COUNTING FACES IN THE EXTENDED SHI ARRANGEMENT

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ABSTRACT. We define the *level* of a subset X of Euclidean space to be the dimension of the smallest subspace such that the distance between each element of X and the subspace is bounded. We prove that the number of faces in the n-dimensional extended Shi arrangement $\widehat{\mathcal{A}}_n^r$ having codimension k and level m is given by $m \cdot \binom{n}{k} \cdot \Delta_r^k \Delta^{m-1} x^{n-1} |_{x=rn-1}$, where Δ is the difference operator and Δ_r is the difference operator of step r, that is, $\Delta_r p(x) = p(x) - p(x-r)$. This generalizes a result of Athanasiadis which counts the number of faces of different dimensions from the Shi arrangement $\widehat{\mathcal{A}}_n$. The proof relies on extending Athanasiadis' result to $\widehat{\mathcal{A}}_n^r$ and applying a multi-variated Abel identity.

RÉSUMÉ. Nous définissons le *niveau* d'un sous-ensemble X d'un espace euclidien comme la dimension du plus petit sous-espace tel que la distance entre chaque element de X et du sous-espace est borné. Nous prouvons que le nombre des faces d'un arrangement de Shi étendu $\widehat{\mathcal{A}}_n^r$ de codimension k et de niveau m est donné par $m \cdot \binom{n}{k} \cdot \Delta_r^k \Delta^{m-1} x^{n-1} |_{x=rn-1}$, où Δ est l'opérateur de différence et Δ_r est l'opérateur de différence d'étape r, c.à.d. que $\Delta_r p(x) = p(x) - p(x-r)$. Ceci généralise un résultat d'Athanasiadis qui compte le nombre de faces de dimensions différentes de l'arrangement de Shi $\widehat{\mathcal{A}}_n$. La démonstration s'appuie sur l'extension du résultat d'Athanasiadis à $\widehat{\mathcal{A}}_n^r$ et sur l'application d'une identité d'Abel multi-variée.

1. Introduction

The *n-dimensional extended Shi arrangement* $\widehat{\mathcal{A}}_n^r$ is defined as

$$x_i - x_j = -r+1, \ldots, r \quad \text{for} \quad 1 \leq i < j \leq n,$$

where r is a given positive integer [17]. When $r = 1$ we have an important special case, namely, the Shi arrangement $\widehat{\mathcal{A}}_n$:

$$x_i - x_j = 0, 1 \quad \text{for} \quad 1 \leq i < j \leq n.$$

Ever since Shi proved that this arrangement has $(n+1)^{n-1}$ number of regions (chambers) [14], there has been a large interest in studying it. Headley [10] computed the characteristic polynomial of the Shi arrangement, namely $\chi(t) = t \cdot (t-n)^{n-1}$. This proof was improved upon by Stanley [16]. Athanasiadis gave a combinatorial proof of this characteristic polynomial [1]. Moreover, he enumerated the faces of the Shi arrangement $\widehat{\mathcal{A}}_n$ according to their dimension. In order to state his result more compactly, let Δ denote the difference operator $\Delta p(x) = p(x) - p(x-1)$.

Theorem 1.1 (Athanasiadis). *The number of faces of codimension k in the n-dimensional Shi arrangement $\widehat{\mathcal{A}}_n$ is given by $\binom{n}{k} \cdot \Delta^k x^{n-1} |_{x=n+1}$.*

We will extend this result to the extended Shi arrangement $\widehat{\mathcal{A}}_n^r$; see Theorem 2.2. Furthermore, we obtain a more refined enumeration of the faces of the extended Shi arrangement, and hence, of the Shi arrangement.

Consider the 2-dimensional line arrangement in Figure 1. It has 4 bounded regions and 12 unbounded regions. These 12 unbounded regions fall into two classes, namely, six of them are unbounded in two directions and the other six are unbounded in only one direction. To make this distinction clear, we define the *level* of a subset of n-dimensional Euclidean space as follows.

Definition 1.2. *The* level *of a subset X of n-dimensional Euclidean space is the smallest non-negative integer m such that there exist a subspace V of dimension m and a positive real number r*

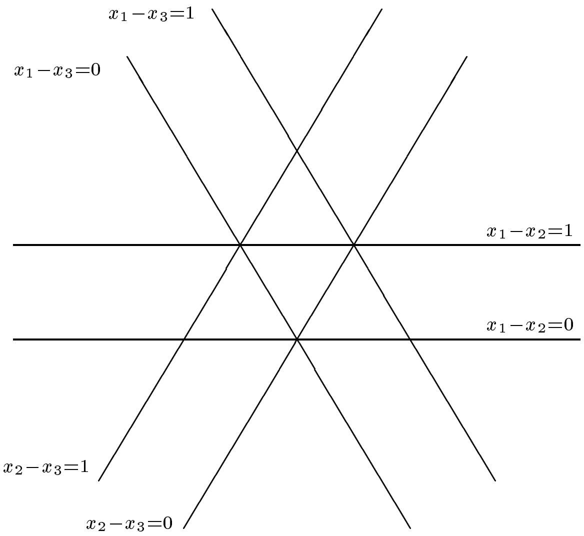

FIGURE 1. The 2-dimensional representation of the 3-dimensional Shi arrangement $\widehat{\mathcal{A}}_3$.

such that

$$X \subseteq \{\mathbf{x} \;:\; d(\mathbf{x}, V) \leq r\}.$$

That is, every element of X is at most distance r from the subspace V.

Informally speaking, the set X has level m if the set viewed from far away looks m-dimensional. Hence in Figure 1 there are 4 regions of level 0, 6 regions of level 1 and 6 regions of level 2.

It is natural to ask how many faces there are in the extended Shi arrangement $\widehat{\mathcal{A}}_n^r$ having codimension k and level m. The surprising answer follows, where we denote the difference operator of step r by Δ_r, that is, $\Delta_r p(x) = p(x) - p(x-r)$.

Theorem 1.3. *The number of faces of the n-dimensional extended Shi arrangement $\widehat{\mathcal{A}}_n^r$ of codimension k and level m is given by $m \cdot \binom{n}{k} \cdot \Delta_r^k \Delta^{m-1} x^{n-1} |_{x=rn-1}$.*

The first ingredient of the proof of Theorem 1.3 is to generalize Athanasiadis' enumeration of the number of faces of the Shi arrangement $\widehat{\mathcal{A}}_n$ to the extended Shi arrangement $\widehat{\mathcal{A}}_n^r$. In the same spirit as Zaslavsky [18] we not only obtain the number of faces of codimension k, but also the number of bounded faces of codimension k. Here, in the extended Shi arrangement a bounded face denotes a face of level one. The second ingredient is a multi-variated Abel identity essentially due to Françon [8]. We sketch a quick proof of this identity in Section 4 using tree enumeration. In the last section we state two corollaries to our main result, describe the connection with parking functions and pose questions for further research.

2. COUNTING THE FACES OF THE EXTENDED SHI ARRANGEMENT

Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be a hyperplane arrangement in $\mathbb{R}^n$ where $\mathbf{n}_i$ is the normal vector to the hyperplane H_i. Let W be the largest subspace of $\mathbb{R}^n$ perpendicular to the normal vectors $\mathbf{n}_1, \ldots, \mathbf{n}_m$. Observe that the arrangement $\mathcal{H}$ is invariant under translation in the direction $\mathbf{w}$, where $\mathbf{w}$ belongs to W. Hence we call the dimension of W the *linear degree of freedom* of the arrangement $\mathcal{H}$. Observe that the arrangement $\mathcal{H}$ is naturally a hyperplane arrangement in the quotient space $\mathbb{R}^n/W$.

Let $\mathcal{L}(\mathcal{H})$ denote the intersection lattice of the hyperplane arrangement $\mathcal{H}$, that is, the lattice of all possible intersections of the hyperplanes $H_1, \ldots, H_m$. The intersection lattice is ordered by reverse inclusion, hence the minimal element $\hat{0}$ is the whole space $\mathbb{R}^n$. The *characteristic polynomial* of the hyperplane arrangement $\mathcal{H}$ is defined by

$$\chi(t) = \sum_{\substack{X \in \mathcal{L}(\mathcal{H}) \\ X \neq \emptyset}} \mu(\hat{0}, X) \cdot t^{\dim(X)},$$

where μ denotes the Möbius function of the intersection lattice.

Call a face of the arrangement $\mathcal{H}$ bounded if the corresponding face in the arrangement in the quotient space $\mathbb{R}^n/W$ is bounded. That is, a face is bounded if it is bounded in all the directions in the span of normal vectors $\mathbf{n}_i$. By Zaslavsky's seminal result [18] the number of regions of $\mathcal{H}$ is given by $(-1)^n \cdot \chi(-1)$ and the number of bounded regions is $(-1)^{n-l} \cdot \chi(1)$, where l is the linear degree of freedom of $\mathcal{H}$ and $\chi(t)$ is the characteristic polynomial of the arrangement $\mathcal{H}$.

Let f_k be the number of k-dimensional faces of the hyperplane arrangement $\mathcal{H}$. Similarly, let b_k denote the number of bounded faces of dimension k.

We call the arrangement $\mathcal{H}$ *rational* if the defining equation of each hyperplane H_i has rational coefficients. From now on the discussion applies to rational arrangements. Hence for a field $\mathbf{k}$ with characteristic 0 or with large enough characteristic, we can view $\mathcal{H}$ as an arrangement in $\mathbf{k}^n$. Let $V_k(\mathcal{H}, \mathbf{k})$ denote the set of points $\mathbf{x}$ in $\mathbf{k}^n$ such that $\mathbf{x}$ belongs to some k-dimensional face of $\mathcal{H}$. That is, $V_k(\mathcal{H}, \mathbf{k})$ is the union of the k-dimensional faces. An extension of the discussion in [1, Section 6] implies the following result.

Theorem 2.1. *Let $\mathcal{H}$ be rational hyperplane arrangement with l as the linear degree of freedom. For large enough prime numbers q the function $q \longmapsto |V_k(\mathcal{H}, \mathbb{Z}_q)|$ is a polynomial in q. Moreover, evaluating this polynomial at $q = -1$ and $q = 1$ one obtains $(-1)^k \cdot f_k$, respectively, $(-1)^{k-l} \cdot b_k$.*

For a quick proof of the $q = -1$ part of this theorem, apply the valuation ν from [6] to the set $V_k(\mathcal{H}, \mathbb{R})$. By setting $t = -1$ the valuation ν reduces to the Euler characteristic and the Euler characteristic of $V_k(\mathcal{H}, \mathbb{R})$ is $(-1)^k \cdot f_k$. Similarly, a quick proof of the $q = 1$ part is obtained by using the bounded Euler characteristic; see [4, 7].

We now turn our attention to the extended Shi arrangement $\widehat{\mathcal{A}}_n^r$. The linear freedom is 1, since the subspace W is spanned by the vector $(1, \ldots, 1)$. Our main theorem of this section is the following result.

Theorem 2.2. *The number of faces of codimension k in the n-dimensional extended Shi arrangement $\widehat{\mathcal{A}}_n^r$ is given by $\binom{n}{k} \cdot \Delta_r^k x^{n-1} |_{x=rn+1}$. Similarly, the number of bounded faces of codimension k in $\widehat{\mathcal{A}}_n^r$ is given by $\binom{n}{k} \cdot \Delta_r^k x^{n-1} |_{x=rn-1}$.*

Let $P(q)$ denote the cardinality of $V_{n-k}(\widehat{\mathcal{A}}_n^r, \mathbb{Z}_q)$. The argument for determining the polynomial $P(q)$ follows the method of the proof of Theorem 1.1; see [1, Theorem 6.5]. For a point $(x_1, \ldots, x_n)$ define a partition τ on the set $\{1, \ldots, n\}$ by letting i and j belong to the same block of τ if $i < j$ and $-r + 1 \leq x_i - x_j \leq r$. The number of blocks of the partition τ is equal to the dimension of the face containing the vector $(x_1, \ldots, x_n)$. Hence $V_{n-k}(\widehat{\mathcal{A}}_n^r, \mathbb{Z}_q)$ consists of those points that correspond to partitions with $n - k$ blocks.

For each congruence class modulo q we have a box and these q boxes are placed in a cyclic order. View a point $(x_1, \ldots, x_n)$ in the vector space $\mathbb{Z}_q^n$ as placement of the integers 1 through n into the q boxes, where the element i is placed into box x_i. When several integers land in the same box, order them increasingly. Thus when removing the boxes completely leaving behind their contents, we obtain a cyclic permutation σ of the elements 1 through n.

Observe that the elements of a block of the partition τ appears as a string of consecutive elements of the cyclic permutation σ. Hence σ is partitioned into $n - k$ partial permutations $\pi_1, \ldots, \pi_{n-k}$, where π_j is a partial permutation on the block B_j, that is, π_j is a linear order on the block B_j.

We will now determine how many points there are in $V_{n-k}(\widehat{\mathcal{A}}_n^r, \mathbb{Z}_q)$ that correspond to a certain cyclic permutation σ. Crucial to the proof is the following observation. Let i and j be adjacent entries in the cyclic permutation σ.

- Assume that $i > j$ forms a descent. If there are at most $r-1$ boxes between i and j then i and j are in the same block, otherwise not.
- Assume that $i < j$ is an ascent and that i and j do not belong to the same box. If there are at most $r-2$ boxes between i and j then the elements i and j are in the same block, otherwise not.

Lemma 2.3. *Let π be a partial permutation on h elements such that the elements of π form a block. Then the generating polynomial for the number of possible boxes between the elements of π is given by*

$$\left(\frac{1-z^r}{1-z}\right)^{h-1} \cdot z^{1+\operatorname{des}(\pi)}.$$

Proof. Begin to insert each entry of π into a distinct box. This is counted by the term z^h. If $\pi(i) > \pi(i+1)$ is a descent in π then the number of boxes between the elements $\pi(i)$ and $\pi(i+1)$ can vary between 0 and $r-1$. Thus we have the factor $1+z+\cdots+z^{r-1}$. If $\pi(i) < \pi(i+1)$ is an ascent, the number of boxes varies between 0 and $r-2$, hence giving the factor $1+z+\cdots+z^{r-2}$. However, there is the extra case of letting $\pi(i)$ and $\pi(i+1)$ belong to the same box. This means removing one box, counted by z^{-1}. Hence we have the factor $z^{-1}+1+z+\cdots+z^{r-2} = (1+z+\cdots+z^{r-1}) \cdot z^{-1}$. Thus the generating polynomial is

$$z^h \cdot (1+z+\cdots+z^{r-1})^{h-1} \cdot z^{-(h-1-\operatorname{des}(\pi))},$$

which is equivalent to the lemma. □

Lemma 2.4. *Let σ be a cyclic permutation on n elements divided into $n-k$ partial permutations $\pi_1, \ldots, \pi_{n-k}$. Then the generating polynomial for the number of boxes that can be inserted is given by*

$$(1-z^r)^k \cdot z^{r \cdot (n-k)} \cdot \frac{z^{\operatorname{des}(\sigma)}}{(1-z)^n}.$$

Proof. Let a be the number of cuts occurring at an ascent of σ and b the number of cuts at a descent. Hence $a+b = n-k$. At an ascent cut the generating function for the number of boxes is $z^{r-1}+z^r+\cdots = z^{-1} \cdot (z^r+z^{r+1}+\cdots)$. Similarly, at a descent cut we have $z^r+z^{r+1}+\cdots$. Hence the generating function for inserting boxes in the $n-k$ cuts is

$$z^{-a} \cdot (z^r+z^{r+1}+\cdots)^{n-k} = z^{-a} \cdot z^{r \cdot (n-k)} \cdot \frac{1}{(1-z)^{n-k}}. \tag{2.1}$$

Assume that π_j is a partial permutation on h_j elements. By Lemma 2.3 the generating function for inserting boxes in the partial permutations $\pi_1, \ldots, \pi_{n-k}$ is

$$\prod_{j=1}^{n-k} \left(\frac{1-z^r}{1-z}\right)^{h_j-1} \cdot z^{1+\operatorname{des}(\pi_j)} = (1-z^r)^k \cdot \frac{z^{a+\operatorname{des}(\sigma)}}{(1-z)^k}, \tag{2.2}$$

since $\sum_{j=1}^{n-k}(1+\operatorname{des}(\pi_j)) = a + \operatorname{des}(\sigma)$. By multiplying equations (2.1) and (2.2), the lemma follows. □

Proof of Theorem 2.2. Recall the identity $\sum_\sigma z^{\operatorname{des}(\sigma)}/(1-z)^n = \sum_{j \geq 0} j^{n-1} \cdot z^j$ where the sum ranges over all cyclic permutations σ. Summing the expression over all cyclic permutations σ in Lemma 2.4 we obtain

$$(1-z^r)^k \cdot z^{r \cdot (n-k)} \cdot \sum_{j \geq 0} j^{n-1} \cdot z^j = (1-z^{-r})^k \cdot (-1)^k \cdot z^{r \cdot n} \cdot \sum_{j \geq 0} j^{n-1} \cdot z^j. \tag{2.3}$$

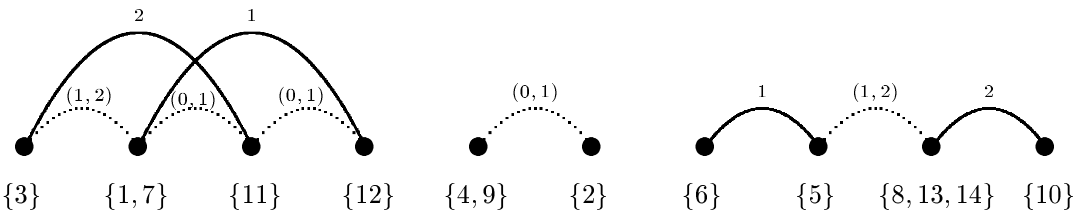

FIGURE 2. The diagram corresponding to a face in $\widehat{\mathcal{A}}^2_{14}$ of dimension 6 and level 3.

Let ∇_r denote the upper difference operator in the variable q, that is, $\nabla_r(p(q)) = p(q+r) - p(q)$. Since the coefficient of z^q in $(-1)^k \cdot z^{r \cdot n} \cdot \sum_{j \geq 0} j^{n-1} \cdot z^j$ is $(-1)^k \cdot (q - r \cdot n)^{n-1}$, the coefficient of z^q in equation (2.3) is $(-1)^k \cdot \nabla_r^k (q - r \cdot n)^{n-1}$. Observe that

$$\begin{aligned} (-1)^k \cdot \nabla_r^k (q - r \cdot n)^{n-1} &= (-1)^{n-k-1} \cdot \nabla_r^k (r \cdot n - q)^{n-1} \\ &= (-1)^{n-k-1} \cdot \Delta_r^k x^{n-1} \mid_{x = rn-q} . \end{aligned}$$

Recall that there are $\binom{n}{k}$ ways to choose the $n-k$ cuts. Moreover, there are q ways to decide where the box labeled 0 will be. Hence the polynomial $P(q)$ is given by

$$P(q) = (-1)^{n-k-1} \cdot \binom{n}{k} \cdot q \cdot \Delta_r^k x^{n-1} \mid_{x = rn-q} . \tag{2.4}$$

Now by considering the values of $(-1)^{n-k} \cdot P(-1)$ and $(-1)^{n-k-1} \cdot P(1)$, the result follows. ☐

By setting $k = 0$ and $q = t$ in equation (2.4) we obtain the following corollary.

Corollary 2.5. *The characteristic polynomial of the extended Shi arrangement $\widehat{\mathcal{A}}^r_n$ is given by $\chi(t) = t \cdot (t - r \cdot n)^{n-1}$.*

3. FACES AND THEIR DIAGRAMS

Let $F^{n,r}_{k,m}$ denote the number of faces of codimension k and level m from the extended Shi arrangement $\widehat{\mathcal{A}}^r_n$. Since the arrangement $\widehat{\mathcal{A}}^r_n$ has linear degree of freedom 1, the level and dimension of a face of $\widehat{\mathcal{A}}^r_n$ are at least 1. In Figure 1 the 2-dimensional representation of the arrangement $\widehat{\mathcal{A}}^1_3 = \widehat{\mathcal{A}}_3$ is displayed. Since there are 12 rays in the figure, there are 12 half-planes in $\widehat{\mathcal{A}}_3$ and thus $F^{3,1}_{1,2} = 12$. Determining all the values of $F^{3,1}_{k,m}$ we have the following table.

$k \backslash m$	1	2	3
0	4	6	6
1	9	12	
2	6		

By Zaslavsky's seminal result [18], $F^{n,r}_{0,1} = |\chi(1)| = (r \cdot n - 1)^{n-1}$. It is straightforward to observe that $F^{n,r}_{0,n} = n!$. Moreover, an equivalent statement of Theorem 2.2 is

$$F^{n,r}_{k,1} + \cdots + F^{n,r}_{k,n-k} = \binom{n}{k} \cdot \Delta_r^k x^{n-1} \mid_{x = rn+1} , \tag{3.1}$$

$$F^{n,r}_{k,1} = \binom{n}{k} \cdot \Delta_r^k x^{n-1} \mid_{x = rn-1} . \tag{3.2}$$

We now describe a representation of the faces of the extended Shi arrangement.

Definition 3.1. *Let F be a face in the Shi arrangement $\widehat{\mathcal{A}}_n^r$. Let $W_1, \ldots, W_p$ be the ordered partition of the set $[n] = \{1, \ldots, n\}$ such that (1) i and j belong to the same block if $x_i = x_j$ for all points $\mathbf{x}$ in the face F and (2) i lies in a block previous to that containing j if $x_i > x_j$.* The diagram D *of the face F consists of this ordered partition together with two types of labeled edges between the blocks of the partition.*

- *Draw a solid edge labeled h from the block W_α to the block W_β if $x_i = x_j + h$ where i belongs to W_α, j belongs to W_β, $i > j$, $\alpha < \beta$ and $1 \leq h \leq r$.*
- *Draw a solid edge labeled h from the block W_α to the block W_β if $x_j = x_i + h$ where i belongs to W_β, j belongs to W_α, $i > j$, $\alpha < \beta$ and $1 \leq h \leq r - 1$.*
- *Draw a dotted edge labeled $(h, h+1)$ from the block W_α to the block W_β if $x_i - x_j$ is contained in the open interval $(h, h+1)$ where i belongs to W_α, j belongs to W_β, $i > j$, $\alpha < \beta$ and $0 \leq h \leq r - 1$.*
- *Draw a dotted edge labeled $(h, h+1)$ from the block W_α to the block W_β if $x_j - x_i$ is contained in the open interval $(h, h+1)$ where j belongs to W_α, i belongs to W_β, $i > j$, $\alpha < \beta$ and $0 \leq h \leq r - 2$.*

A solid edge corresponds to an equality where the label is the difference between the two values of the blocks, whereas a dotted edge allows the difference between the two values to vary. However, for a dotted edge this difference is restricted to belong to the open interval given by the label.

This notion of the diagram of a face extends the notion of the dual diagram of a region in the Shi arrangement $\widehat{\mathcal{A}}_n$ introduced in [3].

As an example, consider the diagram in Figure 2 which corresponds to a face in $\widehat{\mathcal{A}}_{14}^2$. The underlying ordered partition encodes the following string of equalities and inequalities: $x_3 > x_1 = x_7 > x_{11} > x_{12} > x_4 = x_9 > x_2 > x_6 > x_5 > x_8 = x_{13} = x_{14} > x_{10}$. The solid edges describe the equalities $x_3 = x_{11} + 2$, $x_1 = x_{12} + 1$, $x_6 = x_5 + 1$ and $x_8 = x_{10} + 2$, while the dotted edges describe the inequalities $x_3 - x_7 \in (1, 2)$, $x_1 - x_{11} \in (0, 1)$, $x_{11} - x_{12} \in (0, 1)$, $x_4 - x_2 \in (0, 1)$ and $x_5 - x_8 \in (1, 2)$. Observe that the diagram of a face is decomposed into blocks. In Figure 2 there are 3 blocks: $C_1 = \{1, 3, 7, 11, 12\}$, $C_2 = \{2, 4, 9\}$ and $C_3 = \{5, 6, 8, 10, 13, 14\}$.

Proposition 3.2. *The number of blocks of a diagram D is equal to the level of the associated face F.*

Proof. Assume that there are m blocks $C_1, \ldots, C_m$. Observe that $\{C_1, \ldots, C_m\}$ is an ordered partition of $[n]$. Consider the subspace V of the Euclidean space described by the equalities $x_i = x_j$ if i and j belong to the same block C_s. The dimension of the subspace V is m. Let $d(k)$ denote the distance $r \cdot \sqrt{1^2 + \cdots + (k-1)^2}$. Then every point on the face F is at most the distance $\sqrt{d(|C_1|)^2 + \cdots + d(|C_m|)^2}$ away from the subspace V. Hence the level of the face F is at most m. Let B be the subset of the subspace V defined by $B = \{\mathbf{x} \in V \ : \ x_i \geq x_j \text{ if } i \in C_\alpha, j \in C_\beta, \alpha \leq \beta\}$. The set B has level m. Moreover, since any point in the set B can be approximated within a bounded distance by a point from the face F, we obtain that the level of the face F is m. $\square$

Theorem 3.3. *The face numbers $F_{k,m}^{n,r}$ satisfy the identity*

$$F_{k,m}^{n,r} = \sum_{\beta_1 + \cdots + \beta_m = k} \ \sum_{\substack{j_1 + \cdots + j_m = n \\ \forall 1 \leq s \leq m \ \beta_s < j_s}} \binom{n}{j_1, \ldots, j_m} \cdot \prod_{s=1}^{m} F_{\beta_s, 1}^{j_s, r}.$$

Proof. To form the diagram of a face with k equalities and m blocks, proceed in the following manner. First choose how many equalities there will be in each block. Let β_s denote the number of equalities in the sth block. Then choose how many vertices there will be in each block. Let j_s be the cardinality of the sth block. Observe that we necessarily have $\beta_s < j_s$. The multinomial coefficient $\binom{n}{j_1, \ldots, j_m}$ distributes the vertices among the blocks. The diagram on the sth block can now be chosen in $F_{\beta_s, 1}^{j_s, r}$ number of ways. $\square$

4. The Abel identity and the proof of main theorem

Theorems 2.2 and 3.3 offer a method to compute the numbers $F_{k,m}^{n,r}$. Hence to prove the main theorem, Theorem 1.3, all we need to show is that the values $m \cdot \binom{n}{k} \cdot \Delta_r^k \Delta^{m-1} x^{n-1} |_{x=rn-1}$ also satisfy the relation in Theorem 3.3. This is the content of the next theorem.

Theorem 4.1. *Let $Q(n,k,m)$ denote $\Delta_r^k \Delta^{m-1} x^{n-1} |_{n-1}$. Then we have*

$$m \cdot \binom{n}{k} \cdot Q(n,k,m) = \sum_{\beta_1+\cdots+\beta_m=k} \sum_{\substack{j_1+\cdots+j_m=n \\ \forall 1\leq s\leq m \;\; \beta_s<j_s}} \binom{n}{j_1,\ldots,j_m} \cdot \prod_{s=1}^{m} \binom{j_s}{\beta_s} \cdot Q(j_s,\beta_s,1).$$

Before we proceed let us introduce some notation. For a set I of indices let t_I denote the sum of the variables $\sum_{i\in I} t_i$. For sets I, J and K let $I+J=K$ denote that the disjoint union of I and J is the set K. Hence when we sum over the condition $I+J=K$ for a given K, the sum ranges over all compositions of the set K. Similarly let J_S denote the disjoint union $\sum_{s\in S} J_s$. Combining the last two conventions, for a given K a sum over the condition $J_S = K$ means that the sum is over all compositions of the set K into parts which are indexed by the set S. We will also use vector notation and concatenation in the multinomial coefficient. That is, $\binom{k}{\alpha,\vec{\beta}_S}$ denotes $\binom{k}{\alpha,\beta_2,\ldots,\beta_m}$ where $\vec{\beta}_S$ is the vector $(\beta_2,\ldots,\beta_m)$.

The classical Abel identity is the following:

$$(x+y+n\cdot t)^n = \sum_{i+j=n} \binom{n}{i} \cdot (x+i\cdot t)^i \cdot y \cdot (y+j\cdot t)^{j-1}.$$

To proceed we need a generalization of the Abel identity which introduces several y variables and several t variables. This generalization and its proof is essentially due to Françon [8]. See also Section 3.1 in [5]. For completeness, we give a brief sketch of the proof.

Proposition 4.2 (The multi-variated Abel identity)**.** *Let S denote the set $\{2,\ldots,m\}$. Then*

$$(x+y_S+t_{[n]})^n = \sum_{I+J_S=[n]} (x+t_I)^{|I|} \cdot \prod_{s\in S} y_s \cdot (y_s+t_{J_s})^{|J_s|-1}.$$

Proof. Let E be the set $[n+m]$. To each element in E associate a variable z_i by the rule $z_1 = t_1$, $\ldots$, $z_n = t_n$, $z_{n+1} = x$, $z_{n+2} = y_2$, $\ldots$, $z_{m+n} = y_m$. For a function f from the set E to itself, let the weight be the product $\prod_{i\in E} z_{f(i)}$. Now consider the set of all functions such that $n+1$ through $n+m$ are fixed points. The sum of the weight of all such functions is $x\cdot y_2\cdots y_m\cdot(x+y_S+t_{[n]})^n$. The sum can also be evaluated by decomposing the set $[n]$ into m parts I, J_2, $\ldots$, J_m where for $s\in S$ there is an acyclic function on the set $J_s\cup\{n+s\}$ with the fixed point $n+s$. The sum of the weights of all such functions is $y_s^2\cdot(y_s+t_{J_s})^{|J_s|-1}$. Finally on the set $I\cup\{n+1\}$ there could be any function having fixed point $n+1$. These are enumerated by $x\cdot(x+t_I)^{|I|}$. Multiplying these weights together, summing over all compositions $I+J_S=[n]$ and then cancelling the factor $x\cdot y_2\cdots y_m$ from both sides we obtain the result. □

We now have the machinery in place to prove our main result.

Proof of Theorem 4.1. For ease of notation, let $R_{k,m}^n(x)$ denote the polynomial $\Delta_r^k\Delta^m x^n$. Replacing n with $n-1$ in the multi-variated Abel identity, we rewrite it as

$$\begin{aligned} R_{0,0}^{n-1}(x+y_S+t_{[n-1]}) &= \sum_{I+J_S=[n-1]} R_{0,0}^{|I|}(x+t_I)\cdot\prod_{s\in S} y_s\cdot R_{0,0}^{|J_s|-1}(y_s+t_{J_s}) \\ &= \sum_{T\subseteq S} \sum_{\substack{I+J_T=[n-1] \\ \forall s\in T \;\; J_s\neq\emptyset}} R_{0,0}^{|I|}(x+t_I)\cdot\prod_{s\in T} y_s\cdot R_{0,0}^{|J_s|-1}(y_s+t_{J_s}). \end{aligned}$$

By inclusion-exclusion we have

$$\sum_{T \subseteq S} (-1)^{|S-T|} \cdot R_{0,0}^{n-1}(x + y_T + t_{[n-1]}) = \sum_{\substack{I+J_S=[n-1] \\ \forall s \in S \ J_s \neq \emptyset}} R_{0,0}^{|I|}(x + t_I) \cdot \prod_{s \in S} y_s \cdot R_{0,0}^{|J_s|-1}(y_s + t_{J_s}).$$

Setting the variables $y_2 = \cdots = y_m = -1$ and $x = 0$, and observing that the left-hand side is an $(m-1)$st order difference operator gives

$$R_{0,m-1}^{n-1}(t_{[n-1]}) = \sum_{\substack{I+J_S=[n-1] \\ \forall s \in S \ J_s \neq \emptyset}} R_{0,0}^{|I|}(t_I) \cdot \prod_{s \in S} R_{0,0}^{|J_s|-1}(t_{J_s} - 1).$$

Now apply in each of the k variables $t_1, \ldots, t_k$ the difference operator Δ_r. The result is

$$R_{k,m-1}^{n-1}(t_{[n-1]}) = \sum_{A+B_S=[k]} \sum_{\substack{I+J_S=[n-1] \\ \forall s \in S \ B_s \subseteq J_s \neq \emptyset \\ A \subseteq I}} R_{|A|,0}^{|I|}(t_I) \cdot \prod_{s \in S} R_{|B_s|,0}^{|J_s|-1}(t_{J_s} - 1). \tag{4.1}$$

Observe that any term with $B_s = J_s$ for some index s vanishes. Hence the condition $B_s \subseteq J_s$ can be replaced with the strict containment $B_s \subset J_s$. Setting the variables $t_1 = \cdots = t_{n-1} = 1$, we rewrite (4.1) as

$$Q(n,k,m) = \sum_{\alpha+\beta_S=k} \binom{k}{\alpha, \vec{\beta}_S} \cdot \sum_{\substack{i+j_S=n-1 \\ \forall s \in S \ \beta_s < j_s \\ \alpha \leq i}} \binom{n-k-1}{i-\alpha, \vec{\jmath}_S - \vec{\beta}_S} \cdot R_{\alpha,0}^{i}(i) \cdot \prod_{s \in S} Q(j_s, \beta_s, 1),$$

where $\vec{\beta}_S$ denotes the vector $(\beta_2, \ldots, \beta_m)$ and $\vec{\jmath}_S$ the vector $(j_2, \ldots, j_m)$. Let $\alpha = \beta_1$ and $i = j_1 - 1$. Observe now that the conditions $i + j_S = n - 1$ and $\alpha \leq i$ become $j_{[m]} = n$ and $\alpha < j_0$. Hence we have

$$Q(n,k,m) = \sum_{\beta_{[m]}=k} \binom{k}{\beta_1, \vec{\beta}_S} \cdot \sum_{\substack{j_{[m]}=n \\ \forall 1 \leq s \leq m \ \beta_s < j_s}} \binom{n-k-1}{j_1 - \beta_1 - 1, \vec{\jmath}_S - \vec{\beta}_S} \cdot \prod_{s=1}^{m} Q(j_s, \beta_s, 1). \tag{4.2}$$

By symmetry one can obtain $m-1$ identities similar to (4.2), with the only difference being in the multinomial coefficient appearing in the inner summation. Adding these m identities using the classical Pascal recursion for the multinomial coefficient, we have

$$m \cdot Q(n,k,m) = \sum_{\beta_{[m]}=k} \binom{k}{\vec{\beta}} \cdot \sum_{\substack{j_{[m]}=n \\ \forall 1 \leq s \leq m \ \beta_s < j_s}} \binom{n-k}{\vec{\jmath} - \vec{\beta}} \cdot \prod_{s=1}^{m} Q(j_s, \beta_s, 1),$$

where $\vec{\beta} = (\beta_1, \ldots, \beta_m)$ and $\vec{\jmath} = (j_1, \ldots, j_m)$. Multiplying this identity with $\binom{n}{k}$ we obtain the desired identity. □

5. Corollaries and concluding remarks

We now state three immediate corollaries of Theorem 1.3.

Corollary 5.1. *The number of faces in the n-dimensional Shi arrangement $\widehat{\mathcal{A}}_n$ of codimension k and level m is given by $m \cdot \binom{n}{k} \cdot \Delta^{k+m-1} x^{n-1} \big|_{x=n-1}$.*

Corollary 5.2. *The number of regions in the n-dimensional extended Shi arrangement $\widehat{\mathcal{A}}_n^r$ of level m is given by $m \cdot \Delta^{m-1} x^{n-1} \big|_{x=rn-1}$.*

From the fact that $\Delta_r^{n-m} \Delta^{m-1} x^{n-1} = r^{n-m} \cdot (n-1)!$, the next corollary follows.

Corollary 5.3. *The number of faces in the n-dimensional extended Shi arrangement $\widehat{\mathcal{A}}_n^r$ of dimension and level m is given by $r^{n-m} \cdot \binom{n-1}{m-1} \cdot n!$.*

An *r-parking function* f of order n is an n-tuple of non-negative integers $\mathbf{a} = (a_1, \ldots, a_n)$ such that when the tuple $\mathbf{a}$ is rearranged into increasing order $b_1 \leq b_2 \leq \cdots \leq b_n$ then $b_j \leq r \cdot (j-1)$; see Stanley [17]. The case $r = 1$ corresponds to the classical notion of parking functions, however, the notation differs slightly. For the original context of parking functions, see [9, 11]. Define the *level* of an r-parking function $\mathbf{a}$ to be the cardinality of the set $\{j \,:\, 1 \leq j \leq n, b_j = r \cdot (j-1)\}$. For example, parking functions of level 1 are called prime parking functions and were studied by Gessel [3]. Since the number of r-parking functions are $(r \cdot n + 1)^{n-1}$ and the number of r-parking functions of level m satisfy Theorem 3.3 and equation (3.1) (with $k = 0$), we have the following result.

Corollary 5.4. *The number of r-parking functions of order n having level m is the same as the number of regions of the n-dimensional extended Shi arrangement $\widehat{\mathcal{A}}_n^r$ of level m, that is, $m \cdot \Delta^{m-1} x^{n-1} \mid_{x=rn-1}$.*

This result can be proven directly using the fact that the Abel polynomials $x \cdot (x - r \cdot n)^{n-1}$ form a binomial sequence of polynomials [13].

The value of $\Delta^{k+m-1} x^{n-1} \mid_{x=n-1}$ has the natural combinatorial meaning as the number of functions from $[n-1]$ to $[n-1]$ having the set $[k+m-1]$ in the image. That is,

$$\Delta^{k+m-1} x^{n-1} \mid_{x=n-1} \; = \#\{f : [n-1] \longrightarrow [n-1] \;\; : \;\; [k+m-1] \subseteq \mathrm{Im}(f)\}.$$

Hence it is natural to ask if a bijective proof of Corollary 5.1 can be given. This question extends Athanasiadis' question of finding a bijective proof of Theorem 1.1 [3].

One can also ask if Theorem 1.3 can be extended to other root systems. This is in spirit of the work of [1, 10, 15]. There are other generalizations of the Shi arrangement presented in [2, 12]. Are there similar results for these arrangements?

Acknowledgments

The author thanks Margaret Readdy for inspiring discussions. The author also thanks Richard Stanley and Thomas Zaslavsky for bringing the extended Shi arrangement to the author's attention. He also thanks Gábor Hetyei for the translation of the abstract into French. This research was partially supported by Swedish Natural Science Research Council DNR 702-238/98. The author was also supported by National Science Foundation grants DMS 98-00910 and DMS 99-83660, while visiting Cornell University.

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