THE ROBINSON–SCHENSTED–KNUTH BIJECTION, QUANTUM MATRICES AND PIECE–WISE LINEAR COMBINATORICS

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ABSTRACT. We explicitly compute the celebrated Robinson–Schensted–Knuth bijection (RSK) between the set of the matrices with non-negative integer entries, and the set of the plane partitions. More precisely, in suitable linear coordinates on both sets, the RSK is expressed via minima of linear forms, i.e, in piece–wise linear terms. In particular, we answer the following question by C. Greene and G. Viennot: "What shape corresponds to a given matrix under the Robinson–Schensted–Knuth correspondence?" Our main tools in establishing these formulae are the quantum matrices and crystal bases. As a byproduct of our approach, we compute the corresponding crystal equivalence in terms of "generalized" Kazhdan–Lusztig polynomials.

1. INTRODUCTION AND MAIN RESULTS

In what follows, $Mat_{m\times n}(\mathbb{Z})$ will stand for the set of all $m\times n$–matrices with integer entries. Let $M=M_{m\times n}\subset Mat_{m\times n}(\mathbb{Z})$ be the set of all matrices with non–negative integer entries.

Let $P=P_{m\times n}$ be the set of all $m\times n$–*plane partitions*, i.e., the set of all $p=(p_{ij})\in M$ satisfying

$$p_{i-1,j}\le p_{ij}\ge p_{i,j+1} \tag{1.1}$$

for all $1\le i\le m, 1\le j\le n$ (we use the agreement $p_{ij}=0$ whenever $(i,j)\notin[1;m]\times[1;n]$). One can define P equivalently as the set of all order–preserving maps $p:[1;m]\times[1;n]\to\mathbb{Z}_{\ge 0}$ where the partial order on $[1;m]\times[1;n]$ is defined by: $(i,j)\le(i',j')$ if $i\le i', j\ge j'$ (throughout the paper the notation $[a;b]$ stands for the set $\{a,a+1,\ldots,b\}$).

The Robinson–Schensted–Knuth bijection (RSK) is a special bijection $\mathbf{K}=\mathbf{K}_{m\times n}$ of M onto P. In the classical interpretation $\mathbf{K}$ is a bijection of M onto the set of all pairs of semi–standard Young tableaux (in m and n letters respectively) having common shape (see e.g., [11]). It was shown by Bender and Knuth (see [1]) that the so described range is naturally identified with P. In this paper we explicitly compute $\mathbf{K}$ using continuous piece–wise linear (CPL) formulae.

For $(i,j)\in[1;m]\times[1;n]$ let $y_{ij}:P\to\mathbb{Z}$ be defined by the formula:

$$y_{ij}(p)=p_{ij}+p_{i-1,j+1}+\cdots . \tag{1.2}$$

The last summand here is $p_{1,i+j-1}$ if $i+j\le n+1$, and $p_{i+j-n,n}$ if $i+j>n$. One can reconstruct p from the collection of all $y_{ij}=y_{ij}(p)$ by the formula:

$$p_{ij}=y_{ij}-y_{i-1,j+1}$$

for $(i,j)\in[1;m]\times[1;n]$ (we use the agreement $y_{ij}=0$ whenever $(i,j)\notin[1;m]\times[1;n]$).

Now let us consider the set $[1;i]\times[j;n]$ equipped with the standard product partial order (that is, $(k,l)\le(k',l')$ if $k\le k', l\le l'$). For $\mathbf{m}\in\mathbb{Z}_{\ge 0}$ let $S_{ij}^{\mathbf{m}}$ be the set of all weakly increasing maps $s:[1;i]\times[j;n]\to[0;\mathbf{m}]$. In fact, $S_{ij}^{\mathbf{m}}$ can be identified with a finite subset of $P_{i\times(n+1-j)}$.

Theorem 1.1. *The Robinson–Schensted–Knuth bijection* $\mathbf{K}:M\to P$ *is given by the following formula. For* $(i,j)\in[1;m]\times[1;n]$, $d=(d_{kl})\in M$ *one has:*

$$y_{ij}(\mathbf{K}(d))=\min_{s\in S_{ij}^{\mathbf{m}}}\left(\sum_{(k,l)\in[1;i]\times[j;n]} d_{s(k,l)+k,s(k,l)+l-\mathbf{m}}\right), \tag{1.3}$$

where $\mathbf{m} = \mathbf{m}_{ij} = \min(j-1, m-i)$.

Properties of K.

1. Let $m' \geq m, n' \geq n$. Then the following diagram commutes.

$$\begin{array}{ccc} M_{m'\times n'} & \xrightarrow{\mathbf{K}} & P_{m'\times n'} \\ \uparrow f & & \uparrow f \\ M_{m\times n} & \xrightarrow{\mathbf{K}} & P_{m\times n} \end{array} , \tag{1.4}$$

where $f : Mat_{m\times n}(\mathbb{Z}) \hookrightarrow Mat_{m'\times n'}(\mathbb{Z})$ is an inclusion defined by

$$f(d)_{ij} = \begin{cases} d_{i+m-m',j} & \text{if } i \geq m'-m,\ n+1-j \geq m'-m \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq m, 1 \leq j \leq n$. In particular, one can always recover $\mathbf{K}_{m\times n}$ from $\mathbf{K}_{m'\times n'}$ with $m' = n' = \max(m, n)$.

2. Let $\cdot^\tau : Mat_{m\times n}(\mathbb{Z}) \cong Mat_{n\times m}(\mathbb{Z})$ be the reflection about the second diagonal defined by $d \mapsto d^\tau$ where $(d^\tau)_{ij} = d_{m+1-j,n+1-i}$ for $(i,j) \in [1;m] \times [1;n]$. Then the formulae (1.3) imply that $\mathbf{K}$ commutes with τ, that is,

$$\mathbf{K}_{n\times m}(d^\tau) = (\mathbf{K}_{m\times n}(d))^\tau \ . \tag{1.5}$$

for $d \in M_{m\times n}$. This property is well–known (see e.g., [11],[6]).

3. For $i = 1, \ldots, m, j = 1, \ldots, n$ one has

$$y_{i1}(\mathbf{K}(d)) = \sum_{(k,l)\in[1;i]\times[1;n]} d_{kl},\ y_{mj}(\mathbf{K}(d)) = \sum_{(k,l)\in[1;m]\times[j;n]} d_{kl} \ . \tag{1.6}$$

4. We have

$$(\mathbf{K}(d))_{1n} = \min_{1\leq s\leq \min(m,n)} (d_{s,s+(n-m)_+}) \ , \tag{1.7}$$

for $d \in M$, where $(n-m)_+ = \max(0, n-m)$. More generally,

$$(\mathbf{K}(d))_{1j} = \min_{0\leq s_j\leq\cdots\leq s_n\leq\mathbf{m}} (d_{s_j+1,s_j+j-\mathbf{m}} + \cdots + d_{s_n+1,s_n+n-\mathbf{m}}) \tag{1.8}$$

for $j = 1, \ldots n$, where $\mathbf{m} = \min(j-1, m-1)$; and

$$(\mathbf{K}(d))_{in} = \min_{0\leq s_1\leq\cdots\leq s_i\leq\mathbf{n}} (d_{s_1+1,s_1+n-\mathbf{n}} + \cdots + d_{s_i+i,s_i+n-\mathbf{n}})$$

for $i = 1, \ldots, m$, where $\mathbf{n} = \min(m-i, n-1)$.

5. We have

$$(\mathbf{K}(d))_{m1} = \max_\pi (\sum_{(k,l)\in\pi} d_{kl} \ ), \tag{1.9}$$

where the maximum is over all *increasing paths* $\pi \subset [1;m] \times [1;n]$ from $(m,1)$ to $(1,n)$, i.e., over all the sequences

$$\pi = \big((m,1) = (k_1,l_1), (k_2,l_2), \cdots, (k_{m+n-1}, l_{m+n-1}) = (1,n)\big)$$

such that either $(k_{s+1}, l_{s+1}) = (k_s - 1, l_s)$, or $(k_{s+1}, l_{s+1}) = (k_s, l_s + 1)$ for $s = 1, \ldots, m+n-2$. Thus, the number of linear forms involved in the maximum (1.9) is $\binom{m+n-2}{m-1}$.

This property allows for restricting $\mathbf{K}$ to the bijection of finite sets (for every $l \in \mathbb{Z}_{\geq 0}$):

$$M^{(l)} \cong P^{(l)},$$

where $P^{(l)}$ is the set of all $p \in P$ such that $p_{m1} \leq l$, and $M^{(l)}$ is the set of all $d \in M$ such that $\sum_{(k,l)\in\pi} d_{kl} \leq l$ for every path π as above.

6. Now we give such reformulation of Theorem 1.1 that will generalize (1.9). For each $(i,j) \in [1;m] \times [1;n]$ define a linear form $\bar{y}_{ij} : P \to \mathbb{Z}$ by

$$\bar{y}_{ij}(p) = p_{ij} + p_{i+1,j-1} + \cdots ,$$

so that $\bar{y}_{ij}(p) = y_{m,i+j-m} - y_{i-1,j+1}(p)$ if $i+j > m$ and $\bar{y}_{ij}(p) = y_{m+1-i-j,1} - y_{i-1,j+1}(p)$ if $i + j \le m$.

Let us fix $(i,j) \in [1;m] \times [1;n]$ and $\mathbf{m} = \mathbf{m}_{ij} = \min(m-i, j-1)$ as in Theorem 1.1. Define $\mathcal{P}_{ij}^{\mathbf{m}}$ to be the set of all $(\mathbf{m}+1)$-tuples $\bar{\pi} = \pi_0 \sqcup \pi_1 \sqcup \cdots \sqcup \pi_{\mathbf{m}}$ of *non-crossing increasing paths* $\pi_1, \ldots, \pi_m \subset [1; \mathbf{m}+i] \times [1;n]$, where each path π_k starts at $(\mathbf{m}+i, j-\mathbf{m}+k)$ and ends at $(k+1, n)$. Then

$$\bar{y}_{ij}(\mathbf{K}(d)) = \max_{\bar{\pi} \in \mathcal{P}_{ij}^{\mathbf{m}}} \Big(\sum_{(k,l) \in \bar{\pi}} d_{kl} \Big) \tag{1.10}$$

for $d \in M$.

7. Let $\mathbf{1}, \mathbf{J} \in M$ are the matrices defined by $\mathbf{1}_{ij} = 1$, $\mathbf{J}_{ij} = n + i - j$ for $(i,j) \in [1;m] \times [1;n]$. Then

$$\mathbf{K}(d + \mathbf{1}) = \mathbf{K}(d) + \mathbf{J} \ . \tag{1.11}$$

8. The formulae (1.3) naturally extend $\mathbf{K}$ to a continuous piece–wise linear (CPL) isomorphism $\mathbf{K}_{\mathbb{R}} : M_{\mathbb{R}} \to P_{\mathbb{R}}$ of positive real spans of M and P respectively (that is, of real polyhedral cones). Moreover, (1.11) implies that $\mathbf{K}$ uniquely extends to such a CPL–bijection

$$\overline{\mathbf{K}}_{\mathbb{R}} : Mat_{m \times n}(\mathbb{R}) \cong Mat_{m \times n}(\mathbb{R})$$

which preserves $Mat_{m \times n}(\mathbb{Z})$.

9. A formula similar to (1.11) can be stated for $\mathbf{I}$, the the "identity" $m \times n$–matrix defined by $\mathbf{I}_{ij} = (\delta_{i,i+(n-m)_+}) \in M$:

$$\mathbf{K}(d + \mathbf{I}) = \mathbf{K}(d) + \mathbf{1} \ .$$

This formula agrees with (1.7). Combining the latter identity with (1.11), we obtain

$$\mathbf{K}(d + c(\mathbf{1} - \mathbf{I})) = \mathbf{K}(d) + c(\mathbf{J} - \mathbf{1})$$

for $c \in \mathbb{Z}_{\ge 0}$. This implies the following fact (observed in [6]). Let $c : P \to \mathbb{Z}$ be the function defined by $c(d) = \min(d_{ij})$, where the minimum is over all $(i,j) \in [1;m] \times [1;n]$ with $j \ne i + (n-m)_+$. And let $c' : P \to \mathbb{Z}$ be the function defined by $c'(p) = \min(p_{i',j'} - p_{ij})$, where the minimum is over all *adjacent* (in the sense of (1.1)) pairs $(i,j) < (i',j')$ in $[1;m] \times [1;n]$, that is, satisfying either $(i',j') = (i, j-1)$ or $(i',j') = (i+1, j)$. Then

$$c'(\mathbf{K}(d)) = c(d)$$

for $d \in M$.

10. For each $\lambda = (0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{\min(m,n)})$ let $d(\lambda) \in M_{m \times n}$ be the diagonal matrix defined by $d(\lambda)_{ij} := \lambda_i \delta_{j,i+(n-m)_+}$. Then

$$\mathbf{K}(d(\lambda)) = \begin{matrix} \lambda_1 & \lambda_1 & \cdots & \lambda_1 \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_m & \lambda_m & \cdots & \lambda_m \end{matrix} \ . \tag{1.12}$$

This formula can be obtained, e.g., from (1.10) with $\mathbf{m} = 1$ and (1.6).

11. Let $M_{n \times n}^{\mathrm{st}}$ be the set of all "standard" matrices $d \in M_{n \times n}$, that is, having all row and column sums equal to 1, and $P_{n \times n}^{\mathrm{st}}$ be the set of all "standard" plane partitions $p \in P_{n \times n}$, that is, satisfying $y_{i1}(p) = y_{n,n+1-i}(p) = i$ for all i. Then, according to (1.6), $\mathbf{K}_{n \times n}$ restricts to a bijection

$$M_{n \times n}^{\mathrm{st}} \cong P_{n \times n}^{\mathrm{st}}. \tag{1.13}$$

In fact, the latter bijection is the classical Robinson–Schensted correspondence

$$S_n \cong T_n^{(2)}$$

between and the set S_n of all permutation (of the symbols $1,\ldots,n$) and the set $T_n^{(2)}$ of pairs of the standard Young tableaux of the equal shapes (in the same symbols $1,\ldots,n$).

And, second, let us identify $T_n^{(2)}$ with $P_{n\times n}^{\rm st}$ as follows. $(T,T')\mapsto p=(p_{ij})$, where for $i=1,\ldots,n$ the lower diagonal $(p_{ni},p_{n-1,i+1},\ldots,p_{in})$ (resp. the upper diagonal $(p_{i1},p_{i-1,2},\ldots,p_{1i})$) of p is the shape of that subtableau of T (resp. of T') which is formed by the entries $1,2,\ldots,i$. Clearly, this correspondence $(T,T')\mapsto p$ is a bijection $T^{(2)}\cong P_{n\times n}^{\rm st}$.

It turns out that there is an explicit formula for the inverse of $\mathbf{K}$. In order to state the formula we need a bit of notation.

For $(i,j)\in[1;m]\times[1;n]$ let $z_{ij}:M\to\mathbb{Z}$ be defined by the formula:

$$z_{ij}(d)=\sum_{(k,l)\in[1;i]\times[j;n]} d_{kl}\ . \tag{1.14}$$

Note that the collection of all $z_{ij}=z_{ij}(d)$ determines the matrix d uniquely by the formula:

$$d_{ij}=z_{ij}-z_{i-1,j}-z_{i,j+1}+z_{i-1,j+1}$$

(we use the agreement $z_{ij}=0$ whenever $(i,j)\notin[1;m]\times[1;n]$).

For $0\le a<i<b\le m, 1\le j\le n$ define the set $SS_{a;b}^{i;j}=SS_{a;b}^{i;j;m}$ of certain *super–tableaux* of rectangular shape to be the set of all weakly increasing maps $\nu:[a+1;i]\times[i+1;b]\to[0;j+m-i-1]$ such that

$$\text{either } \nu(k,l)+l\le m \text{ or } \nu(k,l)+k>m \tag{1.15}$$

for each $(k,l)\in[a+1;i]\times[i+1;b]$.

Theorem 1.2. *The inverse Robinson–Schensted–Knuth bijection $\mathbf{K}^{-1}:P\to M$ is given by the following formula. For $(i,j)\in[1;m]\times[1;n]$, $p=(p_{ij})\in P$ we have:*

$$\begin{aligned} z_{ij}(\mathbf{K}^{-1}(p)) &= y_{aj}(p)-y_{a,b+j-i}(p)+y_{i,b+j-i}(p)+\\ &+\min_{\nu\in SS_{a;b}^{i;j}}\Big(\sum_{(k,l):\nu(k,l)+l\le m} p_{\nu(k,l)+k,l-k}-p_{\nu(k,l)+k,l+1-k}+\\ &+\sum_{(k,l):\nu(k,l)+k>m} p_{m+1+k-l,\nu(k,l)+l-m}-p_{m+k-l,\nu(k,l)+l-m}\Big)\end{aligned}$$

where both sums in the above minimum are over $(k,l)\in[a+1;i]\times[i+1;b]$; and $a=a_{ij}=\max(0,1+i-j)$, $b=b_{ij}=\min(m,n+i+1-j)$ (we follow the above agreement that $y_{0,j}(p)=y_{i,n+1}=0$).

Properties of $\mathbf{K}^{-1}$.

In the following list of properties we denote $d=(d_{ij}):=K^{-1}(p)$.

1. $z_{i1}(d)=y_{i1}(p)$, $z_{mj}(d)=y_{mj}(x)$ for $(i,j)\in[1;m]\times[1;n]$. This follows from the homogeneity property (1.6) or directly from Theorem 1.2.

2. $d_{1n}=\min(p_{11}-p_{12},\ldots,p_{m-1,1}-p_{m-1,2};p_{m2}-p_{m-1,2},\ldots,p_{mn}-p_{m-1,n})$.

3. $d_{m1}=\min(p_{m1}-p_{m-1,1},p_{m1}-p_{m2})$. More generally, for $1\le i<\min(m,n)$ we have:

$$d_{m+1-i,1}+\cdots+d_{m1}=$$

$$=\min_{s\in[0,i]}\left(\min_{m-i=\nu_0\le\cdots\le\nu_s\le m-s}\left(\sum_{l=0}^{s}p_{m-l,l+1}-p_{m+1-l,l+1}+p_{\nu_l,l}-p_{\nu_l,l+1}\right)\right)\ ,$$

$$d_{m1}+\cdots+d_{mi}=$$

$$= \min_{s\in[0,i]}\left(\min_{i=\nu_0\geq\cdots\geq\nu_s\geq s}\left(\sum_{l=0}^{s} p_{m-l,l+1}-p_{m-l,l}+p_{m+1-l,\nu_l+1}-p_{m-l,\nu_l+1}\right)\right) .$$

Now we consider a couple of examples.

Example 1. Let $2=m\leq n$. For any $d\in M_{2\times n}$ let $p=\mathbf{K}(d)\in P_{2\times n}$. We have (by (1.8)),

$$p_{1j}=\min_{l\in[j-1;n]}((d_{1,j-1}+\cdots+d_{1,l-1})+(d_{2,l+1}+\cdots+d_{2n}))$$

for $j=2,\ldots,n$. The remaining p_{ij} can be found from the above homogeneity conditions (1.6):

$$p_{2j}+p_{1,j+1}=d_{1j}+\cdots+d_{1n}+d_{2j}+\cdots+d_{2n}$$

for $j=0,\ldots,n-1$, and $p_{11}=d_{11}+\cdots+d_{1n}$.

Second, let us compute $d=(\mathbf{K}^{-1}(p))$. We start with the computation of d_{1j} for $j\geq 2$. Let $i=1, j=2,\ldots,n$. Then $a=0, b=2$ and $SS_{1;2}^{1;j}$ is the set of all weakly increasing maps $\nu:[1;1]\times[2;2]\to[0;j]$, such that either $\nu(1,2)+2\leq 2$ or $\nu(1,2)+1>2$, that is, $SS_{1;2}^{1;j}$ is the set $\{0\}\sqcup[2;j]$. Thus for $j=2,\ldots,n$:

$$d_{1j}+\cdots+d_{1n}=p_{1,j+1}+\min(p_{11}-p_{12};\ p_{22}-p_{12},\ldots,p_{2j}-p_{1j}),$$
$$d_{2j}+\cdots+d_{2n}=p_{2j}-\min(p_{11}-p_{12};\ p_{22}-p_{12},\ldots,p_{2j}-p_{1j}).$$

Finally,

$$d_{11}=\max(p_{12}-p_{13},p_{12}-p_{13}+p_{11}-p_{22}),\ d_{21}=\min(p_{21}-p_{22},p_{21}-p_{11}) .$$

Example 2. Let $m=n=3$ and

$$d=\begin{matrix}d_{11}&d_{12}&d_{13}\\d_{21}&d_{22}&d_{23}\\d_{31}&d_{32}&d_{33}\end{matrix},\quad p=\begin{matrix}p_{11}&p_{12}&p_{13}\\p_{21}&p_{22}&p_{23}\\p_{31}&p_{32}&p_{33}\end{matrix}$$

such that $p=\mathbf{K}(d)$. Then p is determined by

$$p_{11}=d_{11}+d_{12}+d_{13},\ p_{33}=d_{13}+d_{23}+d_{33} ,$$
$$p_{21}+p_{12}-p_{11}=d_{21}+d_{22}+d_{23},\ p_{32}+p_{23}-p_{33}=d_{12}+d_{22}+d_{32} ,$$
$$p_{31}+p_{22}+p_{13}=\sum_{(i,j)\in[1;3]\times[1;3]}d_{ij},\ p_{13}=\min(d_{11},d_{22},d_{33}) ,$$
$$p_{12}=\min(d_{11}+d_{12},d_{11}+d_{23},d_{22}+d_{23}) ,$$
$$p_{23}=\min(d_{33}+d_{23},d_{33}+d_{12},d_{22}+d_{12}) ,$$
$$p_{22}+p_{13}=\min(d_{11}+d_{12}+d_{21}+d_{22},d_{11}+d_{12}+d_{21}+d_{33} ,$$
$$d_{11}+d_{23}+d_{21}+d_{33},d_{11}+d_{12}+d_{32}+d_{33} ,d_{11}+d_{23}+d_{32}+d_{33},d_{22}+d_{23}+d_{32}+d_{33}) .$$

Conversely, let $d=\mathbf{K}^{-1}(p)$. Then d is determined by

$$d_{11}+d_{12}+d_{13}=p_{11},\ d_{13}+d_{23}+d_{33}=p_{33} ,$$
$$p_{21}+p_{12}-p_{11}=d_{21}+d_{22}+d_{23},\ d_{12}+d_{22}+d_{32}=p_{32}+p_{23}-p_{33} ,$$
$$=d_{11}+d_{12}+d_{21}+d_{22}+d_{31}+d_{32}=p_{31}+p_{22}+p_{13}-p_{33} ,$$
$$d_{31}=\min(p_{31}-p_{21},p_{31}-p_{32}) ,$$
$$d_{13}=\min(p_{11}-p_{12},p_{21}-p_{22},p_{32}-p_{22},p_{33}-p_{23}) ,$$
$$d_{31}+d_{21}=p_{31}-p_{32}+\min(p_{32}-p_{11},p_{22}-p_{23},p_{22}-p_{12},p_{21}-p_{11}) ,$$
$$d_{31}+d_{32}=p_{31}-p_{21}+\min(p_{21}-p_{33},p_{22}-p_{12},p_{22}-p_{23},p_{32}-p_{33}) .$$

Let $D\subset[1;m]\times[1;n]$ be a union of subsets of the form: $[1;i]\times[j;n]$. We refer to such a D as *shape*. Let $Mat_D(\mathbb{Z})$ be the set of matrices of shape D, that is, the set of all maps $d:D\to\mathbb{Z}$ (denoted again by $d=(d_{ij})$). Similarly, define the set M_D of all *non–negative matrices of shape D* by $M_D := \{d : D \to \mathbb{Z}_{\geq 0}\} \subset Mat_D(\mathbb{Z})$; and the set $P_D \subset M_D$ of *plane partitions of shape* D as the set of all order–preserving maps $p : D \to \mathbb{Z}_{\geq 0}$.

It is well–known (see e.g., [6]) that for any shape D there is a Hillman–Grassl bijection

$$\mathbf{K}_D : M_D \cong P_D \ . \tag{1.16}$$

In the same words as above, we define linear forms $y_{ij} : P_D \to \mathbb{Z}, z_{ij} : M_D \to \mathbb{Z}$ for $(i,j) \in D$. Define the *boundary* ∂D of D to be the set of all $(i,j) \in D$ satisfying either $(i+1,j) \notin D$ or $(i,j-1) \notin D$.

By definition, $\mathbf{K}_D$ satisfies the homogeneity constrains:

$$y_{ij}(\mathbf{K}_D(d)) = z_{ij}(d)$$

for all $(i,j) \in \partial D$.

Next we state a natural conjecture about $\mathbf{K}_D$ generalizing our computation of $\mathbf{K}_{m\times n}$. We keep notation $S_{ij}^{\mathbf{m}}$ of Theorem 1.1 for $(i,j) \in D$, $\mathbf{m} \in \mathbb{Z}_{\geq 0}$.

Conjecture 1.3. *The Hillman–Grassl bijection* $\mathbf{K}_D : M_D \cong P_D$ *is given by the following formula. For each* $(i,j) \in D$, $d = (d_{kl}) \in M_D$ *one has:*

$$y_{ij}(\mathbf{K}(d)) = \min_{s \in S_{ij}^{\mathbf{m}}} \left(\sum_{(k,l)\in[1;i]\times[j;n]} d_{s(k,l)+k,s(k,l)+l-\mathbf{m}} \right) \ , \tag{1.17}$$

where $\mathbf{m} = \mathbf{m}_{ij} = i_0 - i = j - j_0$ *where* $(i_0, j_0) \in \partial D$ *is the only point such that* $i_0 + j_0 = i + j$.

Remark. The above formula admits a reformulation (1.10) in terms of non–crossing paths. We can also conjecture that the inverse Hillman–Grassl bijection $\mathbf{K}_D^{-1}$ is also given by a formula similar to that in Theorem 1.2. More precisely, we conjecture that $z_{ij}(\mathbf{K}_D^{-1}(p))$ is a minimum of certain linear forms in $p \in P_D$ for each $(i,j) \in D$.

We will illustrate the conjectures by the following example.

Example 3. Let $D = [1;3] \times [1;3] \setminus \{(3,1)\}$ and

$$d = \begin{matrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ & d_{32} & d_{33} \end{matrix}, \ p = \begin{matrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ & p_{32} & p_{33} \end{matrix}$$

such that $p = \mathbf{K}(d)$. Then p is determined by

$$p_{11} = d_{11} + d_{12} + d_{13}, \ p_{33} = d_{13} + d_{23} + d_{33},$$

$$p_{21} + p_{12} - p_{11} = d_{21} + d_{22} + d_{23}, \ p_{32} + p_{23} - p_{33} = d_{12} + d_{22} + d_{32},$$

$$p_{22} + p_{13} = d_{12} + d_{13} + d_{22} + d_{23}, \ p_{13} = \min(d_{12}, d_{23}),$$

$$p_{12} = \min(d_{11} + d_{12}, d_{11} + d_{23}, d_{22} + d_{23}), \ p_{23} = \min(d_{33} + d_{23}, d_{33} + d_{12}, d_{22} + d_{12}).$$

Conversely, let $d = \mathbf{K}^{-1}(p)$. Then d is determined by

$$d_{11} + d_{12} + d_{13} = p_{11}, \ d_{13} + d_{23} + d_{33} = p_{33},$$

$$p_{21} + p_{12} - p_{11} = d_{21} + d_{22} + d_{23}, \ d_{12} + d_{22} + d_{32} = p_{32} + p_{23} - p_{33},$$

$$= d_{21} + d_{22} + d_{23} + d_{32} + d_{33} = p_{22} + p_{13} - p_{11},$$

$$d_{21} = \min(p_{21} - p_{11}, p_{21} - p_{22}), \ d_{32} = \min(p_{32} - p_{33}, p_{32} - p_{22}),$$

$$d_{12} + d_{13} = min(p_{11}, p_{22}) + p_{13} - p_{12}, \ d_{23} + d_{13} = min(p_{22}, p_{33}) + p_{13} - p_{23},$$

$$d_{13} = \min(p_{11} - p_{12}, p_{22} - p_{12}, p_{22} - p_{23}, p_{33} - p_{23}).$$

Let us discuss the approach to the proof of both Theorem 1.1 and Theorem 1.2.

Let $\mathcal{M} = \mathcal{M}_{m\times n}$ be the quantized coordinate ring of the space $Mat_{m\times n}(\mathbb{C})$ of complex $m \times n$–matrices. This is an algebra generated by t_{ij}, $1 \le i \le m, 1 \le j \le n$ subject to certain quadratic relations (see [15], [13])

There is a monomial basis $\mathbf{B}_{\text{mon}}$ in this algebra naturally labeled by M. It consists of all monomials

$$t^{[d]} = \prod_{(i,j)\in[1;m]\times[1;n]} t_{ij}^{[d_{ij}]} ,$$

for $d \in M$, where the product is taken in the lexicographic order on the pairs (i,j), and the notation $t_{ij}^{[d_{ij}]}$ stands for $q^{-\binom{d_{ij}}{2}} t_{ij}^{d_{ij}}$.

The algebra $\mathcal{M}_{m\times n}$ is an *integrable* module over the quantized enveloping algebra $U_q(sl_m \times sl_n)$ (this is the deformed dual action the group $SL_m(\mathbb{C}) \times SL_n(\mathbb{C})$ on $Mat_{m\times n}(\mathbb{C})$). Hence there is an upper global crystal basis $\mathbf{B}^{\text{up}}$ for $\mathcal{M}_{m\times n}$ (see [8], [13]). This basis $\mathbf{B}^{\text{up}}$ is naturally labeled by P (see e.g., [13],[9]).

We need the following definition:

Definition 1.4. *Two bases B and B' for a $\mathbb{C}(q)$–vector space V are called crystal equivalent (notation: $B \cong_q B'$) if the transition matrix from B to B' modulo $q\mathbb{Z}[[q]]$ is the identity matrix, or, more precisely, a matrix of a bijection $B \to B'$.*

Note that if V is finite–dimensional or graded by finite–dimensional sub-spaces, this relation $'\cong_q'$ is an equivalence (between all bases for V compatible with the grading).

Our first main result translating the RSK into a crystal equivalence is the following theorem:

Theorem 1.5.

$$\mathbf{B}_{\text{mon}} \cong_{q^{-1}} \mathbf{B}^{\text{up}} \ , \tag{1.18}$$

and the corresponding bijection $M \cong P$ of the labeling sets is the RSK. In other words, the following diagram commutes.

$$\begin{array}{ccc} \mathbf{B}_{\text{mon}} & \xrightarrow{\cong_{q^{-1}}} & \mathbf{B}^{\text{up}} \\ \uparrow & & \uparrow \\ M & \xrightarrow{\mathbf{K}} & P \end{array} \tag{1.19}$$

where the vertical arrows are the labeling bijections.

This theorem was inspired by a remarkable paper [13] by Leclerc and Thibon in which a similar result was proved: there is another monomial basis $\mathbf{B}_\tau$ and another (slightly modified) upper global crystal basis $\widetilde{\mathbf{B}}^{\text{up}}$ such that $\mathbf{B}_\tau \cong_q \widetilde{\mathbf{B}}^{\text{up}}$, and the corresponding bijection on the labeling sets is the RSK.

Our next main result refines Theorem 1.5 as follows:

Theorem 1.6. *Denote by $b : P \to \mathbf{B}^{\text{up}}$ the natural labeling bijection. Then for each $d \in M$,*

$$t^{[d]} = b(\mathbf{K}(d)) + \sum_{p\neq\mathbf{K}(d)} c_p^d \cdot b(p) \ , \tag{1.20}$$

where all $c_p^d = p_{d'}^d(q^{-1}) \in q^{-1}\mathbb{Z}_{\ge 0}[q^{-1}]$.

One can deduce this theorem from Theorem 1.5, and Theorems 1.8, 1.9(a).

Remark. The polynomials c_p^d behave similarly to Kazhdan–Lusztig polynomials for Hecke algebra $\mathbb{C}_q[S_n]$. In fact, for any $(0,1)$-matrix $d \in M_{2\times n}$ they are equal to the parabolic Kazhdan–Lusztig polynomials at q^{-1} (see e.g., [5]). Thus these polynomials $c_p^d(q^{-1})$ deserve the name 'generalized Kazhdan–Lusztig polynomials'. Note also that the above theorem gives an opportunity to define the RSK in terms of these polynomials: $p = \mathbf{K}(d)$ if and only if $c_p^d(q^{-1}) = 1$. We believe that the Hillman–Grassl bijection (1.16) can also be characterized in such terms.

Our next main result is an interpretation of the crystal equivalence (1.18) as a specialization of a dual *Lusztig crystal equivalence* between two bases in the algebra $\mathcal{A}_{m+n-1}$ of quantum unipotent $(m+n) \times (m+n)$-matrices.

More precisely, for any $r \geq 1$ the algebra $\mathcal{A}_r = \mathcal{A} = \mathbb{C}_q[N]$ is the quantized coordinate ring of the group $N = N_r$ of $(r+1)\times(r+1)$ unipotent matrices. The algebra $\mathcal{A}$ is generated by $x_{ij}, 1 \leq i \leq j \leq r$ subject to certain quadratic-linear relations.

Let $M^+ = M_r^+ \subset M_{r\times r}$ of all non-strictly upper triangular $r \times r$–matrices in $M_{r\times r}$ (i.e., those matrices $d = (d_{ij}) \in M_{r\times r}$, $1 \leq i \leq j \leq r$ which satisfy $d_{ij} = 0$ for $i > j$).

There is a monomial basis $\mathbf{B}^+_{\mathrm{mon}}$ for $\mathcal{A}$ that consists of all the monomials of the form

$$x^{[d]} = \prod_{1\leq i\leq j\leq r} x_{ij}^{[d_{ij}]} ,$$

for $d \in M_+^r$, where the product is taken in the *right* lexicographic order on the pairs (i,j) (that is, $(i,j) < (i',j')$ if either $j < j'$ or $j = j'$, $i < i'$).

Let $\mathbf{B}^+ = \mathbf{B}^+_r$ be the *dual canonical basis* for $\mathcal{A}$ (see [2]).

The following fact is a direct consequence of the Lusztig definition of *canonical basis* for $U_q(Lie(N_r))$ (see [14]).

Proposition 1.7. $\mathbf{B}^+_{\mathrm{mon}} \cong_{q^{-1}} \mathbf{B}^+$. *This crystal equivalence determines, in particular, the labeling bijection*

$$\mathbf{b} : M^+ \cong \mathbf{B}^+ \tag{1.21}$$

We refine Proposition 1.7 below in Theorem 1.9.

The next crucial step is based on the following (presumably) well–known inclusion of algebras $\varphi : \mathcal{M}_{m\times n} \hookrightarrow \mathcal{A}_{m+n-1}$ defined by

$$\varphi(t_{ij}) = x_{i,j+m-1} \tag{1.22}$$

for $1 \leq i \leq m, 1 \leq j \leq n$. By definition, φ embeds $\mathbf{B}_{\mathrm{mon}}$ into $\mathbf{B}^+_{\mathrm{mon}}$.

The following result is very natural.

Theorem 1.8. *Under the inclusion (1.22), the upper global crystal basis $\mathbf{B}^{\mathrm{up}}$ becomes a certain part of the dual canonical basis $\mathbf{B}^+_{m+n-1}$ so that the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{B}^+_{\mathrm{mon}} & \xrightarrow{\cong_{q^{-1}}} & \mathbf{B}^+_{m+n-1} \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\varphi} \\ \mathbf{B}_{\mathrm{mon}} & \xrightarrow{\cong_{q^{-1}}} & \mathbf{B}^{\mathrm{up}} \end{array} . \tag{1.23}$$

Despite the naturality of Theorem 1.8 we have failed to find precise references.

Thus, in order to compute the RSK we have only to interpret both natural labeling of $\mathbf{B}^{\mathrm{up}}$ as restrictions (to the image of $\mathbf{B}^{\mathrm{up}}$ in $\mathbf{B}^+_{m+n-1}$) of some other two natural labeling of $\mathbf{B}^+_{m+n-1}$.

Now we present our next main result that introduces such a labeling and refines Proposition 1.7.

We need a bit of notation. Recall that a map ν of $\mathcal{A} \setminus \{0\}$ to a linearly ordered semigroup $\mathcal{C}$, is called *valuation of* $\mathcal{A}$ if $\nu(xy) = \nu(x) + \nu(y)$ and $\nu(x+y) \leq \max(\nu(x), \nu(y))$ for all $x, y \in \mathcal{A} \setminus \{0\}$; and, additionally, $\nu(x+y) = \max(\nu(x), \nu(y))$ whenever $\nu(x) \neq \nu(y)$).

Let $'\prec'$ be a linear semigroup order on M^+ defined by: $d' = (d'_{ij}) \prec d$ if and only if there exists a (unique) pair (i,j) such that

$$d'_{1j} + \cdots + d'_{ij} < d_{1j} + \cdots + d_{ij}$$

and

$$d'_{1j'} + \cdots + d'_{i'j'} = d_{1j'} + \cdots + d_{i'j'}$$

whenever $j' < j$ or $j' = j$, $i' > i$.

Theorem 1.9. (a) *For $d \in M^+$ one has*

$$x^{[d]} = \mathbf{b}(d) + \sum_{d' \prec d} c^d_{d'} \mathbf{b}(d') \ , \tag{1.24}$$

where all $c^d_{d'} \in q^{-1}\mathbb{Z}_{\geq 0}[q^{-1}]$ and $\mathbf{b}$ is the labeling bijection defined in (1.21).

(b) *There exists a unique (surjective) valuation $\mathbf{d} = \mathbf{d}_r : \mathcal{A} \setminus \{0\} \to M^+$ (with respect to the above " $\prec$ " order) such that*

$$\mathbf{d}(x^{[d]}) = d$$

for $d \in M^+$.

(c) *$\mathbf{d}(\mathbf{b}(d)) = d$ for any $d \in M^+$, that is, the restriction of $\mathbf{d}$ to $\mathbf{B}^+$ is the bijection $\mathbf{B}^+ \cong M^+$ inverse to $\mathbf{b}$.*

The valuation $\mathbf{d}$ can be easily obtained out of the *string* valuation $a(\mathbf{i}_0, \bullet)$ constructed in [2].

Now we define the second valuation $\mathbf{d}^* : \mathcal{A} \setminus \{0\} \to M^+$ by $\mathbf{d}^*(x) = \mathbf{d}(x^*)$, where $^* : \mathcal{A} \to \mathcal{A}$ is a unique anti–automorphism of $\mathcal{A}$ determined by $x^*_{ii} = x_{ii}$ for $i = 1, \ldots, r$. (see [8]). It is well–known (see e.g., [14]) that this anti–automorphism * preserves the dual canonical basis $\mathbf{B}^+$. Thus, according to Theorem 1.9, the restriction of $\mathbf{d}^*$ to $\mathbf{B}^+$ is also a bijection $\mathbf{B}^+ \cong M^+$.

Let $\delta = \delta_{mn} : P_{m \times n} \to M^+_{m+n-1}$ be a linear injective map defined by the following formula ($1 \leq i \leq j \leq m+n-1$): if $j - i \geq \min(m,n)$ then $z_{ij}(\delta(p)) = 0$, and if $j - i < \min(m,n)$ then

$$z_{ij}(\delta(p)) = \begin{cases} y_{i,j+1-i}(p) & \text{if } i \leq j \leq m \\ y_{m+i-j,j+1-i}(p) & \text{if } i \leq m \leq j \\ y_{m+i-j,j+1-m}(p) & \text{if } m \leq i \leq j \end{cases} . \tag{1.25}$$

(Recall that y_{ij} and z_{ij} are defined in (1.2) and (1.14)).

Theorem 1.10. *The following diagram commutes.*

$$\begin{array}{ccc} \mathbf{B}^+_{m+n-1} & \xrightarrow{\mathbf{d}^*} & M^+_{m+n-1} \\ \uparrow \varphi & & \uparrow \delta \\ \mathbf{B}^{\mathrm{up}} & \longrightarrow & P_{m \times n} \end{array} \tag{1.26}$$

where the lower horizontal arrow is the inverse of the natural labeling.

The following proposition provides the desirable link between the valuations $\mathbf{d}$ and $\mathbf{d}^*$.

Proposition 1.11. *([2]) There exists a continuous piece–wise linear involution $\zeta = \zeta_r : M^+ \to M^+$ such that for $b \in \mathbf{B}^+$*

$$\mathbf{d}(b^*) = \zeta(\mathbf{d}(b)).$$

Fortunately, this ζ is given by an explicit CPL–formula obtained by Knight and Zelevinsky in [10].

In order to state our next main result we need a bit of notation.

Let $\mathbf{j}$ be an injective map $M_{m \times n} \hookrightarrow M^+_{m+n-1}$ determined by the left vertical arrow in the diagram (1.23), that is, for $d \in M_{m \times n}$

$$(\mathbf{j}(d))_{ij} = \begin{cases} d_{i,j+1-m} & \text{if } 1 \leq i \leq m \leq j \leq m+n-1 \\ 0 & \text{otherwise} \end{cases} .$$

Thus Theorem 1.10 together with Theorem 1.5 imply the following corollary:

Corollary 1.12. *The following diagram commutes:*

$$\begin{array}{ccc} M^+_{m+n-1} & \xrightarrow{\zeta} & M^+_{m+n-1} \\ \uparrow{\scriptstyle \mathbf{j}} & & \uparrow{\scriptstyle \delta} \\ M_{m\times n} & \xrightarrow{\mathbf{K}} & P_{m\times n} \end{array} \quad . \tag{1.27}$$

In other words, for $(i,j)\in[1;m]\times[1;n]$*:*

$$y_{ij}(\mathbf{K}(d)) = \begin{cases} z_{i,i+j-1}(\zeta(\mathbf{j}(d))) & \text{if } i+j\le m+1 \\ z_{i+j-1,m+j-1}(\zeta(\mathbf{j}(d))) & \text{if } i+j\ge m+1 \end{cases} \tag{1.28}$$

$$z_{ij}(\mathbf{K}^{-1}(p)) = z_{i,j+m-1}(\zeta(\delta(p))) \ . \tag{1.29}$$

Thus, based on the explicit formula for ζ, one derives both Theorems 1.1 and 1.2 from Corollary 1.12.

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