOM WALK IN AN ALCOVE OF AN AFFINE WEYL GROUP, AND NON-COLLIDING RANDOM WALKS ON AN INTERVAL
MARCHES ALÉATOIRES DANS UNE ALCÔVE D'UN GROUPE DE WEYL AFFINE, ET MARCHES ALÉATOIRES SANS COLLISION SUR UN INTERVALLE DONNÉ

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Abstract. We use a reflection argument, introduced by Gessel and Zeilberger, to count the number of k-step walks between two points which stay within a chamber of a Weyl group. We apply this technique to walks in the alcoves of the classical affine Weyl groups. In all cases, we get determinant formulas for the number of k-step walks. One important example is the region $m > x_1 > x_2 > \cdots > x_n > 0$, which is a rescaled alcove of the affine Weyl group $\tilde{C}_n$. If each coordinate is considered to be an independent particle, this models n non-colliding random walks on the interval $(0, m)$.

Résumé. On utilise un principe de réflexion, introduit par Gessel et Zeilberger, pour compter le nombre de marches de k-pas entre deux points qui demeurent à l'intérieur d'une chambre d'un groupe de Weyl. On applique cette technique aux marches dans les alcôves des groupes classiques affines de Weyl. Dans tous les cas, on obtient des formules déterminantales pour le nombre de marches de k-pas. Un exemple important est celui de la région $m > x_1 > x_2 > \cdots > x_n > 0$ qui est une alcôve agrandie du groupe de Weyl affine $\tilde{C}_n$. Si l'on considère chacune des coordonnées comme une particule indépendante on obtient alors un modèle pour les marches aléatoires sans collision sur l'intervalle $(0, m)$.

1. Introduction

The *ballot problem*, a classical problem in random walks, asks how many ways there are to walk from the origin to a point $(\lambda_1, \ldots, \lambda_n)$, taking k unit-length steps in the positive coordinate directions while staying in the region $x_1 \geq x_2 \geq \cdots \geq x_n$. The solution is known in terms of the hook-length formula for Young tableaux; a combinatorial proof, using a reflection argument, is given in [13, 15].

The same reflection argument has also been applied to the case of n independent diffusions, or n discrete processes which cannot pass each other without first colliding. Using this method, Karlin and McGregor [10, 11] give a determinant formula for the probability or measure for the n particles, starting at known positions, not to have collided up to time t and to be in given positions. Hobson and Werner [9] generalize this argument to n independent Brownian motions in an interval or on a circle.

Gessel and Zeilberger [6], and independently Biane [1], consider a more general question, for which some of the same techniques apply. For certain "reflectable" walk-types, we can count the number of k-step walks between two points of a lattice, staying within a Weyl chamber. The argument generalizes naturally to "reflectable" diffusions [7]; we can compute the density function for the diffusion started at a point η to stay within the Weyl chamber up to time t and be at a point λ.

Grabiner and Magyar [8] classify all reflectable random walks for finite Weyl groups, and compute determinant formulas for many important cases, including walks with steps $\pm e_i$ or with steps $\pm\frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_n$ in all of the classical Weyl groups.

We prove analogous results for the alcoves of affine Weyl groups. In contrast to the chambers of classical Weyl groups, these are bounded regions, such as $m > x_1 > \cdots > x_n > 0$. The reflectable

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random walks for the affine Weyl groups are the same as for the corresponding classical Weyl groups. We use these reflection arguments to find determinant formulas for the number of walks of length k which stay within the alcoves of the classical affine Weyl groups. We then simplify the determinant of infinite sums to get a determinant of finite sums of cosines or of exponentials of cosines, depending on the random walk.

One important case is the model of n non-colliding particles in discrete random walks in the interval $(0, m)$. Equivalently, we can consider a single random walk in n dimensions in the region $m > x_1 > \cdots > x_n > 0$, with each coordinate of the n-dimensional walk corresponding to one of the n particles, and permitted steps in the positive and negative coordinate directions. We then simplify the determinant of infinite sums, computing the exponential generating function in the number of steps. This gives the number of walks for one particle to go from η to λ while staying in this region, or for n particles to go from η_i to λ_i while staying in the interval $(0, m)$ and not colliding. The exponential generating function is

$$(1)\quad \sum_{k=0}^{\infty} \frac{b_{\eta\lambda,k} x^k}{k!} = \det_{n\times n} \left| \frac{1}{m} \sum_{r=0}^{m-1} \big(\cos(2\pi r(\lambda_j - \eta_i)/m) - \cos(2\pi r(\lambda_j + \eta_i)/m)\big) \cdot \exp(2x \cos(2\pi r/m)) \right|.$$

Using a reflection argument of Gessel and Krattenthaler [5] which uses the methods of [10] and [9], we can also give a formula for the model of n independent discrete random walks on the circle; as a one-particle model, this is a variation of the affine Weyl group $\tilde{A}_{n-1}$. Gessel and Krattenthaler [5] give a q-analogue of the formula obtained from the reflection argument.

Many results are known in the $\tilde{A}_{n-1}$ cases, in which the region in $\mathbb{R}^2$ is $x_1 > x_2 > \cdots > x_n > x_1 - m$. The $\tilde{A}_1$ case (or equivalently $\tilde{B}_1$, region $m > x > 0$) is a single random walk on an interval, and the formula in this case goes back to Lagrange [3]. The n-dimensional case with steps in the positive coordinate directions only was solved by Filaseta [4], and a q-analogue was proved by a reflection argument by Krattenthaler [12].

2. Reflectable random walks

A walk-type is defined by a lattice L, a set S of allowable steps between lattice points, and a region C to which the walks are confined. Without affecting the walk problems, we may restrict L and C to the linear span of S, so that L, S, and C have the same linear span.

Let W be a finite Weyl group, generated by a root system Φ with positive roots Φ^+. The corresponding affine Weyl group $\tilde{W}$ is the semidirect product of W and the translation group of the coroot lattice (translations by vectors which have integer dot product with every root); that is, it is generated by reflections

$$r_{\alpha,k} : \vec{x} \mapsto \vec{x} - \frac{2(\alpha, \vec{x}) - k}{(\alpha, \alpha)} \alpha$$

for all roots α and all integers k. Again, the group $\tilde{W}$ contains the reflections $r_{\alpha,k}$ not only for simple α but for all α, and these hyperplanes of reflection partition space into *alcoves*. The alcoves are the regions bounded by $|\Phi^+|$ simultaneous inequalities $k_\alpha < (\lambda, \alpha) < k_\alpha + 1$, as α runs over all roots in Φ^+, for k_α in $\mathbb{Z}$. The *principal alcove* A is bounded by the inequalities $0 < (\lambda, \alpha) < 1$ for all positive roots α; this can be shown to be non-empty.

Example. Let W be the symmetric group S_n permuting the n coordinates in $\mathbb{R}^n$; this is generated by reflections in the simple roots $\Delta = \{e_i - e_{i+1}\ ,\ 1 \le i \le n-1\}$, (where e_i is the ith coordinate vector), which gives Weyl chamber $x_1 > x_2 > \cdots > x_n$. The positive roots are $e_i - e_j$ for $i < j$; their hyperplanes $x_i = x_j$ give $n!$ Weyl chambers, one for each permutation of the coordinates. The corresponding affine Weyl group contains a reflection in the hyperplane $e_i - e_j = k$ for any integer k; the principal alcove is thus $x_1 > x_2 > \cdots > x_n > x_1 - 1$. (While this alcove is unbounded, W really acts only on the subspace in which $\sum x_i = 0$, and the alcove is bounded in that subspace.)

We would like to classify those Weyl chambers and alcoves which allow us to reflect a walk from the point at which it hits a wall. The definition of a reflectable walk from [8] generalizes easily to the affine case.

Definition. A walk-type (L, S, C) is *reflectable* with respect to the finite or affine Weyl group W if the steps S are symmetric under the finite Weyl group, and the following equivalent conditions hold:

(1) Any step $s \in S$ from any lattice point in the interior of C will not exit C.
(2) For each simple root α_i, there is a real number k_i such that $(\alpha_i, s) = \pm k_i$ or 0 for all steps $s \in S$; (α_i, λ) is an integer multiple of k_i for all $\lambda \in L$; and if C is an alcove of an affine Weyl group, then $1/k_i$ must be an integer.

To see that these conditions are equivalent, note that if α_j is any root which is symmetric to α_i, then $k_j = k_i$. Thus the second condition guarantees than L contains only points whose dot-product with α_j is an integer multiple of k_j, and a single step can change the dot product by at most k_j. Therefore, the walk cannot go from one side of the wall $(\alpha_j, \lambda) = 0$ to the other without stopping on a wall. Likewise, since each alcove wall is the set of points with $(\alpha_j, \lambda) = m$ for some m, and m is a multiple of k_j, the walk cannot go from one side to the other in a single step.

Example. In the example above, in which W is the symmetric group, the steps $\pm e_i/t$ on $L = \mathbb{Z}^n$ give a reflectable random walk, with each $k_i = 1/t$. This is reflectable for the affine alcove as well provided that t is an integer.

By a similar argument, all of the reflectable random walks on the finite Weyl groups give reflectable walks on the corresponding affine Weyl groups. These steps are enumerated in [8]; they turn out to be precisely the Weyl group images of the *minuscule weights* [2], those weights with dot-product 0 or ± 1 with every root.

In the natural cases we study later, it will be clear from the definitions that the first reflectability condition is satisfied.

3. The reflection argument of Gessel and Zeilberger

In a reflectable random walk problem, we want to compute $b_{\eta\lambda,k}$, the number of walks from η to λ of length k which stay in the interior of a Weyl chamber or alcove. For example, the ballot problem can be converted to this form by starting at the point $(n-1, n-2, \ldots, 0)$ instead of the origin, and requiring the coordinates to remain strictly ordered.

Let $\chi(\vec{u}) = \sum_{s \in S} \vec{u}^s$, the generating function for the steps in the formal monomials $u^{(x_1,\ldots,x_n)} = u_1^{x_1} \cdots u_n^{x_n}$. Let $c_{\gamma,k}$ denote the number of random walks of length k, with steps in S, from the origin to γ, but *unconstrained* by a chamber. Then we have

$$c_{\gamma,k} = \chi(\vec{u})^k \Big|_{\vec{u}^\gamma},$$

where $\Big|_{\vec{u}^\gamma}$ denotes the coefficient of $\vec{u}^\gamma$ in the polynomial.

The fundamental result of Gessel and Zeilberger [6] (also proved independently by Biane [1] for finite Weyl groups), is

Theorem 1. *If the walk from η to λ is reflectable, then*

$$b_{\eta\lambda,k} = \sum_{w \in W} \operatorname{sgn}(w) c_{w(\lambda)-\eta,k}, . \tag{2}$$

If W is an affine group, this is an infinite sum, but only finitely many terms are nonzero for any fixed k.

To prove this theorem, reflect the part of any walk from η to any $w(\lambda)$ after it first touches a wall in that wall. These walks will cancel out in the sum, and the only walks which do not cancel out are those which touch no walls and thus go from η to λ.

The specific case of the theorem in which W is the symmetric group S_n was proved by Karlin and McGregor [10]. We can view the S_n process as n separate walks which are not allowed to collide, rather than one walk restricted to the region $x_1 > \cdots > x_n$, and interchange two particles when they collide. If n_{ij} is the number of walks from η_i to λ_j, then the formula (2) becomes

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) n_{i,\sigma(i)} = \det_{n\times n} |n_{ij}| \,. \tag{3}$$

For general Weyl groups, two natural choices of step sets are the positive and negative coordinate directions $\pm e_i$, or the 2^n diagonals $\pm\frac{1}{2}e_1 \cdots \pm \frac{1}{2}e_n$. Both cases give reflectable walks for all the classical Weyl chambers. To get a non-trivial walk for the alcoves of the affine Weyl groups, we have to re-scale either the steps or the alcoves. It is more natural to state the problem if we leave the steps alone and re-scale the alcoves by a factor of m or $2m$, so that we study the steps $\pm e_i$ in the alcoves of the Weyl group mW or $2mW$.

For the diagonals, $c_{\gamma k}$ is easy to compute, and for the coordinate directions, the exponential generating function

$$h_\gamma(x) = \sum_{k=0}^{\infty} c_{\gamma k} x^k/k! = \exp(x\chi(\vec{u}))\Big|_{\vec{u}^\gamma}$$

is easy to compute.

If the steps S are $\pm e_i$, the positive and negative coordinate directions, then

$$h_\gamma(x) = \sum_{k=0}^{\infty} \prod_{i=1}^{n} \exp(x(u_i + u_i^{-1}))\Big|_{\vec{u}^\gamma}.$$

This infinite sum can be written as a product of hyperbolic Bessel functions, using the generating-function definition of the Bessel functions [14]. We have

$$\begin{aligned}
\exp(x(u+u^{-1})) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=-k}^{k} \binom{k}{j} u^{k-2j} \\
&= \sum_{m=-\infty}^{\infty} u^m \sum_{k=0}^{\infty} \frac{x^k}{k!} \binom{k}{(k+m)/2} \\
&= \sum_{m=-\infty}^{\infty} u^m \sum_{t=0}^{\infty} \frac{x^{2t+m}}{t!(t+m)!} \\
&= \sum_{m=-\infty}^{\infty} u^m I_m(2x),
\end{aligned}$$

Thus, in this case, the exponential generating function for the unconstrained walks is

$$h_\gamma(x) = \prod_{i=1}^{n} I_{\gamma_i}(2x). \tag{4}$$

If, instead, the steps S are the diagonals $\pm\frac{1}{2}e_1 \cdots \pm \frac{1}{2}e_n$, then each step involves a step of $\pm\frac{1}{2}$ in each coordinate direction, and thus the walk will go forwards a distance of γ_i if there are γ_i more forward steps than backward steps. That is,

$$c_{\gamma k} = \prod_{i=1}^{n} \binom{k}{(k/2)+\gamma_i}. \tag{5}$$

4. Basic techniques, and formulas for $\tilde{A}_{n-1}$

The basic techniques are the same for all the classical affine Weyl groups; we will do $\tilde{A}_{n-1}$ in detail because it is the simplest case.

The Weyl group $m\tilde{A}_{n-1}$ acts on the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ with $x_1+x_2+\cdots+x_n = 0$. It contains all permutations of the coordinates, along with translations of one coordinate by $m(n-1)/n$ and all others by $-m/n$; this gives alcove $x_1 > x_2 > \cdot > x_n > x_1 - m$. The sign of a translation is zero (it is the product of reflections in $x_1 = x_j$ and $x_1 = x_j + m$), so the sign of an element of the affine Weyl group is the sign of the underlying permutation.

We can extend the Weyl group action to all of $\mathbb{R}^n$ by allowing the group to act trivially on the orthogonal complement. The alcove is still defined by the same inequalities; it is now unbounded along that line. The Weyl group now allows translations of one coordinate by m. This is more natural because it allows us to study the walk with steps in the coordinate directions or diagonals of $\mathbb{R}^n$. The problem remains equivalent because we can project our walk down to the hyperplane. (Note that the diagonal steps $\frac{1}{2}e_1+\cdots+\frac{1}{2}e_n$ and $-\frac{1}{2}e_1-\cdots-\frac{1}{2}e_n$ both project to the zero step when we do this, so this walk will have the zero step with weight 2 when projected to the hyperplane.)

Several results are known in this case for the step set $+e_i$, which gives a reflectable walk for $\tilde{A}_{n-1}$ but not for the other affine Weyl groups. This is a generalization of the classical ballot problem in which n candidates must stay in non-increasing order, and the first candidate can never lead the last candidate by $m-(n-1)$ votes. Translating the starting point from the origin to $(n-1, n-2, \ldots, 0)$ then makes this a walk in the region $x_1 > x_2 > \cdots > x_n > x_1 - m$. Filaseta [4] gives a formula in this case by a non-reflection argument, and Krattenthaler [12] uses a reflection argument to prove a q-analogue.

We will first consider the walk for the diagonal steps $\pm\frac{1}{2}e_1 \cdots \pm \frac{1}{2}e_n$. We write an element of the Weyl group as the product of translations by mt_i in the coordinate directions with σ in the symmetric group. Thus Theorem 1 gives

$$b_{\eta\lambda,k} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{t_i\in\mathbb{Z}} c_{(\lambda_{\sigma(1)}+mt_1,\ldots,\lambda_{\sigma(n)}+mt_n)-\eta,k}. \tag{6}$$

Using the formula (5) gives

$$b_{\eta\lambda,k} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{t_i\in\mathbb{Z}} \prod_{i=1}^{n} \binom{k}{(k/2)+\lambda_\sigma(i)-\eta_i+mt_i}. \tag{7}$$

We interchange the inner sum and the product to get

$$b_{\eta\lambda,k} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \sum_{t_i\in\mathbb{Z}} \binom{k}{(k/2)+\lambda_\sigma(i)-\eta_i+mt_i}. \tag{8}$$

And the signed sum over permutations is the definition of a determinant, which gives

$$b_{\eta\lambda,k} = \det_{n\times n}\left|\sum_{t_i\in\mathbb{Z}} \binom{k}{(k/2)+\lambda_j-\eta_i+mt_i}\right|. \tag{9}$$

This is essentially the same as the argument of [10, 11] for the case of $A_{n-1} = S_n$.

The infinite periodic sum of binomial coefficients can be written as a finite sum of powers of roots of unity, and thus of cosines. Let $\omega = e^{2\pi i/m}$; then for fixed r and s, we have

$$\omega^{-rs}(\omega^r+\omega^{-r})^k = \sum_j \omega^{r(2j-s)}\binom{k}{(k/2)+j} \tag{10}$$

Thus, if we take the sum over all r from 0 to $m-1$, all terms for which $2j \not\equiv s \pmod m$ will cancel out because of the roots of unity; that is,

$$\frac{1}{m}\sum_{r=0}^{m-1} \omega^{-rs}(\omega^r + \omega^{-r})^k = \sum_{2j \equiv s \pmod m} \binom{k}{(k/2)+j}. \tag{11}$$

Since the original sum was real, we can eliminate the roots of unity in this sum by taking the real part of ω^{-rs} and writing everything in terms of cosines. This gives

$$\frac{1}{m}\sum_{r=0}^{m-1} \cos(2\pi rs/m)(2\cos(2\pi r/m))^k = \sum_{2j \equiv s \pmod m} \binom{k}{(k/2)+j}. \tag{12}$$

Substituting this formula into (9) gives our final formula for this case.

$$b_{\eta\lambda,k} = \det_{n\times n}\left|\frac{2^k}{m}\sum_{r=0}^{m-1}\cos(2\pi r(\lambda_j - \eta_i)/m)\cos^k(2\pi r/m)\right|. \tag{13}$$

The procedure for the steps $\pm e_i$ is similar. The same argument that gave us the determinant (9) of periodic sums of binomial coefficients for the case of diagonals shows that the exponential generating function is a product of periodic sums of Bessel functions. We get

$$g_{\eta\lambda}(x) = \det_{n\times n}\left|\sum_{t_i\in\mathbb{Z}} I_{\lambda_j-\eta_i+mt_i}(2x)\right|. \tag{14}$$

We simplify this periodic sum of Bessel functions by again using the generating-function definition for the hyperbolic Bessel functions

$$\exp(x(z+z^{-1})) = \sum_{j\in\mathbb{Z}} z^j I_j(2x).$$

If we let $z = \omega^r$, where $\omega = e^{2\pi i/m}$ as before, then we have

$$\omega^{-rs}\exp(x(\omega^r + \omega^{-r})) = \sum_{j\in\mathbb{Z}} \omega^{r(j-s)} I_j(2x). \tag{15}$$

As before, we now take $1/m$ times the sum over all values of r to eliminate the terms for which $j \not\equiv s \pmod m$, and then take the real part of ω^{-rs} to write everything in terms of cosines. This gives

$$\frac{1}{m}\sum_{r=0}^{m-1}\cos(2\pi rs/m)\exp(2x\cos(2\pi r/m)) = \sum_{j\equiv s \pmod m} I_j(2x). \tag{16}$$

Substituting this into our determinant gives the final formula for the exponential generating function for this random walk.

$$g_{\eta\lambda}(x) = \det_{n\times n}\left|\frac{1}{m}\sum_{r=0}^{m-1}\cos(2\pi r(\lambda_j-\eta_i)/m)\exp(2x\cos(2\pi r/m))\right|. \tag{17}$$

5. Non-colliding particles on an interval, and formulas for $\tilde{C}_n$

The Weyl chamber for $2m\tilde{C}_n$ is $m > x_1 > x_2 > \cdots > x_n > 0$ because one of the roots is $2x_1$. The walk in $\mathbb{R}^n$ is thus equivalent to a walk of n independent particles in the interval $(0,m)$. The particles cannot change order in the walk; thus, if η and λ are both in the Weyl chamber, then for any legal walk of n discrete particles, particle i must move from η_i to λ_i.

We will work first in the case in which the steps are $\pm x_i$. We write an element of the affine Weyl group as a product of σ in the symmetric group, $\epsilon_i = \pm 1$ which change the signs of coordinates, and translations by mt_i. Thus we have

$$g_{\eta\lambda}(x) = \sum_{\sigma \in S_n} \sum_{\epsilon_i = \pm 1} \sum_{t_i \in \mathbb{Z}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \epsilon_i I_{\epsilon_i \lambda_{\sigma(i)} - \eta_i + mt_i}(2x). \tag{18}$$

Since this is a model of n non-colliding particles, this formula could also be derived from the argument of Karlin and McGregor [10, 11]; the possibility of a walk restricted to an interval is not mentioned there, but the argument is still valid, and the formula above could be derived from the sum of Bessel functions for the one-dimensional walks.

We combine the terms of $\epsilon_i = \pm 1$ in each factor of (18) to get

$$g_{\eta\lambda}(x) = \sum_{\sigma \in S_n} \sum_{t_i \in \mathbb{Z}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(I_{\lambda_{\sigma(i)} - \eta_i + mt_i}(2x) - I_{-\lambda_{\sigma(i)} - \eta_i + mt_i}(2x) \right). \tag{19}$$

This is again a determinant. For notational elegance, we will get rid of the minus signs on $I_{-\lambda_j - \eta_i + mt_i}$ by noting that $I_t = I_{-t}$ and thus

$$\sum_{t_i \in \mathbb{Z}} I_{-\lambda_j - \eta_i + mt_i}(2x) = \sum_{t_i \in \mathbb{Z}} I_{\lambda_j + \eta_i - mt_i}(2x) = \sum_{t_i \in \mathbb{Z}} I_{\lambda_j + \eta_i + mt_i}(2x).$$

This gives the determinant

$$g_{\eta\lambda}(x) = \det_{n \times n} \left| \sum_{t_i \in \mathbb{Z}} \left(I_{\lambda_j - \eta_i + mt_i}(2x) - I_{\lambda_j + \eta_i + mt_i}(2x) \right) \right|. \tag{20}$$

And these periodic sums of Bessel functions simplify just as before, giving the final formula

$$g_{\eta\lambda}(x) = \det_{n \times n} \left| \frac{1}{m} \sum_{r=0}^{m-1} \left(\cos(2\pi r(\lambda_j - \eta_i)/m) - \cos(2\pi r(\lambda_j + \eta_i)/m) \right) \cdot \exp(2x \cos(2\pi r/m)) \right|. \tag{21}$$

Similarly, for the diagonal walk, or for n particles moving simultaneously by steps of $\frac{1}{2}$ in the interval $(0, m)$, we have

$$b_{\eta\lambda,k} = \det_{n \times n} \left| \frac{2^k}{m} \sum_{r=0}^{m-1} \left(\cos(2\pi r(\lambda_j - \eta_i)/m) - \cos(2\pi r(\lambda_j + \eta_i)/m) \right) \cdot \cos^k(2\pi r/m) \right|. \tag{22}$$

6. Other cases

We can also consider the case of n non-colliding random walks on the circle of size m (the interval $[0, m]$ with endpoints identified). This problem is more naturally studied in terms of the individual non-colliding walks, rather than a single n-dimensional walk; however, we will use the same notation, as this is analogous to a single n-dimensional walk in $m\tilde{A}_{n-1}$. The desired number of walks $b'_{\eta\lambda,k}$ represents the number of ways that particle i goes from η_i to λ_i in k steps and no two particles collide. (In the analogous walk problem on $m\tilde{A}_{n-1}$, the particles could change order by a circular permutation without colliding; the final positions would still give an image of λ under $m\tilde{A}_{n-1}$.)

We use a technique used in [9] for the analogous problem for Brownian motion. We assume the coordinates of η are in decreasing order; let λ_s be the smallest coordinate of λ. If we look at each particle moving in $\mathbb{R}$ (and then project the walks to the circle by working modulo m), then the only paths in $\mathbb{R}$ which correspond to proper walks on the circle are those in which, for a fixed t, the ith particle goes from η_i to $\lambda_i + tm$ if $i < s$, and from η_i to $\lambda_i + (t+1)m$ if $i \geq s$; this is the only way for the particles to end in the correct places.

We can now get our formula by using a reflection formula closely related to Theorem 1. This was developed in [5] for non-intersecting lattice paths, and in [10] for Brownian motion. We take the signed sum over permutations of walks, but only over those permutations which have the correct number of revolutions around the circle. We reflect walks when two particles collide by interchanging the particles themselves for the remainder of the walk. Again, this causes walks with collisions to cancel out in the signed sum. This gives the formula

$$b'_{\eta\lambda,k} = \sum_{\sigma\in S_n} \sum_{\sum t_i \equiv s \pmod{n}} \operatorname{sgn}(\sigma) c'_{(mt_1,\ldots,mt_n)+\sigma(\lambda)-\eta,k}. \tag{23}$$

We use an argument analogous to the previous argument to restrict to the terms $\sum t_i \equiv s \pmod{n}$. We take the sum over all t_i, and then add a term involving roots of unity which cancels out unless the congruence holds. We cannot eliminate all of the complex terms by taking the real part, but we can get a formula which still contains the complex roots of unity.

Let $\zeta = e^{2\pi i/mn}$. For the walk with steps $\pm e_i$, corresponding to particles moving independently, we get

$$g'_{\eta\lambda}(x) = \frac{1}{n}\sum_{u=0}^{n-1} \zeta^{-ums} \det_{n\times n}\left|\frac{1}{m}\sum_{r=0}^{n-1} \zeta^{-nr(\lambda_j-\eta_i)} \exp(2\cos(2\pi(u+nr)/(mn)))\right|. \tag{24}$$

The formula for the diagonal walk is similar.

The affine Weyl group $2m\tilde{B}_n$ gives the chamber $x_1 > x_2 > \cdots > x_n > 0, x_1 + x_2 < 2m$. The affine Weyl group $2m\tilde{D}_n$ gives the chamber $x_1 > x_2 > \cdots > x_n, x_1+x_2 < 2m, x_{n-1} > -x_n$. Neither one gives a natural model for n independent particles in one dimension.

The Weyl group $2m\tilde{B}_n$ contains all permutations with any number of sign changes, and with translations of coordinates by multiples of m such that the total translation is a multiple of $2m$ (since $\tilde{B}_n$ does not have $2x_i$ as a root and thus does not have $\check{x}_i/2$ as a coroot). The Weyl group $2m\tilde{D}_n$ contains all permutations with an even number of sign changes, and with translations of coordinates by multiples of m such that the total translation is a multiple of $2m$.

We can find the formulas for these cases by using the fact that $2m\tilde{B}_n$ and $2m\tilde{D}_n$ are subgroups of $2m\tilde{C}_n$, of index 2 and 4. We can thus modify the formula (18) by including a term which is 1 if the potential endpoint of the walk is in the Weyl group image of the appropriate subgroup, and 0 otherwise; this is analogous to the technique used to get D_n formulas from B_n formulas in [8] and [7]. The formulas are sums of two or four determinants analogous to the $\tilde{C}_n$ determinant formulas (21) and (22).

7. Open questions

The number of reflectable random walks in a classical Weyl chamber is of interest in representation theory. Let the step set be the set of weights of a representation of the corresponding Lie group. Let the starting point η be ρ, half the sum of the positive roots, and the end point λ be $\rho+\mu$. If the walk is reflectable, then the number of walks from ρ to $\rho+\mu$ is the multiplicity of the representation with highest weight μ in the kth tensor power of the representation whose weights are the step set [8]. Does the number of walks in an alcove of an affine Weyl chamber have a similar meaning in representation theory?

We have a formula for the probability that n particles in an interval or on a circle will not collide in k steps. Can we get a single general formula, or an asymptotic, for the total probability that there will be no collision?

There are formulas and interpretations of the q-analogue of the case with Weyl group $\tilde{A}_{n-1}$ and steps $+e_i$ [12]. Can q-analogues of the other cases be defined, and are they of interest?

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