DESCENT STATISTICS FOR THE HYPEROCTAHEDRAL GROUP

RON M. ADIN, FRANCESCO BRENTI, AND YUVAL ROICHMAN

Abstract. Three new statistics on the hyperoctahedral group B_n are introduced. It is shown that they give two generalizations of Carlitz's identity for the descent number and major index over S_n. This answers a question posed by Foata. These statistics are then extended to a family of refined "descent statistics", leading to multivariate identities.
Résumé. C'est le résumé français.

1. Introduction

A well known classical result due to MacMahon (see [15]) asserts that the inversion number and major index of a permutation are equidistributed over the symmetric group S_n. The joint distribution of major index and descent number was studied by Carlitz [6], Gessel [13], and others. Despite the fact that an increasing number of enumerative results of this nature have been generalized to the hyperoctahedral group B_n (see, e.g., [5], [11], [18], [19], [20]) and that several "major index" statistics have been introduced and studied for B_n (see, e.g., [7], [8], [9], [16], [17], [22]), no generalization of MacMahon's result has been found until the recent paper [3]. There a new statistic, the *flag major index* (denoted here *fmaj*), was introduced; and it was shown to be equidistributed with length, which is the natural analogue of inversion number from a Coxeter group theoretic point of view. No corresponding "descent statistic" has been found that allows the generalization to B_n of the well known Carlitz identity for descent number and major index on S_n (see [6], and also Theorem 2.2 below), a problem first posed by Foata [10].

Problem 1.1. *(Foata) : Extend the ("Euler-Mahonian") bivariate distribution of descent number and major index to the hyperoctahedral group B_n.*

The purpose of this paper is to introduce three new statistics on B_n: the *negative descent* (denoted *ndes*); the *negative major* (*nmaj*); and the *flag descent* (*fdes*). When restricted to S_n they reduce to descent number, major index, and twice descent number, respectively, and they solve the above problem. More precisely, we show that *nmaj* is equidistributed with length on B_n, and that *ndes* is the "right" corresponding descent statistic needed to extend Carlitz's result to B_n, thus answering Foata's question. Then we prove the surprising result that the pair of statistics (*fdes*, *fmaj*) is equidistributed with (*ndes*, *nmaj*) over B_n, thus obtaining a second generalization of Carlitz's identity to B_n. Finally, we generalize these results to multivariate statistics and identities.

2. Notation, Definitions, and Preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the ring of integers, and $\mathbf{Q}$ be the field of rational numbers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). Given $n, m \in \mathbf{Z}$, $n \le m$, we let $[n, m] \stackrel{\text{def}}{=} \{n, n+1, \ldots, m\}$. For $S \subset \mathbf{N}$ we write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \ldots < a_r$. The cardinality of a set A will be denoted by $|A|$. More generally, given a multiset $M = \{1^{a_1}, 2^{a_2}, \ldots, r^{a_r}\}$ we denote by $|M|$ its cardinality, so $|M| = \sum_{i=1}^{r} a_i$.

Research of all authors was supported in part by the Israel Science Foundation, administered by the Israel Academy of Sciences and Humanities.

Given a variable q and a commutative ring R we denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in q with coefficients in R. For $i \in \mathbf{N}$ we let, as customary, $[i]_q \stackrel{\rm def}{=} 1 + q + q^2 + \ldots + q^{i-1}$ (so $[0]_q = 0$).

Given a sequence $\sigma = (a_1, \ldots, a_n) \in \mathbf{Z}^n$ we say that a pair $(i,j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $a_i > a_j$. We say that $i \in [n-1]$ is a *descent* of σ if $a_i > a_{i+1}$. We denote by $inv(\sigma)$ (respectively, $des(\sigma)$) the number of inversions (respectively, descents) of σ. We also let

$$maj(\sigma) \stackrel{\rm def}{=} \sum_{\{i:\ a_i > a_{i+1}\}} i$$

and call it the *major index* of σ.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\rm def}{=} S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$.

We denote by B_n the group of all bijections π of the set $[-n, n] \setminus \{0\}$ onto itself such that

$$\pi(-a) = -\pi(a)$$

for all $a \in [-n, n] \setminus \{0\}$, with composition as the group operation. This group is usually known as the group of "signed permutations" on $[n]$, or as the *hyperoctahedral group* of rank n. We identify S_n as a subgroup of B_n, and B_n as a subgroup of S_{2n}, in the natural ways.

If $\pi \in B_n$ then we write $\pi = [a_1, \ldots, a_n]$ to mean that $\pi(i) = a_i$ for $i = 1, \ldots, n$, and we let

$$\begin{aligned} des_A(\pi) &\stackrel{\rm def}{=} des(a_1, \ldots, a_n), \\ inv(\pi) &\stackrel{\rm def}{=} inv(a_1, \ldots, a_n), \\ maj_A(\pi) &\stackrel{\rm def}{=} maj(a_1, \ldots, a_n), \\ Neg(\pi) &\stackrel{\rm def}{=} \{i \in [n] :\ a_i < 0\}, \end{aligned} \tag{1}$$

and

$$neg(\pi) \stackrel{\rm def}{=} |Neg(\pi)|.$$

It is well known (see, e.g., [4, Proposition 8.1.3]) that B_n is a Coxeter group with respect to the generating set $\{s_0, s_1, s_2, \ldots, s_{n-1}\}$, where

$$s_0 \stackrel{\rm def}{=} [-1, 2, \ldots n]$$

and

$$s_i \stackrel{\rm def}{=} [1, 2, \ldots, i-1, i+1, i, i+2, \ldots n]$$

for $i = 1, \ldots, n-1$. This gives rise to two other natural statistics (defined for any Coxeter group) on B_n, namely

$$l(\pi) \stackrel{\rm def}{=} min\{r \in \mathbf{N} :\ \pi = s_{i_1} \ldots s_{i_r} \quad \text{for some} \quad i_1, \ldots, i_r \in [0, n-1]\}$$

(known as the *length* of π) and

$$des_B(\pi) \stackrel{\rm def}{=} |\{i \in [0, n-1] : l(\pi s_i) < l(\pi)\}|.$$

There is a well known direct combinatorial way to compute these two statistics for $\pi \in B_n$ (see, e.g., [4, Propositions 8.1.1 and 8.1.2] or [5, Proposition 3.1 and Corollary 3.2]), namely

$$l(\pi) = inv(\pi) - \sum_{i \in Neg(\pi)} \pi(i) \tag{2}$$

and

$$des_B(\pi) = |\{i \in [0, n-1] :\ \pi(i) > \pi(i+1)\}|, \tag{3}$$

where $\pi(0) \stackrel{\rm def}{=} 0$. For example, if $\pi = [-3, 1, -6, 2, -4, -5] \in B_6$ then $des_A(\pi) = 3$, $inv(\pi) = 9$, $maj_A(\pi) = 11$, $neg(\pi) = 4$, $l(\pi) = 27$, and $des_B(\pi) = 4$.

As customary, given a variable t we define an operator $\delta_t : \mathbf{Q}[q,t] \to \mathbf{Q}[q,t]$ by

$$\delta_t(P(q,t)) = \frac{P(q,qt) - P(q,t)}{qt - t},$$

for all $P \in \mathbf{Q}[q,t]$.

For $n \in \mathbf{P}$ we let

$$A_n(t,q) \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} t^{des_A(\sigma)} q^{maj_A(\sigma)},$$

and $A_0(t,q) \stackrel{\text{def}}{=} 1$. The following two results are due to Carlitz [6] and Gessel [13], and a proof of them can also be found in [12].

Theorem 2.1. *Let $n \in \mathbf{P}$. Then*

$$A_n(t,q) = (1 + tq[n-1]_q)\, A_{n-1}(t,q) + q(t - t^2)\delta_t(A_{n-1}(t,q)).$$

Theorem 2.2. *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{A_n(t,q)}{\prod_{i=0}^{n} (1 - tq^i)}.$$

in $\mathbf{Z}[q][[t]]$.

3. The "Negative" Statistics

In this section we define and study a new "descent set" for the elements of B_n. This gives rise, in a very natural way, to new "major index" and "descent number" statistics for B_n. We then show that these two statistics give a generalization of Carlitz's identity to B_n, and that one of them is equidistributed with length.

For $\pi \in B_n$ we define

$$Des_A(\pi) \stackrel{\text{def}}{=} \{i \in [n-1] : \pi(i) > \pi(i+1)\}$$

and the *negative-descent multiset*

$$NDes(\pi) \stackrel{\text{def}}{=} Des_A(\pi) \biguplus \{-\pi(i) : i \in Neg(\pi)\}, \tag{4}$$

where $Neg(\pi)$ is the set of positions of negative entries in π defined in (1), and $\biguplus$ denotes disjoint union (of multisets).

For example, if $\pi = [-3, 1, -6, 2, -4, -5] \in B_6$ then $Des_A(\pi) = \{2,4,5\}$ and $NDes(\pi) = \{2, 3, 4^2, 5^2, 6\}$. Note that if $\pi \in S_n$ then $NDes(\pi)$ is a set and coincides with the usual descent set of π. Also note that $NDes(\pi)$ can be defined rather naturally also in purely Coxeter group theoretic terms. In fact, for $i \in [n]$ let $\eta_i \in B_n$ be defined by

$$\eta_i \stackrel{\text{def}}{=} [1, 2, \ldots, i-1, -i, i+1, \ldots, n],$$

so $\eta_1 = s_0$. Then $\eta_1, \ldots, \eta_n$ are reflections (in the Coxeter group sense; see, e.g., [14]) of B_n and it is clear from (2) that

$$NDes(\pi) = \{i \in [n-1] : l(\pi s_i) < l(\pi)\} \biguplus \{i \in [n] : l(\pi^{-1}\eta_i) < l(\pi^{-1})\}.$$

These considerations explain why it is natural to think of $NDes(\pi)$ as a "descent set". With this in mind, the following definitions are also natural. For $\pi \in B_n$ we let

$$ndes(\pi) \stackrel{\text{def}}{=} |NDes(\pi)|$$

and

$$nmaj(\pi) \stackrel{\text{def}}{=} \sum_{i \in NDes(\pi)} i.$$

For example, if $\pi = [-3, 1, -6, 2, -4, -5] \in B_6$ then $ndes(\pi) = 7$ and $nmaj(\pi) = 29$.

Note that from (4) it follows that

$$nmaj(\pi) = maj_A(\pi) - \sum_{i \in Neg(\pi)} \pi(i) \qquad (\forall \pi \in B_n). \tag{5}$$

This formula is also one of the motivations behind our definition of $nmaj(\pi)$, because of the corresponding formula (2).

The pair of statistics $ndes$ and $nmaj$ gives a generalization of Theorem 2.2 to B_n.

Theorem 3.1. *[1, Theorem 3.2] Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\pi \in B_n} t^{ndes(\pi)} q^{nmaj(\pi)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

4. The Flag Statistics

In this section we introduce yet another pair of statistics on B_n and show that it also gives a solution to Foata's problem. We then derive some consequences of this result.

4.1. The Flag Major Index.

For $i = 0, 1, \ldots, n-1$ let $t_i \stackrel{\text{def}}{=} s_i s_{i-1} \cdots s_0$, or equivalently

$$t_i = [-i-1, 1, 2, \ldots, i, i+2, \ldots, n].$$

It is then not hard to show (see also [3]) that for any $\pi \in B_n$ there exist unique integers $k_0, \ldots, k_{n-1}$, with $0 \leq k_i \leq 2i+1$ for $i = 0, \ldots, n-1$, such that

$$\pi = t_{n-1}^{k_{n-1}} \ldots t_2^{k_2} t_1^{k_1} t_0^{k_0}.$$

The *flag major index* of π (see [3]) is then defined by

$$fmaj(\pi) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} k_i.$$

There is a simple way to compute the flag major index of a signed permutation π.

Theorem 4.1. *[3, Theorem 3.1] Let $\pi \in B_n$. Then*

$$fmaj(\pi) = 2\, maj_A(\pi) + neg(\pi).$$

For example, if $\pi = [-3, 1, -6, 2, -4, -5]$ then $fmaj(\pi) = 2 \cdot 11 + 4 = 26$.

In [3] it was shown that $fmaj$ appears naturally in the Hilbert series of (diagonal action) invariant algebras; note that this property is not shared by $nmaj$.

4.2. The Flag Descent Number.

For $\pi \in B_n$ let

$$fdes(\pi) \stackrel{\text{def}}{=} 2\, des_A(\pi) + \varepsilon_1(\pi), \tag{6}$$

where des_A is as in Section 2 and

$$\varepsilon_1(\pi) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \pi(1) < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

This definition is motivated by Theorem 4.1, and because of this similarity we call $fdes(\pi)$ the *flag descent number* of π. For example, if $\pi = [-3, 1, -6, 2, -4, -5]$ then $fdes(\pi) = 2 \cdot 3 + 1 = 7$. Note that from (3), (1) and (7) it follows immediately that

$$fdes(\pi) = des_A(\pi) + des_B(\pi) \tag{8}$$

and also

$$fdes(\pi) = des(\pi(-n), \ldots, \pi(-1), \pi(1), \ldots, \pi(n))$$

for all $\pi \in B_n$.

4.3. **Main Theorems.** Let

$$S_n(t,q) \stackrel{\text{def}}{=} \sum_{\pi \in B_n} t^{fdes(\pi)} q^{fmaj(\pi)} \tag{9}$$

for all $n \in \mathbf{P}$, and set $S_0(t,q) \stackrel{\text{def}}{=} 1$. Then

Theorem 4.2. *[1, theorem 4.3] Let $n \in \mathbf{P}$. Then*

$$S_n(t,q) = (1 + qt + t^2q^2[2n-2]_q)S_{n-1}(t,q) + q(t-t^2)(1+tq)\delta_t(S_{n-1}(t,q)).$$

This Theorem has several interesting consequences. The first one is that the descent statistic *fdes* yields a solution to Foata's problem.

Theorem 4.3. *[1, Theorem 4.2] Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\pi \in B_n} t^{fdes(\pi)} q^{fmaj(\pi)}}{(1-t)\prod_{i=1}^{n}(1-t^2q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

This in turn implies, together with Theorem 3.1, that the two pairs of statistics $(fdes, fmaj)$ and $(ndes, nmaj)$ are equidistributed in B_n.

Corollary 4.4. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in B_n} t^{ndes(\pi)} q^{nmaj(\pi)} = \sum_{\pi \in B_n} t^{fdes(\pi)} q^{fmaj(\pi)}.$$

Finally,

Corollary 4.5. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in B_n} q^{l(\pi)} = \sum_{\pi \in B_n} q^{nmaj(\pi)} = \sum_{\pi \in B_n} q^{fmaj(\pi)}.$$

5. Multivariate Statistics

For any signed permutation $\pi \in B_n$ let

$$d_i(\pi) \stackrel{\text{def}}{=} |\{j \in Des_A(\pi) : \ j \geq i\}|$$

be the number of descents in π from position i on;

$$\nu_i(\pi) \stackrel{\text{def}}{=} |\{j \in Neg(\pi) : \ j \geq i\}|$$

be the number of positions with negative entries in π from position i on; and

$$\varepsilon_i(\pi) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \pi(i) < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

By direct combinatorial methods we prove

Theorem 5.1. *[2] Let $n \in \mathbf{P}$. Then*

$$\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} \binom{n}{m_0(\lambda), m_1(\lambda), \ldots} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\pi \in B_n} \prod_{i=1}^{n} q_i^{d_i(\pi)+\nu_i(\pi^{-1})}}{\prod_{i=1}^{n} (1 - q_1^2 \cdots q_i^2)}$$

in $\mathbf{Z}[q][[t]]$, where the sum in the left-hand side is taken over all partitions with n nonnegative parts and

$$m_j(\lambda) := |\{1 \le i \le n \mid \lambda_i = j\}| \qquad (j \ge 0).$$

Using the Hilbert series of a new family of B_n-modules we prove

Theorem 5.2. *[2] With the above notations*

$$\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} \binom{n}{m_0(\lambda), m_1(\lambda), \ldots} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\pi \in B_n} \prod_{i=1}^{n} q_i^{2d_i(\pi)+\varepsilon_i(\pi)}}{\prod_{i=1}^{n} (1 - q_1^2 \cdots q_i^2)}$$

in $\mathbf{Z}[q][[t]]$.

Corollary 5.3. *Let $n \in \mathbf{P}$. Then*

$$\sum_{\pi \in B_n} \prod_{i=1}^{n} q_i^{d_i(\pi)+\nu_i(\pi^{-1})} = \sum_{\pi \in B_n} \prod_{i=1}^{n} q_i^{2d_i(\pi)+\varepsilon_i(\pi)}.$$

Acknowledgments. This paper grew out of stimulating discussions with Dominique Foata, Ira Gessel and Doron Zeilberger during the conference "Classical Combinatorics" in honor of Foata's 65^{th} birthday. The work on it was carried out while the second author was visiting the other authors at Bar-Ilan University.

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Department of Mathematics and Computer Science, Bar-Ilan University, Ramat Gan 52900, Israel.
E-mail address: `radin@math.biu.ac.il`

Dipartimento di Matematica, Universitá di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: `brenti@mat.uniroma2.it`

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat Gan 52900, Israel.
E-mail address: `yuvalr@math.biu.ac.il`