A DIRECT ALGORITHM TO CONSTRUCT ZEILBERGER'S RECURRENCES FOR RATIONAL FUNCTIONS

H. Q. LE

ABSTRACT. We present a direct algorithm to construct Zeilberger's recurrences for rational functions. We also describe a Maple implementation of the algorithm and show timing comparisons between this algorithm and other related algorithms.

RÉSUMÉ. Nous présentons un algorithme direct pour la construction de récurrences de Zeilberger dans le cas de fractions rationnelles. Nous en décrivons aussi une implémentation en Maple, et comparons les temps d'éxecution obtenus par cet algorithme avec ceux obtenus par d'autres algorithmes.

1. PRELIMINARIES

Zeilberger's algorithm [6, 10, 13], also known as the method of *creative telescoping,* provides very useful tools for finding closed-form solutions of definite hypergeometric summation, and for certifying the truth for large classes of identities that occur in combinatorics and in the theory of special functions. Given a function $T(n,k)$ as input, the algorithm (we name it hereafter as $\mathcal{Z}$) tries to construct for $T(n,k)$ a *Z-pair* (L,G) which consists of a linear difference operator with coefficients which are polynomials in n over $\mathbb{C}$

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0 \tag{1}$$

and a function $G(n,k)$ such that

$$LT(n,k) = (E_k - 1)G(n,k). \tag{2}$$

(E_n is the shift operator w.r.t. n, defined by $E_nT(n,k) = T(n+1,k)$. Similarly E_k is the shift operator w.r.t. k, defined by $E_kT(n,k) = T(n,k+1)$.) The operator L, which we call a *telescoper,* is *k-free.* Note that the algorithm can only be applied to $T(n,k)$ which is a *hypergeometric term* in both arguments, i.e., there exist first order operators $L_1 \in \mathbb{C}[n,k,E_n]$, $L_2 \in \mathbb{C}[n,k,E_k]$ such that $L_1T = L_2T = 0$. It is proven in [13] that if there exist Z-pairs for $T(n,k)$, then $\mathcal{Z}$ terminates with one of the Z-pairs and the telescoper L in the returned Z-pair is of minimal possible order. Note that L is unique up to a factor $P(n) \in \mathbb{C}[n]$, and we name it *the minimal telescoper.*

The algorithm uses an *item-by-item examination* of the order ρ of L. It starts with the value of 0 for ρ and increases ρ until it is successful in finding a Z-pair (L,G). The question for what hypergeometric terms the Z-pairs do exist is not conclusively answered although a sufficient condition is known via the "fundamental theorem" (see [6, 10, 12]). For a given *proper* hypergeometric term [12], an upper bound for ρ can be computed. This upper bound, however, is much too large in practice. This is the main reason for applying item-by-item examination strategy. As a consequence, when applying $\mathcal{Z}$ to a hypergeometric term, we waste resources either trying to compute a Z-pair in the case when no such Z-pair exists or trying to compute without success a telescoper with $\operatorname{ord} L < \rho$ in the case when the Z-pairs exist and the order of the minimal telescoper is ρ.

Let $T(n,k)$ be a rational function. The problem of determining a necessary and sufficient condition for the termination of $\mathcal{Z}$ on $T(n,k)$ is completely solved and presented in [4]. First, it provides a decision procedure as to whether or not one should apply $\mathcal{Z}$ to the input $T(n,k)$. Secondly, in the case when the termination of $\mathcal{Z}$ is guaranteed, it helps to speed up the construction of a Z-pair by

Supported in part by Natural Sciences and Engineering Research Council of Canada Grant No. CRD215442-98.

reducing the size of the problem to be solved. Note that an item-by-item examination technique is still used in this case [4].

In this paper we present a *direct algorithm* to construct a Z-pair (L, G) for a given rational function $T(n,k)$ where L is the minimal telescoper. By direct algorithm, we mean an algorithm which computes a Z-pair directly, without using an item-by-item examination. The algorithm is based on a special form of representation of $T(n,k)$, on direct construction of minimal telescopers for each member of this representation. A Z-pair for $T(n,k)$ can then be obtained using Least Common Left Multiple (LCLM) computation. We also describe a Maple implementation of the algorithm and show timing comparisons between this algorithm, the original Zeilberger's algorithm [13], and the modified version of $\mathcal{Z}$ [4].

2. The Decomposition Problem and the Existence of a Z-pair

In this section, we recall some known results relating to the decomposition problem [1, 3, 8, 11] and the existence of a Z-pair [4, 5] for rational functions. They are needed in subsequent sections.

2.1. **The Decomposition Problem.** Let $T(k)$ be a rational function over a field K of characteristic 0. For the decomposition problem, one constructs a rational function $S(k)$ and a proper rational function $F(k)$ over K such that

$$T(k) = (E_k - 1)S(k) + F(k) \tag{3}$$

where the denominator $g(k)$ of $F(k)$ has the lowest possible degree.

If $F(k) = 0$, then $T(k)$ is said to be *rational summable.*

Consider the complete partial fraction decomposition of $T(k)$ over an algebraic closure $\overline{K}$.

$$T(k) = \sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{\beta_{ij}}{(k-\alpha_i)^j},\ \alpha_i, \beta_{ij} \in \overline{K}.$$

Define α_i to be shift equivalent to α_j iff $\alpha_i - \alpha_j \in \mathbb{Z}$. It is shown in [3] that $T(k)$ can be written as

$$\sum_{i=1}^{s} \sum_{j=1}^{l_i} M_{ij} \frac{1}{(k-\alpha_i)^j}$$

where $M_{ij} \in \overline{K}[E_k], 0 < s \le m, l_i > 0$; α_i is the representative, which is the maximal element, for the i-th shift equivalence class of the set $\{\alpha_1, \ldots, \alpha_m\}$; and $T(k)$ is rational summable iff

$$M_{ij} = L_{ij} \circ (E_k - 1),\ L_{ij} \in \overline{K}[E_k],\ i = 1, \ldots, s,\ j = 1, \ldots, l_i. \tag{4}$$

Let us represent $F(k)$ in the reduced form $F(k) = f(k)/g(k)$, where $f(k), g(k) \in K[k]$. By [1], $g(k)$ has the following property:

P1. If $p_1(k), p_2(k)$ are factors of $g(k)$ irreducible over $\overline{K}$ then $p_1(k+h) \neq p_2(k)$ for all $h \in \mathbb{Z} \setminus \{0\}$.

On the other hand, suppose that $T(k)$ is rational summable. Let $T(k) = a(k)/b(k)$ where $a(k), b(k)$ are relatively prime elements of $K[k]$. Then $b(k)$ has the following property:

P2. If $q_1(k)$ is a factor of $b(k)$ irreducible over $\overline{K}$, then there exist a factor $q_2(k)$ irreducible over $\overline{K}$ of $b(k)$ and a non-zero integer h such that $q_1(k+h) = q_2(k)$.

2.2. **The Existence of a *Z*-pair.** Let $T(n,k) \in \mathbb{C}(n,k)$. Apply an algorithm to solve the decomposition problem with $K = \mathbb{C}(n)$ to rewrite $T(n,k)$ in the form (3). Let $g(n,k)$ be the denominator of $F(n,k)$. It is shown in [4] that a Z-pair for $T(n,k)$ exists iff each factor of $g(n,k)$ irreducible over $\overline{\mathbb{C}(n)}$ has (up to a factor from $\mathbb{C}$) the form

$$an + bk + c,\ a, b \in \mathbb{Z}, c \in \mathbb{C}. \tag{5}$$

An algorithm for using the above criterion as well as the verification of the algorithm are presented in [4, 5]. Note that one can always choose a and b such that $\gcd(a,b) = 1$ and $b > 0$.

3. On the Minimal Telescoper of a Sum of Rational Functions

Lemma 1. *Let (L, G) be a Z-pair for a hypergeometric term $F(n, k)$ where L is the minimal telescoper. If L_1 is a telescoper for F, then L_1 is right divisible by L in $\mathbb{C}(n)[E_n]$.*

Proof. We have

$$LF = (E_k - 1)G. \tag{6}$$

Since L_1 is a telescoper for F, there exists a hypergeometric term $G_1(n, k)$ such that

$$L_1 F = (E_k - 1)G_1. \tag{7}$$

Applying the right Euclidean division of L_1 by L yields

$$L_1 = Q \circ L + R,\ Q, R \in \mathbb{C}(n)[E_n],\ R = 0 \text{ or } \operatorname{ord} R < \operatorname{ord} L. \tag{8}$$

From (8), one obtains

$$L_1 F = Q(LF) + RF. \tag{9}$$

The substitution of (6) and (7) into (9) yields $RF = (E_k - 1)(G_1 - QG)$. Since L is the minimal telescoper, this implies $R = 0$. □

Theorem 1. *Let $F_1, \ldots, F_s$ be rational functions such that if L^* is a telescoper for $F = F_1 + \cdots + F_s$ then L^* is a telescoper for every $F_i, i = 1, \ldots, s$. Let $L_1, \ldots, L_s$ be the minimal telescopers for $F_1, \ldots, F_s$, respectively. Then* $\operatorname{lclm}(L_1, \ldots, L_s)$ *is the minimal telescoper for $F_1 + \cdots + F_s$.*

Proof. First we need to show that $\operatorname{lclm}(L_1, \ldots, L_s)$ is a telescoper for F. A proof for the case when $s = 2$ is shown in [4] (Lemma 1, Section 2) and we simply follow the same approach.

Since L_i is the minimal telescoper for F_i, there exists $G_i \in \mathbb{C}(n, k)$ such that

$$L_i F_i = (E_k - 1)G_i,\ 1 \le i \le s. \tag{10}$$

Set $L = \operatorname{lclm}(L_1, \ldots, L_s)$. This implies there exist $L'_1, \ldots, L'_s \in \mathbb{C}(n)[E_n]$ such that

$$L = \operatorname{lclm}(L_1, \ldots, L_s) = L'_1 \circ L_1 = \cdots = L'_s \circ L_s.$$

Then

$$LF = LF_1 + \cdots + LF_s = L'_1(L_1 F_1) + \cdots + L'_s(L_s F_s) = (E_k - 1)(L'_1 G_1 + \cdots + L'_s G_s).$$

Since $L'_1 G_1 + \cdots + L'_s G_s \in \mathbb{C}(n, k)$, $(L, L'_1 G_1 + \cdots + L'_s G_s)$ is a Z-pair for F.

Now let L^* be any telescoper for F. From the hypothesis, L^* is also a telescoper for each $F_i, 1 \le i \le s$. Since L_i is the minimal telescoper for F_i and since every rational function is a hypergeometric term, one can apply Lemma 1 and deduces that L^* is right divisible by $L_i, 1 \le i \le s$. Consequently, L^* is right divisible by $\operatorname{lclm}(L_1, \ldots, L_s)$. □

Notice that if $G_1, \ldots, G_s$ in (10) are hypergeometric terms that are not rational functions then $L'_1 G_1, \ldots, L'_s G_s$ are also hypergeometric terms but $L'_1 G_1 + \cdots + L'_s G_s$, in general, is not. But if $G_1, \ldots, G_s$ are hypergeometric terms such that $G_1 + \cdots + G_s$ is also a hypergeometric term, then $L'_1 G_1 + \cdots + L'_s G_s$ is also a hypergeometric term. In this case, Theorem 1 still holds if $F_1, \ldots, F_s$ are hypergeometric terms.

4. The Basic Case

We will show in this section how to compute directly a Z-pair (L, G) for MF where $M \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}]$, and

$$F = \frac{1}{(an + bk + c)^m},\ a, b \in \mathbb{Z},\ b > 0,\ c \in \mathbb{C},\ m \in \mathbb{N} \setminus \{0\},\ \gcd(a, b) = 1. \tag{11}$$

Lemma 2. *Let $F(n, k)$ have the form (11). Then there exists a* Z*-pair (L, G) for $F(n, k)$ where $L = E_n^b - 1$ is the minimal telescoper.*

Proof. Case 1 ($a \neq 0$)*:* Let

$$L = E_n^b - 1, M = E_k^a - 1 = (E_k - 1) \circ U, \text{ where } U = \begin{cases} \sum_{i=0}^{a-1} E_k^i & a > 0 \\ \sum_{i=1}^{-a} -E_k^{-i} & \text{otherwise} . \end{cases}$$

It is easy to check that

$$LF = MF = (E_k - 1)(UF), \; UF \in \mathbb{C}(n, k).$$

Now suppose there exists a Z-pair (L_1, G_1) for F such that $\operatorname{ord} L_1 < \operatorname{ord} L = b$. Without loss of generality we can assume that the coefficient of E_n^0 in L_1 is a non-zero element of $\mathbb{C}[n]$. Otherwise, choose a new Z-pair for F

$$\left(E_n^{-\lambda} \circ L_1, G_1(n - \lambda, k)\right)$$

where λ is the minimal positive integer such that the coefficient of E_n^λ in L_1 is not zero. Set

$$H = L_1 F = \frac{s(n,k)}{t(n,k)}, \; s(n,k), t(n,k) \in \mathbb{C}[n,k].$$

Since H is rational summable w.r.t. k, $t(n,k)$ has property **P2**, i.e., for the factor $an + bk + c$ of $t(n,k)$, there exists a non-zero integer h such that $an + b(k+h) + c$ is also a factor of $t(n,k)$. Since all the irreducible factors of $t(n,k)$ have the form $a(n+i) + bk + c$, $i = 0, 1, \ldots, \operatorname{ord} L_1$, this means $(an + b(k+h) + c) - (a(n+i) + bk + c)) = bh - ai$ is the zero polynomial for some i. Equivalently,

$$h = \frac{a}{b} i.$$

If $i = 0$, then $h = 0$. This contradicts the assumption on h. Otherwise, since $\gcd(a,b) = 1$, $b | i$. This is impossible since $0 < i < b$. Therefore, L is the minimal telescoper for F.

Note that when b is set to a high enough value, and we use $\mathcal{Z}$ to compute a Z-pair (L, G) for F, we simply waste resources (or even exhaust resources) trying to find a pair with $\operatorname{ord} L < b$ (see Example 3).

Case 2 ($a = 0$)*:* Since $\gcd(a,b) = 1, b = 1$. By following the same argument as that used in case 1, we have $(E_n - 1, 0)$ as a Z-pair for F. ☐

Lemma 3. *Let F be of the form (11), $M \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}]$. Let*

$$M = (E_k - 1) \circ U + V, \; V \in \mathbb{C}(n)[E_n]. \tag{12}$$

Then a telescoper for MF is also a telescoper for VF and vice versa.

Proof. Suppose there exist $L \in \mathbb{C}[n, E_n], G \in \mathbb{C}(n,k)$ such that

$$L(MF) = (E_k - 1)G. \tag{13}$$

The substitution of (12) into (13) yields $L(VF) = (E_k - 1)(G - L \circ UF)$. Hence, L is a telescoper for VF. On the other hand, let $L \in \mathbb{C}[n, E_n], G \in \mathbb{C}(n,k)$ be such that

$$L(VF) = (E_k - 1)G. \tag{14}$$

By substituting $V = M - (E_k - 1) \circ U$ into (14), we obtain $L(MF) = (E_k - 1)(G + L \circ UF)$. Hence, L is a telescoper for MF. ☐

From Lemma 3, in order to find the minimal telescoper for MF, it is sufficient to apply the left Euclidean division of M by $(E_k - 1)$ to obtain $V \in \mathbb{C}(n)[E_n]$ and then find the minimal telescoper for VF.

Lemma 4. *Let F be of the form (11), L_F the minimal telescoper for F, $V \in \mathbb{C}(n)[E_n]$ and*

$$\operatorname{lclm}(V, L_F) = L_1 \circ V = L_2 \circ L_F, \; L_1, L_2 \in \mathbb{C}(n)[E_n]. \tag{15}$$

Then L_1 is the minimal telescoper for VF.

Proof. From Lemma 2, the existence of a *Z*-pair (L_F, G_F) for F is guaranteed. Now we have

$$L_1(VF) = L_2(L_F F) = L_2 \circ (E_k - 1)G_F = (E_k - 1)(L_2 G_F).$$

Therefore, L_1 is a telescoper for VF.

Let L^* be any telescoper for VF, i.e., there exists $G^* \in \mathbb{C}(n,k)$ such that $L^*(VF) = (E_k - 1)G^*$, or equivalently, $(L^* \circ V)F = (E_k - 1)G^*$. Therefore, $(L^* \circ V, G^*)$ is a *Z*-pair for F. Since L_F is the minimal telescoper for F, from Lemma 1 one deduces that $L^* \circ V$ is right divisible by L_F. Additionally, $L^* \circ V$ is right divisible by V and $L_1 \circ V = \mathrm{lclm}(V, L_F)$. Therefore, $L^* \circ V$ is right divisible by $L_1 \circ V$. Consequently, L_1 is the minimal telescoper for VF. □

Theorem 2. *One can directly compute a* Z*-pair* (L, G) *for a rational function* MF*, where* F *is of the form (11) and* $M \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}]$*. Moreover, the telescoper* L *in the returned* Z*-pair is the minimal telescoper.*

Proof. The proof follows from Lemmas 2, 3 and 4. □

Example 1 Consider the rational function

$$(E_n^{10}E_k^{10} - 1)\frac{1}{-5n + 4k - 4}.$$

Here, $M = E_n^{10}E_k^{10} - 1, F = 1/(-5n + 4k - 4)$. Let (L_F, G_F) be a *Z*-pair for F. Applying Lemma 2 yields

$$(L_F, G_F) = \left(E_n^4 - 1, \left(\sum_{i=1}^{5} -E_k^{-i}\right) F\right).$$

Applying the left Euclidean division of M by $(E_k - 1)$ yields (12) where

$$U = E_n^{10} \circ \sum_{i=0}^{9} E_k^i,\ V = E_n^{10} - 1.$$

From Lemma 4, we obtain (15) where $L_1 = -E_n^2 - 1$, $L_2 = -E_n^8 - E_n^6 - E_n^4 - E_n^2 - 1$. Hence, a *Z-pair* (L_V, G_V) for VF is $(L_V, G_V) = (L_1, L_2 G_F)$. Finally, applying Lemma 3, we obtain a *Z*-pair (L, G) for MF where L is the minimal telescoper:

$$(L, G) = (L_V, G_V + L_V \circ UF),$$

which can be written explicitly as

$$\left(-E_n^2 - 1, \left((-E_n^8 - E_n^6 - E_n^4 - E_n^2 - 1) \circ \left(\sum_{i=1}^{5} -E_k^{-i}\right) + (-E_n^2 - 1) \circ \left(E_n^{10} \circ \sum_{i=0}^{9} E_k^i\right)\right) \frac{1}{-5n + 4k - 4}\right).$$

5. The Generic Case

Lemma 5. *Let* $F(n,k) \in \mathbb{C}(n,k)$*, and*

$$F(n,k) = \sum_{i=1}^{t}\sum_{j=1}^{m_i} \frac{r_{ij}(n)}{(a_i n + b_i k + c_i)^j},\ r_{ij}(n) \in \mathbb{C}(n),\ a_i, b_i \in \mathbb{Z}, b_i > 0, c_i \in \mathbb{C},\ \gcd(a_i, b_i) = 1. \tag{16}$$

Then $F(n,k)$ *can be represented in the form*

$$M_1 F_1 + \cdots + M_s F_s, \tag{17}$$

where $M_i \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}]$*, and* $F_i = 1/(a_i n + b_i k + c_i)^{m_i}$ *is such that*

$$a_i, b_i \in \mathbb{Z}, b_i > 0, c_i \in \mathbb{C}, \gcd(a_i, b_i) = 1, \tag{18}$$

and for all $i \neq j$*, at least one of the following four relations is not valid:*

$$m_i = m_j, a_i = a_j, b_i = b_j, c_i - c_j \in \mathbb{Z} \setminus \{0\}. \tag{19}$$

Proof. Let R_1, R_2 be two simple fractions in (16). Define $R_1 \sim R_2$ iff the four relations in (19) hold. $\sim$ is an equivalence relation on the set of simple fractions in (16).

For any pair of simple fractions R_1 and R_2 in the same class, i.e., R_1 and R_2 can be written as

$$R_1 = \frac{r_1(n)}{(an+bk+c_1)^m}, R_2 = \frac{r_2(n)}{(an+bk+c_2)^m},\ r_1(n), r_2(n) \in \mathbb{C}(n), c_1 - c_2 \in \mathbb{Z} \setminus \{0\}, \tag{20}$$

consider the relation

$$a(n+i) + b(k+j) + c_1 = an + bk + c_2,\ i \in \mathbb{N}, j \in \mathbb{Z}. \tag{21}$$

This implies

$$ai + bj = c_2 - c_1. \tag{22}$$

Case 1 ($a \neq 0$): Since $c_2 - c_1 \in \mathbb{Z} \setminus \{0\}$ and $\gcd(a, b) = 1$, the diophantine equation (22) is consistent and has an infinite number of solutions. Therefore, there exist $i \in \mathbb{N} \setminus \{0\}, j \in \mathbb{Z}$ such that relation (21) holds. As a consequence, one can combine R_1 and R_2 into

$$R_1 + R_2 = M \frac{1}{(an+bk+c_1)^m} \text{ where } M = r_1(n) + r_2(n) E_n^i E_k^j \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}].$$

Case 2 ($a = 0$): Since $\gcd(a, b) = 1, b = 1$. Consequently,

$$R_1 + R_2 = M \frac{1}{(k+c_1)^m} \text{ where } M = r_1(n) + r_2(n) E_k^{c_2 - c_1} \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}].$$

By using induction on the number of elements in each equivalence class, the sum of the elements in each class can be represented as

$$M_i \frac{1}{(a_i n + b_i k + c_i)^{m_i}},\ M_i \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}].$$

□

Lemma 6. *Let $F(n, k)$ have the form (17) where $M_i \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}]$, F_i is such that (18) takes place and if $i \neq j$ then at least one of the four relations in (19) is not valid. Let L be a telescoper for $F(n, k)$. Then L is a telescoper for each $M_i F_i, 1 \leq i \leq s$.*

Proof. Let $L \in \mathbb{C}[n, E_n]$ be a telescoper for $F(n, k)$. Consider

$$M_i F_i = M_i \frac{1}{(a_i n + b_i k + c_i)^{m_i}},\ 1 \leq i \leq s.$$

It is easy to see that the application of $L \in \mathbb{C}[n, E_n]$ to a simple fraction of $M_i F_i$ yields a sum of simple fractions such that each of them is in the same equivalence class (w.r.t. the relation $\sim$ as defined in the proof of Lemma 5) as that of F_i.

Let R_1, R_2 be any two simple fractions of $LF(n, k)$. Define $R_1 \sim' R_2$ iff in addition to the four relations in (19), i.e., R_1 and R_2 can be written as

$$R_1 = \frac{1}{(a_i n + b_i k + c_i)^{m_i}}, R_2 = \frac{1}{(a_i n + b_i k + c_j)^{m_i}},\ c_i - c_j \in \mathbb{Z} \setminus \{0\}, \tag{23}$$

the relation $b_i | (c_i - c_j)$ also holds. It is easy to check that $\sim'$ is an equivalence relation, and if $R_1 \sim' R_2$ then $R_1 \sim R_2$. Consequently, it is impossible that $L \circ M_i F_i$ and $L \circ M_j F_j$ both have simple fractions from the same class w.r.t. $\sim'$ for $i \neq j$.

Considering any $L \circ M_i F_i$, we distribute the corresponding simple fractions to the classes w.r.t. $\sim'$. For every pair of simple fractions R_1, R_2 as defined in (23) in each class, since b_i divides $(c_i - c_j)$ and $c_i \neq c_j$, there exists a non-zero integer h such that $c_i - c_j = b_i h$. Equivalently,

$$a_i n + b_i (k + h) + c_j = a_i n + b_i k + c_i.$$

This means the irreducible factors of the denominator of the sum of simple fractions in each class are shift-equivalent. Since $LF(n, k)$ is rational summable w.r.t. k, from the necessary and sufficient condition as specified in (4), Subsection 2.1, the sum of all fractions from such a class is rational summable. This implies that $L \circ M_i F_i$ is rational summable. □

Theorem 3. *For any rational function*

$$F(n,k)=\sum_{i=1}^{t}\sum_{j=1}^{m_i}\frac{r_{ij}(n)}{(a_i n+b_i k+c_i)^j},\ r_{ij}(n)\in\mathbb{C}(n),\ a_i,b_i\in\mathbb{Z},b_i>0,c_i\in\mathbb{C},\gcd(a_i,b_i)=1, \tag{24}$$

one can directly compute a Z*-pair* (L,G) *for* $F(n,k)$ *where* L *is the minimal telescoper.*

Proof. Lemma 5 allows one to rewrite $F(n,k)$ in the form (17). Theorem 2 allows one to compute a *Z*-pair (L_i,G_i) for each term M_iF_i, $1\le i\le s$, where L_i is the minimal telescoper. One then uses Theorem 1 to compute a *Z*-pair (L,G) for $F(n,k)$ where $L=\operatorname{lclm}(L_1,\dots,L_s)$. Lemma 6 and Theorem 1 guarantee that L is indeed the minimal telescoper for $F(n,k)$. □

6. A Direct Algorithm to Construct Z-pairs for Rational Functions

For a given rational function $T(n,k)$, a direct algorithm to construct a *Z*-pair for $T(n,k)$ can be decomposed into two steps. In the first step, check for the existence of a *Z*-pair and if the existence is guaranteed, rewrite $F(n,k)$ in (3) into the form where each factor of the denominator of $F(n,k)$ has the form (5). In the second step, apply the results shown in Sections 3, 4 and 5 to obtain a *Z*-pair (L_1,G_1) for $F(n,k)$. Then a *Z*-pair (L,G) for the given rational function $T(n,k)$ where L is the minimal telescoper is $(L_1, L_1S(n,k)+G_1(n,k))$ where $S(n,k)$ is defined in (3).

An algorithm to perform the first step is presented in [4]. Its correctness is verified in [5], and we will make a minor modification to this algorithm to complete the step. Represent $T(n,k)$ in the form (3) and rewrite $F(n,k)$ as the quotient $f(n,k)/g(n,k)$ of two relatively prime polynomials from $\mathbb{C}[n,k]$. First extract from $g(n,k)$ the maximal factors $v_1(n)\in\mathbb{C}[n]$ and $v_2(k)\in\mathbb{C}[k]$. Set $w(n,k)=g(n,k)/(v_1(n)v_2(k))\in\mathbb{C}[n,k]$. Now it remains to investigate whether $w(n,k)$ can be decomposed into factors of the form

$$k+cn+\gamma,\quad c\in\mathbb{Q}\setminus\{0\},\ \gamma\in\mathbb{C} \tag{25}$$

or not. Substitute $k-cn$ into $w(n,k)$ for k (this gives us a polynomial $\tilde{w}(c,n,k)$) and compute all nonzero rational values of c such that $\tilde{w}(c,n,k)$ has a non-constant factor from $\mathbb{C}[k]$. To attain this goal we represent $\tilde{w}(c,n,k)$ as a polynomial in n with coefficients in $\mathbb{C}[c,k]$ and find all nonzero rational values of c such that these coefficients have a non-constant greatest common divisor (a polynomial from $\mathbb{C}[k]$). This can be achieved by using resultant or subresultant approaches [2]. Let $c_0,\dots,c_m$ be the computed non-zero rational values and $p_0(k),\dots,p_m(k)$ be the corresponding polynomials from $\mathbb{C}[k]$. Set $\delta_i=\deg p_i(k)$. A *Z*-pair for $T(n,k)$ exists iff

$$\delta_0+\cdots+\delta_m=\deg_k w(n,k). \tag{26}$$

If relation (26) holds, we proceed to find the irreducible factors of the form (5) of $g(n,k)$. First, rewrite $v_2(k)\in\mathbb{C}[k]$ in the form $(k-\alpha_1)\cdots(k-\alpha_s)$, $\alpha_i\in\mathbb{C}$. Then for each factor $p_i(k)\in\mathbb{C}[k]$, rewrite it in the form

$$(k-\gamma_{i_1})\cdots(k-\gamma_{i_t}),\ \gamma_{i_j}\in\mathbb{C}, \tag{27}$$

and substitute $k+c_in$ for k into (27) to obtain the factors of the form (25). $F(n,k)$ now can be written in the desired form.

In the second step, we apply the partial fraction decomposition w.r.t. k to $F(n,k)$. Note that the denominator of $F(n,k)$ is already in the desired factored form, and no factorization needs to be done. This gives a representation of $F(n,k)$ in the form (24). We then use Theorem 3 to construct a *Z*-pair (L_1,G_1) for $F(n,k)$ where L_1 is the minimal telescoper.

Example 2 Consider the rational function

$$T(n,k)=\frac{25n^2+100n-40nk+3k+16k^2+6}{(n+11k-5)(-5n+4k-14)(-5n+4k-4)}.$$

(For clarity we keep the denominator of $T(n,k)$ in the desired factored form.)

In the first step, applying an algorithm to solve the decomposition problem yields $T(n,k) = F(n,k)$. It is easy to see from the denominator of $F(n,k)$ that a Z-pair for $T(n,k)$ exists.

In the second step, applying Lemma 5 yields

$$F(n,k) = M_1F_1 + M_2F_2 \text{ where}$$
$$M_1 = 1, F_1 = \frac{1}{n+11k-5}, M_2 = E_n^{10}E_k^{10} - 1, F_2 = \frac{1}{-5n+4k-4}.$$

From Lemma 2, we obtain a Z-pair (L_1, G_1) for M_1F_1 :

$$(L_1, G_1) = \left(E_n^{11} - 1, \frac{1}{n+11k-5}\right).$$

From Example 1, we obtain a Z-pair (L_2, G_2) for M_2F_2 :

$$(L_2, G_2) = \left(-E_n^2 - 1, \left(\left(-E_n^8 - E_n^6 - E_n^4 - E_n^2 - 1\right) \circ \sum_{i=1}^{5} -E_k^{-i} + (-E_n^2 - 1) \circ E_n^{10} \sum_{i=0}^{9} E_k^i\right) F_2\right).$$

From Theorem 1, set $L = \text{lclm}(L_1, L_2) = L_1' \circ L_1 = L_2' \circ L_2$. We obtain $L = E_n^{13} + E_n^{11} - E_n^2 - 1$, $L_1' = E_n^2 + 1, L_2' = -E_n^{11} + 1$, and a Z-pair for $F(n,k)$ which is also a Z-pair for $T(n,k)$ is

$$(L, G) = \left(E_n^{13} + E_n^{11} - E_n^2 - 1, L_1'G_1 + L_2'G_2\right).$$

7. Implementation

The algorithm is implemented in Maple 6 [7]. The main procedure, named `ZpairDirect`, has the following calling sequence

```
ZpairDirect(T,n,k,E_n,E_k,dE_k,Z_pair);
```

where `T` is a rational function in `n` and `k`, and `E_n, E_k, dE_k` denote the operators E_n, E_k, E_k^{-1}, respectively. If a Z-pair for T does not exist, then `ZpairDirect` returns `false`; otherwise, it returns `true`. In the latter case, if the optional argument `Z_pair` is given, the computed Z-pair (L,G) is assigned to `Z_pair`. The auxiliary function `ZpairVerify` with the calling sequence

```
ZpairVerify(T,[Z_pair],E_n,n,E_k,k);
```

where `Z_pair` is the result from `ZpairDirect` verifies that $LT = (E_k - 1)G$.

Note that there exist different Maple implementations of the original $\mathcal{Z}$ such as `zeil` in the EKHAD package [10], and `sumrecursion` in the `sumtools` package. A Mathematica implementation is presented in [9]. In our experimentation, we use the implementation of $\mathcal{Z}$ and the modified version of $\mathcal{Z}$ (which we call $\mathcal{Z}$-modified) as described in [4].

Example 3 Consider the rational function

```
> T := 1/(3*n+20*k+2)^3;
```

$$T := \frac{1}{(3n+20k+2)^3}$$

We invoke `ZpairDirect` which first checks if a Z-pair for T exists; and if it does, then returns a pair.

```
> ZpairDirect(T,n,k,E_n,E_k,dE_k,Z_pair);
```

$$true$$

The computed Z-pair for T is

```
> (L,G) := Z_pair;
```

$$(L, G) = E_n^{20} - 1, \frac{1}{(3n+20k+2)^3} + \frac{1}{(3n+20k+22)^3} + \frac{1}{(3n+20k+42)^3}$$

```
> ZpairVerify(T,[L,G],E_n,n,E_k,k);
```

$$true$$

The result from `ZpairVerify` shows that the computed Z-pair from `ZpairDirect` is correct. It takes `ZpairDirect` 0.1 seconds and 600,416 bytes to verify that a Z-pair exists and to compute one [1].

```
> Z_pair := Zeilberger(T,n,k,E_n);
> Error, No recurrence of order 6 was found
```

The invocation of the original $\mathcal{Z}$ as shown above returns the inconclusive answer in 4.1 seconds, using 24,188,040 bytes. Since the terminating condition that allows a hypergeometric term to have a Z-pair is unknown, $\mathcal{Z}$ uses an item-by-item examination on the order ρ of L starting with $\rho = 0$ until it either succeeds in finding a Z-pair or reaches a maximum value of ρ which can be set in advance. This value depends on a given hypergeometric term. In general it cannot be set high enough.

We now use $\mathcal{Z}$-modified (see [4]) on the given rational function.

```
> is_Z_applicable(T,E_n,n,k,'Zpair');
```

$$true$$

```
> Zpair[1];
```

$$E_n^{20} - 1$$

The program verifies that there does exist a Z-pair for T. Consequently, it does not need to set an upper bound for ρ. However, since it still uses an item-by-item examination technique, it wastes resources trying to compute a Z-pair with $\operatorname{ord} L < 20$. It takes the program 4889.9 seconds and 7,553,018,568 bytes to return the desired answer.

Example 4 Consider the rational function

```
> T := 1/expand((k^3-5*n-2)*(k^2+n+3));
```

$$T := \frac{1}{k^5 + k^3 n + 3k^3 - 5nk^2 - 2k^2 - 5n^2 - 17n - 6}$$

```
> ZpairDirect(T,n,k,E_n,E_k,dE_k);
```

$$false$$

In this example $T(n,k) = F(n,k)$. The denominator of F can be written in the form $(k-5n-3)(k^2+n+3)$ and hence there does not exist a Z-pair for T. It takes `ZpairDirect` and `is_Z_applicable` approximately 0.2 seconds and 468,196 bytes to return the desired answer. The original $\mathcal{Z}$, however, does not determine if a Z-pair for T exists. It tries to compute one and returns the inconclusive answer `Error, No recurrence of order 6 was found` in 9.7 seconds, using 39,465,248 bytes.

```
> Z_pair := Zeilberger(T,n,k,E_n);
> Error, No recurrence of order 6 was found
```

Example 5 In this example, we randomly generate three set of tests. Each rational function (where the numerator and the denominator are in expanded forms) in the test is of the form (17). We run `ZpairDirect` (Z_1), `is_Z_applicable` (Z_2), `Zeilberger` (Z_3) on these tests, and collect resource requirements. We also enforce a limit of 1000 seconds on each input rational function in the tests. Note that we only record the time and space requirements for the tests that run under this time limit.

The first set of tests (S_1) consists of 20 rational functions each of which is of the form $T = M_1F_1 + M_2F_2$ where F_l is of the form (11), and

$$M_l = (r_{l_1}(n)/r_{l_2}(n))E_n^iE_k^j,\ i \in \{1,2,3\}, j \in \{-3,-2,-1,0,1,2,3\}, \deg r_{l_s}(n) = 1, 1 \le s \le 2.$$

For instance

$$T = \left(\frac{n}{n+1}E_k^{-2}\right)\frac{1}{(-3n+4k-1)^2} + \left(\frac{n+1}{2n-3}E_nE_k\right)\frac{1}{-4n+3k-2}.$$

Row 1 in Table 1 shows the time and space requirements to run S_1 using Z_1, Z_2, Z_3. All the tests are completed using Z_1. As for Z_2 and Z_3, 7 tests fail the time limit (1000 seconds) for each of them.

[1] All the reported timings were obtained on a 400Mhz SUN SPARC SOLARIS with 1Gb RAM.

The second set of tests (S_2) consists of 20 rational functions each of which is of the form $T = MF$ where F is of the form (11), and

$$M = (r_1(n)/r_2(n))E_n^i E_k^{-j} + (r_3(n)/r_4(n))E_n^u E_k^v, i, j, u, v \in \mathbb{N}, i + j = u + v = 2,$$
$$\deg r_1(n) = \deg r_3(n) = 1, \deg r_2(n) = \deg r_4(n) = 2.$$

All the tests are completed using Z_1. As for Z_2 and Z_3, 3 tests fail the time limit for each of them.

The third set of tests (S_3) consists of 20 rational functions. They are of the same format as those in S_1 except that each input rational function consists of three terms, i.e., $T = M_1F_1 + M_2F_2 + M_3F_3$, instead of two. All the tests are completed using Z_1, and all the tests fail the time limit using Z_2 and Z_3.

TABLE 1. Time and space requirements of Z_1, Z_2 and Z_3.

	Completed			Timing (seconds)			Memory (bytes)		
	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3	Z_1	Z_2	Z_3
S_1	20	13	13	19.0	583.5	563.9	9,4347,084	2,460,521,648	2,583,044,740
S_2	20	17	17	22.4	1239.1	2208.7	108,432,868	3,737,465,860	6,490,051,300
S_3	20	0	0	2210.3	–	–	9,442,235,992	–	–

ACKNOWLEDGEMENTS

The author is very grateful to Professor S.A. Abramov of the Computer Center of the Russian Academy of Science for his valuable advice and comments. He also would like to thank Professor K.O. Geddes of the University of Waterloo for his encouragement and support.

REFERENCES

[1] S.A. Abramov, Rational component of the solutions of a first-order linear recurrence relation with a rational right-hand side. *USSR Comput. Math. Phys.* Transl. from *Zh. vychisl. mat. mat. fyz.* **14** 1035–1039, 1975.

[2] S.A. Abramov, K.Yu. Kvashenko, On the greatest common divisor of polynomials which depend on a parameter. *Proceedings ISSAC'93*, 1993.

[3] S.A. Abramov, Indefinite sums of rational functions. *Proceedings ISSAC'95*, 303–308, 1995.

[4] S.A. Abramov, H.Q. Le, Applicability of Zeilberger's algorithm to rational functions. *Proceedings FPSAC'2000*, Springer–Verlag LNCS, 2000, 91–102.

[5] S.A. Abramov, H.Q. Le, A criterion for the applicability of Zeilberger's algorithm to rational functions. Submitted to *Discrete Math.*

[6] R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics.* Addison Wesley, Reading, MA, 1994, Second Edition.

[7] M.B. Monagan, K.O. Geddes, K.M. Heal, G. Labahn, S.M. Vorkoetter, J. McCarron, *Maple 6 Programming Guide.* 2000, ISBN 1-894511-01-8.

[8] P. Paule, Greatest Factorial Factorization and Symbolic Summation. *J. Symbolic Comput.* **20**, 1995, 235–268.

[9] P. Paule, M. Schorn, A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. *J. Symb. Comput.* **20**, 1995, 673–698.

[10] M. Petkovšek, H. Wilf, D. Zeilberger, *A=B.* A.K. Peters, Wellesley, Massachusetts, 1996.

[11] R. Pirastu, V. Strehl, Rational summation and Gosper-Petkovšek representation. *J. Symb. Comput.* **20**, 1995, 617–635.

[12] H. Wilf, D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Inventiones Mathematicae* **108**, 1992, 575–633.

[13] D. Zeilberger, The method of creative telescoping. J. Symb. Comput. **11**, 1991, 195–204.

E-mail address: `hqle@scg.math.uwaterloo.ca`

SYMBOLIC COMPUTATION GROUP, DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF WATERLOO, WATERLOO, N2L 3G1, CANADA