INCIDENCE COMBINATORICS OF RESOLUTIONS

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ABSTRACT. For an arbitrary meet-semilattice we introduce notions of combinatorial blowups, building sets, and nested sets. This gives a common abstract framework for the incidence combinatorics occurring in the context of DeConcini-Procesi models of subspace arrangements and resolutions of singularities in toric varieties. Our main theorem states that a sequence of combinatorial blowups, prescribed by a building set in linear extension compatible order, gives the face poset of the simplicial complex of nested sets. As applications we trace the incidence combinatorics through every step of the DeConcini-Procesi model construction, and we introduce the notions of building sets and nested sets to the context of toric varieties.

1. INTRODUCTION

For an arbitrary meet-semilattice we introduce notions of *combinatorial blowups, building sets, and nested sets.* The definitions are given on order-theoretic level without any reference to geometry. This provides a common abstract framework for the incidence combinatorics occurring in at least two different situations in algebraic geometry: the construction of DeConcini-Procesi models of subspace arrangements [4], and the resolution of singularities in toric varieties.

The various parts of this abstract framework have received different emphasis within different situations: while the notion of a combinatorial blowup clearly specializes to the stellar subdivision of a defining fan in the context of toric varieties, building sets and nested sets were introduced in the context of model constructions by DeConcini & Procesi [4], earlier and in a more special setting by Fulton & MacPherson [8].

It was proved in [4] that a sequence of blowups within an arrangement of complex linear subspaces leads from the intersection stratification of complex space given by the maximal subspaces of the arrangement to an arrangement model stratified by divisors with normal crossings. In the context of toric varieties, there exist many different procedures for stellar subdivisions of a defining fan that result in a simplicial fan, so-called simplicial resolutions.

The purpose of our Main Theorem 3.4 is the unification of these two situations on the combinatorial level: a sequence of combinatorial blowups of a (combinatorial) building set in linear extension compatible order transforms the initial semilattice to a semilattice where all intervals are boolean algebras, more precisely to the face poset of the simplicial complex of nested sets. In particular, the structure of the resulting semilattice can be fully described by the initial data of nested sets.

We adopt our terminology from the work of Fulton & MacPherson and DeConcini & Procesi cited above. However, there is a subtle difference between the notions: the building sets in [4, 8] are not canonical, they depend on the geometry, while ours do not. See Section 4.1 for further details.

This extended abstract of our paper [7] is organized as follows: In **Section 2** we define building sets and nested sets for meet-semilattices in purely order-theoretic terms and list some of their properties. In **Section 3** we define combinatorial blowups of meet-semilattices, and study their effect on building sets and nested sets. This section contains our Main Theorem 3.4 which describes the result of blowing up the elements of a building set in terms of the initial nested set complex. **Section 4** is devoted to relating our abstract framework to two different contexts in algebraic geometry: In 4.1 we

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The second author acknowledges support by the Forschungsinstitut für Mathematik, ETH Zürich, and the Swedish National Science Foundation.

MSC Classification: primary 06A07; secondary 52C35, 06A12, 05E99.

describe the change of incidence combinatorics during the step-by-step construction of arrangement models in [4] as combinatorial blowups in the semilattice of strata. In 4.2 we draw the connection to simplicial resolutions of toric varieties: we recognize stellar subdivisions as combinatorial blowups of the face posets of defining fans and discuss building and nested sets in the toric context.

For proofs, respectively further details of proofs, we refer to our paper [7].

2. Building sets and nested sets of meet-semilattices

2.1. **Building sets.**

Let us first recall some poset terminology (for more standard concepts we refer to [13]). A poset is called *irreducible* if it is not a direct product of two other posets, both consisting of at least two elements. For a poset $\mathcal{L}$ with a unique minimal element $\hat{0}$, we call $I(\mathcal{L}) = \{x \in \mathcal{L} \,|\, [\hat{0}, x] \text{ is irreducible}\}$ the *set of irreducible elements* in $\mathcal{L}$. In particular, the minimal element $\hat{0}$ and all atoms of $\mathcal{L}$ are irreducible elements in $\mathcal{L}$. For $x \in \mathcal{L}$, we call $D(x) := \max(I(\mathcal{L})_{\leq x})$ the *set of elementary divisors* of x. We will be mostly concerned with meet-semilattices, calling them *semilattices* for short.

Definition 2.1. *Let $\mathcal{L}$ be a semilattice. A subset $\mathcal{G}$ in $\mathcal{L}$ is called a* **building set** *of $\mathcal{L}$ if for any $x \in \mathcal{L}$ and $\max \mathcal{G}_{\leq x} = \{x_1, \ldots, x_k\}$ there is an isomorphism of posets*

$$\varphi_x : \prod_{j=1}^{k} [\hat{0}, x_j] \xrightarrow{\cong} [\hat{0}, x] \tag{2.1}$$

with $\varphi_x(\hat{0}, \ldots, x_j, \ldots, \hat{0}) = x_j$ for $j = 1, \ldots, k$. We call $F(x) := \max \mathcal{G}_{\leq x}$ the **set of factors** *of x in $\mathcal{G}$.*

Proposition 2.2. *For a semilattice $\mathcal{L}$ and a subset $\mathcal{G}$ of $\mathcal{L}$ the following are equivalent:*

1. *$\mathcal{G}$ is a building set of $\mathcal{L}$;*
2. *$\mathcal{G}$ contains $I(\mathcal{L})$, and for every $x \in \mathcal{L}$ with $D(x) = \{y_1, \ldots, y_l\}$ the elementary divisors of x, there exists a partition $\pi_x = \pi_1 | \ldots | \pi_k$ of $[l]$ with blocks $\pi_t = \{i_1, \ldots, i_{|\pi_t|}\}$ for $t = 1, \ldots, k$, such that the elements in $\max \mathcal{G}_{\leq x} = \{x_1, \ldots, x_k\}$ are of the form $x_t = \varphi_x^{\text{el}}(\hat{0}, \ldots, \hat{0}, y_{i_1}, \hat{0}, \ldots, \hat{0}, y_{i_2}, \hat{0}, \ldots, \hat{0}, y_{i_{|\pi_t|}}, \hat{0})$.*
   Informally speaking, the factors of x in $\mathcal{G}$ are products of disjoint sets of elementary divisors of x.
3. *$\mathcal{G}$ generates $\mathcal{L}$ by $\vee$, and for any $x \in \mathcal{L}$, any $\{y, y_1, \ldots, y_t\} \subseteq \max \mathcal{G}_{\leq x}$, and $z \in \mathcal{L}$ with $z < y$, we have $\mathcal{G}_{\leq y} \cap \mathcal{G}_{\leq z \vee y_1 \vee \cdots \vee y_t} = \mathcal{G}_{\leq z}$.*
4. *$\mathcal{G}$ generates $\mathcal{L}$ by $\vee$, and for any $x \in \mathcal{L}$, any $\{y, y_1, \ldots, y_t\} \subseteq \max \mathcal{G}_{\leq x}$, and $z \in \mathcal{L}$ with $z < y$, the following two conditions are satisfied:*

   i) $\mathcal{G}_{\leq y} \cap \mathcal{G}_{\leq y_1 \vee \cdots \vee y_t} = \{\hat{0}\}$ *"disjointness,"*
   ii) $z \vee y_1 \vee \cdots \vee y_t < y \vee y_1 \vee \cdots \vee y_t$ *"necessity."*

Remark 2.3. The definition of building sets and of irreducible elements, as well as the characterization of building sets in Proposition 2.2 (2), are independent of the existence of a join operation and can be formulated for any poset with a unique minimal element.

Proposition 2.4. *For a building set $\mathcal{G}$ of $\mathcal{L}$, the following holds:*

1. *Let $x \in \mathcal{L}$, $F(x) = \{x_1, \ldots, x_k\}$ the set of factors of x in $\mathcal{G}$, and $\hat{0} \neq y \in \mathcal{G}$ with $y \leq x$. Then there exists a unique $j \in \{1, \ldots, k\}$ such that $y \leq x_j$; i.e., $F(x) = \max \mathcal{G}_{\leq x}$ induces a partition of $\mathcal{G}_{\leq x} \setminus \{\hat{0}\}$.*
2. *For $x \in \mathcal{L}$ and $x_0 \in F(x)$,*
   $$\bigvee (F(x) \setminus \{x_0\}) \;<\; \bigvee F(x) = x \,,$$
   i.e., each factor of x in $\mathcal{G}$ is needed to generate x.
3. *If $h_1, \ldots, h_k$ in $\mathcal{G}$ are such that $(h_i, \bigvee_{j=1}^{k} h_j] \cap \mathcal{G} = \emptyset$ for $i = 1, \ldots, k$, then $F(\bigvee_{j=1}^{k} h_j) = \{h_1, \ldots, h_k\}$.*

2.2. **Nested Sets.**

Definition 2.5. *Let $\mathcal{L}$ be a semilattice and $\mathcal{G}$ a building set of $\mathcal{L}$. A subset N in $\mathcal{G}$ is called* **nested** *if, for any set of incomparable elements $x_1, \dots, x_t$ in N, $t \geq 2$, the join $x_1 \vee \cdots \vee x_t$ exists and does not belong to $\mathcal{G}$. The nested sets in $\mathcal{G}$ form an abstract simplicial complex, denoted $\mathcal{N}(\mathcal{G})$.*

Note that the elements of $\mathcal{G}$ are the vertices of the complex of nested sets $\mathcal{N}(\mathcal{G})$. Moreover, the order complex of $\mathcal{G}$ is a subcomplex of $\mathcal{N}(\mathcal{G})$, since linearly ordered subsets of $\mathcal{G}$ are nested.

Proposition 2.6. *For a given semilattice $\mathcal{L}$ and a subset N of a building set $\mathcal{G}$ of $\mathcal{L}$, the following holds: N is nested if and only if for any $x_1, \dots, x_t$ noncomparable in N, the join $x_1 \vee \cdots \vee x_t$ exists, and $F(x_1 \vee \cdots \vee x_t) = \{x_1, \dots, x_t\}$.*

3. Sequences of Combinatorial Blowups

3.1. **Combinatorial Blowups.**

Definition 3.1. *For a semilattice $\mathcal{L}$ and an element $\alpha \in \mathcal{L}$ we define a poset $\mathrm{Bl}_{\alpha}\mathcal{L}$, the* **combinatorial blowup of $\mathcal{L}$ at** *α, as follows:*

- *elements of $\mathrm{Bl}_{\alpha}\mathcal{L}$:*
  (1) *$y \in \mathcal{L}$, such that $y \not\geq \alpha$;*
  (2) *$[\alpha, y]$, for $y \in \mathcal{L}$, such that $y \not\geq \alpha$ and $(y \vee \alpha)_{\mathcal{L}}$ exists*
  (in particular, $[\alpha, \hat{0}]$ can be thought of as the result of blowing up α);
- *order relations in $\mathrm{Bl}_{\alpha}\mathcal{L}$:*
  (1) *$y > z$ in $\mathrm{Bl}_{\alpha}\mathcal{L}$ if $y > z$ in $\mathcal{L}$;*
  (2) *$[\alpha, y] > [\alpha, z]$ in $\mathrm{Bl}_{\alpha}\mathcal{L}$ if $y > z$ in $\mathcal{L}$;*
  (3) *$[\alpha, y] > z$ in $\mathrm{Bl}_{\alpha}\mathcal{L}$ if $y \geq z$ in $\mathcal{L}$; where in all three cases $y, z \not\geq \alpha$.*

Note that the atoms in $\mathrm{Bl}_{\alpha}\mathcal{L}$ are the atoms of $\mathcal{L}$ together with the element $[\alpha, \hat{0}]$. It is easy, albeit tedious, to check that the class of (meet-)semilattices is closed under combinatorial blowups.

Lemma 3.2. *Let $\mathcal{L}$ be a semilattice and $\alpha \in \mathcal{L}$, then $\mathrm{Bl}_{\alpha}\mathcal{L}$ is a semilattice.*

For the proof note that joins in $\mathrm{Bl}_{\alpha}\mathcal{L}$ are defined by the rules

$$\begin{aligned}([\alpha, y] \vee [\alpha, z])_{\mathrm{Bl}_{\alpha}\mathcal{L}} &= [\alpha, (y \vee z)_{\mathcal{L}}],\\ ([\alpha, y] \vee z)_{\mathrm{Bl}_{\alpha}\mathcal{L}} &= [\alpha, (y \vee z)_{\mathcal{L}}],\\ (y \vee z)_{\mathrm{Bl}_{\alpha}\mathcal{L}} &= (y \vee z)_{\mathcal{L}},\end{aligned}$$

which are applicable only if $(y \vee z)_{\mathcal{L}}$ exists and $(y \vee z)_{\mathcal{L}} \not\geq \alpha$. Otherwise the corresponding joins do not exist. Observe that it is possible that $(x \vee y)_{\mathcal{L}}$ exists, while $(x \vee y)_{\mathrm{Bl}_{\alpha}\mathcal{L}}$ does not.

3.2. **Blowing up building sets.**

In this subsection we prove that if one combinatorially blows up a building set of a semilattice in any chosen linear extension order, then one ends up with the face poset of the simplicial complex of nested sets of this building set. The following proposition provides the essential step for the proof.

Proposition 3.3. *Let $\mathcal{L}$ be a semilattice, $\mathcal{G}$ a building set of $\mathcal{L}$, $\alpha \in \max \mathcal{G}$. Then, $\widetilde{\mathcal{G}} = (\mathcal{G} \setminus \{\alpha\}) \cup \{[\alpha, \hat{0}]\}$ is a building set of $\mathrm{Bl}_{\alpha}\mathcal{L}$. Furthermore, the nested subsets of $\widetilde{\mathcal{G}}$ are precisely the nested subsets of $\mathcal{G}$ with α replaced by $[\alpha, \hat{0}]$.*

Proof. It is easy to see that $\widetilde{\mathcal{G}}$ is a building set of $\mathrm{Bl}_{\alpha}\mathcal{L}$. Indeed, given $x \in \mathcal{L} \setminus \mathcal{L}_{\geq \alpha}$, (2.1) is obvious for $x \in \mathrm{Bl}_{\alpha}\mathcal{L}$, and, if $(x \vee \alpha)_{\mathcal{L}}$ exists, it follows for $[\alpha, x] \in \mathrm{Bl}_{\alpha}\mathcal{L}$ from the identity

$$[\hat{0}, [\alpha, x]]_{\mathrm{Bl}_{\alpha}\mathcal{L}} \;=\; [\hat{0}, x]_{\mathrm{Bl}_{\alpha}\mathcal{L}} \times B_1\,,$$

where B_1 is the subposet consisting of the two comparable elements $\hat{0} < [\alpha, \hat{0}]$.

Let us now see that the sets of nested subsets of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are the same when replacing α by $[\alpha,\hat{0}]$: Let N be a nested set in $\mathcal{G}$, not containing α. For incomparable elements $x_1,\ldots,x_t$ in N, $\bigvee_{i=1}^t x_i \not\geq \alpha$, since otherwise we had $\alpha \in \max \mathcal{G}_{\leq \bigvee x_i} = F(\bigvee_{i=1}^t x_i) = \{x_1,\ldots,x_t\}$ by Proposition 2.6. Thus, $\bigvee_{i=1}^t x_i$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$ and $\bigvee_{i=1}^t x_i \notin \widetilde{\mathcal{G}}$. Hence, N is nested in $\widetilde{\mathcal{G}}$. A nested subset in $\widetilde{\mathcal{G}}$ not containing $[\alpha,\hat{0}]$ is obviously nested in $\mathcal{G}$.

Let now N be nested in $\mathcal{G}$ containing α, and set $\widetilde{N} = (N\setminus\{\alpha\})\cup\{[\alpha,\hat{0}]\}$. Subsets of incomparable elements in $\widetilde{N}$ not containing $[\alpha,\hat{0}]$ can be dealt with as above. Thus assume that $[\alpha,\hat{0}],x_1,\ldots,x_t$ are incomparable in $\widetilde{N}$. Then, $x_1,\ldots,x_t$ are incomparable in the nested set N, and, as above, we conclude that $\bigvee_{i=1}^t x_i$ exists and $\bigvee_{i=1}^t x_i \not\geq \alpha$. Moreover, $\alpha \vee \bigvee_{i=1}^t x_i$ exists in $\mathcal{L}$ (joins of nested sets always exist!), thus, $[\alpha, \bigvee_{i=1}^t x_i] = [\alpha,\hat{0}] \vee \bigvee_{i=1}^t x_i$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$ and is obviously not contained in $\widetilde{\mathcal{G}}$. We conclude that $\widetilde{N}$ is nested in $\widetilde{\mathcal{G}}$.

Vice versa, let $\widetilde{N}$ be nested in $\widetilde{\mathcal{G}}$ containing $[\alpha,\hat{0}]$, and set $N = (\widetilde{N}\setminus\{[\alpha,\hat{0}]\})\cup\{\alpha\}$. Again it suffices to consider subsets of incomparable elements $\alpha,x_1,\ldots,x_t$ in N. With $[\alpha,\hat{0}],x_1,\ldots,x_t$ incomparable in $\widetilde{N}$, $[\alpha,\hat{0}] \vee \bigvee_{i=1}^t x_i = [\alpha, \bigvee_{i=1}^t x_i]$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$, thus $\alpha \vee \bigvee_{i=1}^t x_i$ exists in $\mathcal{L}$. Incomparability implies that $\alpha \vee \bigvee_{i=1}^t x_i > \alpha$, and thus $\alpha \vee \bigvee_{i=1}^t x_i \notin \mathcal{G}$. We conclude that N is nested in $\mathcal{G}$. $\square$

By iterating the combinatorial blowup described in Proposition 3.3 through all of $\mathcal{G}$, we obtain the following theorem, which serves as a motivation for the entire development.

Theorem 3.4. *Let $\mathcal{L}$ be a semilattice and $\mathcal{G}$ a building set of $\mathcal{L}$ with some chosen linear extension: $\mathcal{G} = \{G_1,\ldots,G_t\}$, with $G_i > G_j$ implying $i < j$. Let $\mathrm{Bl}_k\mathcal{L}$ denote the result of subsequent blowups $\mathrm{Bl}_{G_k}(\mathrm{Bl}_{G_{k-1}}(\ldots \mathrm{Bl}_{G_1}\mathcal{L}))$. Then the final semilattice $\mathrm{Bl}_t\mathcal{L}$ is equal to the face poset of the simplicial complex $\mathcal{N}(\mathcal{G})$.*

Proof. The building set $\mathcal{G}_t$ of $\mathrm{Bl}_t\mathcal{L}$ that results from iterated application of Proposition 3.3 obviously is the set of atoms $\mathfrak{A}$ in $\mathrm{Bl}_t\mathcal{L}$. Every element $x \in \mathrm{Bl}_t\mathcal{L}$ is the join of atoms below it: $x = \bigvee \mathfrak{A}_{\leq x}$. The subset $\mathfrak{A}_{\leq x}$ of $\mathcal{G}_t$ is nested, in particular, it is the set of factors of x in $\mathrm{Bl}_t\mathcal{L}$ with respect to $\mathcal{G}_t$ (Proposition 2.6). Proposition 2.4(2) implies that the interval $[\hat{0},x]$ in $\mathrm{Bl}_t\mathcal{L}$ is boolean. We conclude that $\mathrm{Bl}_t\mathcal{L}$ is the face poset of a simplicial complex with faces in one-to-one correspondence with the nested sets in $\mathcal{G}_t$, which in turn correspond to the nested sets in $\mathcal{G}$ by Proposition 3.3. $\square$

4. Instances of combinatorial blowups

4.1. **DeConcini-Procesi models of subspace arrangements.** Let $\mathcal{A}$ denote an arrangement of linear subspaces in complex space $\mathbb{C}^d$. Much effort has been spent on describing the cohomology of the complement $\mathcal{M}(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup \mathcal{A}$ of such an arrangement and, in particular, on answering the question whether the cohomology algebra is completely determined by the combinatorial data of the arrangement. Here, combinatorial data is understood as the lattice $\mathcal{L}(\mathcal{A})$ of intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion together with the complex codimensions of the intersections. A major step towards the solution of this problem (for a complete answer see [2, 3]) was the construction of a smooth model for the complement $\mathcal{M}(\mathcal{A})$ by DeConcini & Procesi [4] that allowed for an explicit description of a rational model for $\mathcal{M}(\mathcal{A})$ following [10]. The DeConcini-Procesi model for arrangements in turn is one instance in a sequence of model constructions reaching from compactifications of symmetric spaces [5, 6], over the Fulton-MacPherson compactifications of configuration spaces [8] to the general framework of wonderful conical compactifications proposed by MacPherson & Procesi [11].

Given a complex subspace arrangement $\mathcal{A}$ in $\mathbb{C}^d$, DeConcini & Procesi describe a smooth irreducible variety Y together with a proper map $\pi : Y \longrightarrow \mathbb{C}^d$ such that π is isomorphism over $\mathcal{M}(\mathcal{A})$, and the complement of the preimage of $\mathcal{M}(\mathcal{A})$ is a union of irreducible divisors with normal crossings in Y. The model Y can be constructed by a sequence of blowups of smooth subvarieties that is prescribed by the stratification of complex space induced by the arrangement.

4.1.1. *Building sets for subspace arrangements.* In order to enumerate the strata in the intersection stratification of Y given by the irreducible divisors, DeConcini & Procesi introduced the notions of building sets, nested sets and irreducible elements as follows:

Definition 4.1. ([4, §2]) *Let $\mathcal{L}(\mathcal{A})$ be the intersection lattice of an arrangement $\mathcal{A}$ of linear subspaces in a finite dimensional complex vector space. Consider the lattice $\mathcal{L}(\mathcal{A})^*$ formed by the orthogonal complements of intersections ordered by inclusion.*

(1) *For $U \in \mathcal{L}(\mathcal{A})^*$, $U = \oplus_{i=1}^k U_i$ with $U_i \in \mathcal{L}(\mathcal{A})^*$, is called a* **decomposition** *of U if for any $V \subseteq U$, $V \in \mathcal{L}(\mathcal{A})^*$, $V = \oplus_{i=1}^k (U_i \cap V)$ and $U_i \cap V \in \mathcal{L}(\mathcal{A})^*$ for $i = 1, \ldots, k$.*
(2) *Call $U \in \mathcal{L}(\mathcal{A})^*$* **irreducible** *if it does not admit a non-trivial decomposition.*
(3) *$\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})^*$ is called a* **building set** *for $\mathcal{A}$ if for any $U \in \mathcal{L}(\mathcal{A})^*$ and $G_1, \ldots, G_k$ maximal in $\mathcal{G}$ below U, $U = \oplus_{i=1}^k G_i$ is a decomposition (the $\mathcal{G}$-decomposition) of U.*
(4) *A subset $\mathcal{S} \subseteq \mathcal{G}$ is called* **nested** *if for any set of non-comparable elements $U_1, \ldots, U_k$ in $\mathcal{S}$, $U = \oplus_{i=1}^k U_i$ is the $\mathcal{G}$-decomposition of U.*

Note that $\mathcal{L}(\mathcal{A})^*$ coincides with $\mathcal{L}(\mathcal{A})$ as abstract lattices. We will therefore talk about irreducible elements, building sets and nested sets in $\mathcal{L}(\mathcal{A})$ without explicitly referring to the dual setting of the preceding definition.

The notions of Definition 4.1 are in part based on the earlier notions introduced by Fulton & MacPherson in [8] to study compactifications of configuration spaces. Our terminology is naturally adopted from [8, 4]. Building sets and nested sets in the sense of DeConcini & Procesi are building and nested sets for the intersection lattices of subspace arrangements in our combinatorial sense (see Proposition 4.4 (1) below). However, there are differences. The opposite is not true: A combinatorial building set for the intersection lattice of a subspace arrangement is not necessarily a building set for this arrangement in the sense of DeConcini & Procesi, neither are irreducible elements in the sense of DeConcini & Procesi irreducible in our sense.

Example. Consider the following arrangement of 3 subspaces in $\mathbb{C}^4$:

$$A_1 : \ z_4 = 0 \,, \quad A_2 : \ z_1 = z_2 = 0 \,, \quad A_3 : \ z_1 = z_3 = 0 \,.$$

The intersection lattice is a boolean algebra on 3 elements. $\{A_1, A_2, A_3\} \subseteq \mathcal{L}(\mathcal{A})$ is a combinatorial building set, in fact, it is the set of irreducibles in the abstract lattice. However, the minimal building set in the sense of DeConcini & Procesi is $\{A_1, A_2, A_3, A_2 \cap A_3\}$.

The main difference can be formulated in the following way: our constructions are order-theoretically canonical for a given semilattice. The set of combinatorial building sets, in particular the set of irreducible elements, depends only on the semilattice itself and not on the geometry of the subspace arrangement which it encodes. See Proposition 4.4 for the complete explanation.

4.1.2. *Local subspace arrangements.* In order to trace the DeConcini-Procesi construction step by step we need the more general notion of a local subspace arrangement.

Definition 4.2. *Let M be a smooth complex manifold and $\mathcal{A}$ a union of finitely many smooth submanifolds of M such that all non-empty intersections of submanifolds in $\mathcal{A}$ are connected smooth submanifolds. $\mathcal{A}$ is called a* **local subspace arrangement** *if for any $x \in \mathcal{A}$ there exists an open set N in M with $x \in N$, a subspace arrangement $\widetilde{\mathcal{A}}$ in $\mathbb{C}^n$, and a biholomorphic map $\phi : N \to \mathbb{C}^n$, such that $\phi(N \cap \mathcal{A}) = \widetilde{\mathcal{A}}$.*

Given a subspace arrangement $\mathcal{A}$, the initial ambient space $\mathbb{C}^d$ of $\mathcal{M}(\mathcal{A})$ carries a natural stratification by the subspaces of $\mathcal{A}$ and their intersections, the poset of strata being the intersection lattice $\mathcal{L}(\mathcal{A})$ of the arrangement. For a local subspace arrangement $\mathcal{A} = \{A_1, \ldots, A_t\}$ in a manifold M we consider analogously the stratification of M by all possible intersections of A_i's, with poset of strata called the intersection semilattice and denoted by $\mathcal{L}(\mathcal{A})$.

Definition 4.3. *Let $\mathcal{A}$ be a local subspace arrangement and $\mathcal{L}(\mathcal{A})$ its intersection semilattice. For $U \in \mathcal{L}(\mathcal{A})$, $U_1, \ldots, U_k \in \mathcal{L}(\mathcal{A})$ are said to form a* **decomposition** *of U if for any $x \in U$ there exists* *an open set N with $x \in N$ and a biholomorphic map $\phi : N \to \mathbb{C}^n$, such that $\phi(N \cap U_1), \dots, \phi(N \cap U_k)$ form a decomposition of $\phi(N \cap U)$ in the sense of* Definition 4.1(1).

As in the global case, $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a **building set** *for $\mathcal{A}$ if for any $U \in \mathcal{L}(\mathcal{A})$, the set of strata $\max \mathcal{G}_{\leq U}$ gives a decomposition of U.*

We shall refer to these building sets as *geometric* building sets. The difference between combinatorial building sets and geometric ones is contained in the dimension function as is explained in the following proposition.

Proposition 4.4. *Let $\mathcal{A}$ be a local subspace arrangement with intersection semilattice $\mathcal{L}(\mathcal{A})$.*

(1) *If $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a geometric building set of $\mathcal{A}$, then it is a combinatorial building set.*
(2) *If $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a combinatorial building set of $\mathcal{L}(\mathcal{A})$, and for any $x \in \mathcal{L}(\mathcal{A})$ the sum of codimensions of its factors is equal to the codimension of x, then $\mathcal{G}$ is a geometric building set.*

4.1.3. *Intersection stratification of local arrangements after blowup.* Let a space X be given with an intersection stratification induced by a local subspace arrangement, and let G be a stratum in X. In the blowup of X at G, $\mathrm{Bl}_G X$, we find the following maximal strata:

- maximal strata in X that do not intersect with G,
- blowups of maximal strata V at $G \cap V$, $\mathrm{Bl}_{G\cap V} V$, where V is maximal in X and intersects G,
- the exceptional divisor $\widetilde{G}$ replacing G.

We consider the intersection stratification of $\mathrm{Bl}_G X$ induced by these maximal strata. We will later see (cf. proof of Proposition 4.6, and the Remark 4.5) that in case G is maximal in a building set for the local arrangement in X, then the union of maximal strata in $\mathrm{Bl}_G X$ is again a local arrangement with induced intersection stratification.

For ease of notation, let us agree here that formally blowing up an empty (non-existing) stratum has no effect on the space. We think about a stratum Y in X, intersection of all maximal strata $V_1, \dots, V_t$ that contain Y, as being replaced by the intersection of corresponding maximal strata in $\mathrm{Bl}_G X$:

$$\mathrm{Bl}_{G\cap V_1} V_1 \ \cap \ \dots \ \cap \ \mathrm{Bl}_{G\cap V_t} V_t \,, \tag{4.1}$$

(recall that $\mathrm{Bl}_{G\cap V_j} V_j = V_j$ for $G \cap V_j = \emptyset$). The intersection (4.1) being empty means that the stratum Y vanishes under blowup of G. For notational convenience, we most often retain names of strata under blowups, thereby referring to the replacement of strata described above.

Remark 4.5. Let us mention here that blowing up a stratum in a local subspace arrangement does not necessarily result in a local subspace arrangement again. Consider the following arrangement of 2 planes and 1 line in $\mathbb{C}^3$:

$$A_1 : \ y - z = 0 \,, \quad A_2 : \ y + z = 0 \,, \quad L : \ x = y = 0 \,.$$

After blowing up L, the planes A_1 and A_2 are replaced by complex line bundles over $\mathbb{C}\mathrm{P}^1$, which have a common zero section Z and a complex line Y; L is replaced by a direct product of $\mathbb{C}$ and $\mathbb{C}\mathrm{P}^1$, which intersects both line bundles in Z. The new maximal strata fail to form a local subspace arrangement in the point $Z \cap Y$.

4.1.4. *Tracing incidence structure during arrangement model construction.* We now give a more detailed description of the model construction by DeConcini & Procesi via successive blowups, and then proceed with linking our notion of combinatorial blowups to the context of arrangement models.

Let $\mathcal{A}$ be a complex subspace arrangement, $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ a building set for $\mathcal{A}$, and $\{G_1, \dots, G_t\}$ some linear extension of the partial containment order on associated strata in $\mathbb{C}^d$ such that $G_k \supset G_l$ implies $l < k$. The DeConcini-Procesi model $Y = Y_{\mathcal{G}}$ of $\mathcal{M}(\mathcal{A})$ is the result of blowing up the strata indexed by elements of $\mathcal{G}$ in the given order. Note that the linear order was chosen so that at each step the stratum which is to be blown up does *not* contain any other stratum indexed by an element of $\mathcal{G}$. At each step we consider intersection stratifications as described above, and we denote the poset of strata after blowup of G_i with $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$. For the case of a stratum G_i being empty after previous blowups remember our agreement of considering blowups of $\emptyset$ as having no effect on a space. The later Proposition 4.6 however shows that strata indexed by elements in $\mathcal{G}$ do not disappear during the sequence of blowups.

Let us remark that the combinatorial data of the initial stratification, i.e., of the arrangement, prescribes much of the geometry of $Y_{\mathcal{G}}$: the complement $Y_{\mathcal{G}} \setminus \mathcal{M}(\mathcal{A})$ is a union of smooth irreducible divisors indexed by elements of $\mathcal{G}$, and these divisors intersect if and only if the set of indices is nested in $\mathcal{G}$ [4, Thm 3.2].

Proposition 4.6. *Let $\mathcal{A}$ be an arrangement of complex subspaces, $\mathcal{G}$ a building set for $\mathcal{A}$ in the sense of DeConcini & Procesi, and $\{G_1,\dots,G_t\}$ some linear extension of the partial containment order on associated strata as described above. Let $\mathrm{Bl}\,_i^{\mathcal{G}}(\mathcal{A})$ denote the geometric result of successively blowing up strata $G_1,\dots,G_i$, for $1\le i\le t$. Then,*

1. *The poset of strata $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$ of $\mathrm{Bl}\,_i^{\mathcal{G}}(\mathcal{A})$ can be described as the result of a sequence of combinatorial blowups of the intersection lattice $\mathcal{L}=\mathcal{L}(\mathcal{A})$:*
$$\mathcal{L}_i^{\mathcal{G}}(\mathcal{A}) \;=\; \mathrm{Bl}\,_i(\mathcal{L})\,, \qquad \textit{for } 1\le i\le t\,.$$
(Recall that $\mathrm{Bl}\,_i(\mathcal{L})=\mathrm{Bl}\,_{G_i}(\mathrm{Bl}\,_{G_{i-1}}(\dots\mathrm{Bl}\,_{G_1}\mathcal{L}))$ for $1\le i\le t$.)
2. *The union of maximal strata $\mathcal{A}_i^{\mathcal{G}}$ in $\mathrm{Bl}\,_i^{\mathcal{G}}(\mathcal{A})$ is a local subspace arrangement, with $\mathcal{G}$ in $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$ being a building set for $\mathcal{A}_i^{\mathcal{G}}$ in the sense of* Definition 4.3. *(Recall that $\mathcal{G}$ here refers to the preimages of the original strata in $\mathcal{G}\subseteq\mathcal{L}(\mathcal{A})$ under the sequence of blowups.)*

Sketch of the Proof. We proceed by induction on the number of blowups. The induction start is obvious: $\mathcal{L}_0^{\mathcal{G}}(\mathcal{A})=\mathrm{Bl}\,_0(\mathcal{L})$. The union of maximal strata is the arrangement $\mathcal{A}$ itself with its given building set $\mathcal{G}$.

Assume that $\mathcal{L}_{i-1}^{\mathcal{G}}(\mathcal{A})=\mathrm{Bl}\,_{i-1}(\mathcal{L})$ for some $1\le i\le t$, the union of maximal strata $\mathcal{A}_{i-1}^{\mathcal{G}}$ in $\mathrm{Bl}\,_{i-1}^{\mathcal{G}}(\mathcal{A})$ being a local arrangement, and $\mathcal{G}$ a building set for $\mathcal{L}_{i-1}^{\mathcal{G}}(\mathcal{A})$. Let $G=G_i$ be the next stratum to be blown up. First, we proceed in 4 steps to show that $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})=\mathrm{Bl}\,_i(\mathcal{L})$. In 2 further steps we then verify the claims in (2).

Step 1: *Assign strata of* $\mathrm{Bl}\,_i^{\mathcal{G}}(\mathcal{A})$ *to elements in* $\mathrm{Bl}\,_i(\mathcal{L})$.
We distinguish two types of elements in $\mathrm{Bl}\,_i(\mathcal{L})$:
$$\begin{array}{lrl} \text{Type I}: & Y & \text{with } Y\in\mathrm{Bl}\,_{i-1}(\mathcal{L}) \text{ and } Y\not\geq G\,,\\ \text{Type II}: & [G,Y] & \text{with } Y\in\mathrm{Bl}\,_{i-1}(\mathcal{L})\,,\ Y\not\geq G\,,\\ & & \text{and } Y\vee G \text{ exists in } \mathrm{Bl}\,_{i-1}(\mathcal{L})\,. \end{array}$$

To $Y\in\mathrm{Bl}\,_i(\mathcal{L})$ of type I, assign $\mathrm{Bl}\,_{G\cap Y}Y$ (recall that blowing up an empty stratum does not change the space). Note that $\dim\mathrm{Bl}\,_{G\cap Y}Y=\dim Y$.
To $[G,Y]\in\mathrm{Bl}\,_i(\mathcal{L})$ of type II, assign $(\mathrm{Bl}\,_{G\cap Y}Y)\,\cap\,\widetilde{G}$, where $\widetilde{G}$ denotes the exceptional divisor that replaces G in $\mathrm{Bl}\,_i^{\mathcal{G}}(\mathcal{A})$. This description comprises $\widetilde{G}$ being assigned to $[G,\hat{0}]$. Note that $\dim(\mathrm{Bl}\,_{G\cap Y}Y)\,\cap\,\widetilde{G}=\dim Y-1$.
Step 2: *Reverse inclusion order on the assigned spaces coincides with the partial order on* $\mathrm{Bl}\,_i(\mathcal{L})$.
Step 3: *Each of the assigned spaces is the intersection of maximal strata in* $\mathrm{Bl}\,_i^{\mathcal{L}}(\mathcal{A})$.
Step 4: *Any intersection of maximal strata in* $\mathrm{Bl}\,_i^{\mathcal{G}}(\mathcal{A})$ *occurs as an assigned space.*
Step 5: $\mathcal{A}_i^{\mathcal{G}}$ *is a local subspace arrangement in* $\mathrm{Bl}\,_i^{\mathcal{G}}(\mathcal{A})$.
Step 6: $\mathcal{G}$ *is a building set for* $\mathcal{A}_i^{\mathcal{G}}$ *in the sense of* Definition 4.3. ☐

4.2. **Simplicial resolutions of toric varieties.**

The study of toric varieties has proved to be a field of fruitful interplay between algebraic and convex geometry: toric varieties are determined by rational polyhedral fans, and many of their algebraic geometric properties are reflected by combinatorial properties of their defining fans. We relate our general framework of building sets, nested sets and combinatorial blowups to the context of toric varieties. For background material we refer to the standard sources [1, 9].

Let X_Σ be a toric variety defined by a rational polyhedral fan Σ. Any subdivision of Σ gives rise to a proper, birational toric morphism between the associated toric varieties (cf. [1, 5.5.1]). In particular, simplicial subdivisions yield toric morphisms from quasi-smooth toric varieties to the initial variety – so-called *simplicial resolutions*. Quasi-smooth toric varieties being rational homology manifolds, such morphisms can replace smooth resolutions for (co)homological considerations.

We recall the definition of stellar subdivisions and recognize their influence on the incidence combinatorics of the fan as a combinatorial blowup in its face poset.

Definition 4.7. *Let $\Sigma = \{\sigma\}_{\sigma\in\Sigma} \subseteq \mathbb{R}^d$ be a polyhedral fan. The* stellar subdivision $\mathrm{sd}(\Sigma, x)$ *of Σ in x, $x \in \operatorname{relint}\tau$, $\tau \in \Sigma$, is given by the collection of cones*

$$(\,\Sigma \setminus \mathrm{star}(\tau,\Sigma)\,) \;\cup\; \{\,\mathrm{cone}(x,\rho)\,|\,\rho \subseteq \sigma \;\; \textit{for some} \;\; \sigma \in \mathrm{star}(\tau,\Sigma)\,\}\,,$$

where $\mathrm{star}(\tau,\Sigma) = \{\sigma \in \Sigma \,|\, \tau \subseteq \sigma\}$, and $\mathrm{cone}(x,\rho)$ the closed polyhedral cone spanned by ρ and x. For brevity (and combinatorial purposes) we also talk about stellar subdivision of Σ in τ, $\mathrm{sd}(\Sigma,\tau)$, meaning any stellar subdivision in x for $x \in \operatorname{relint}\tau$.

Proposition 4.8. *Let $\mathcal{F}(\Sigma)$ be the face poset of a polyhedral fan Σ. For $\tau \in \Sigma$, the face poset of the stellar subdivision of Σ in τ can be described as the combinatorial blowup of τ in $\mathcal{F}(\Sigma)$:*

$$\mathcal{F}(\mathrm{sd}(\Sigma,\tau)) \;=\; \mathrm{Bl}_{\,\tau}\mathcal{F}(\Sigma)\,.$$

We apply our Main Theorem to the present context.

Theorem 4.9. *Let Σ be a polyhedral fan in $\mathbb{R}^d$ with face poset $\mathcal{F}(\Sigma)$. Let $\mathcal{G} \subseteq \mathcal{F}(\Sigma)$ be a building set of $\mathcal{F}(\Sigma)$ in the sense of Definition 2.1, $\mathcal{N}(\mathcal{G})$ the complex of nested sets in $\mathcal{G}$ (cf. Definition 2.5). Then, the consecutive application of stellar subdivision in every cone $G \in \mathcal{G}$ in a non-increasing order yields a simplicial subdivision of Σ with face poset equal to the face poset of $\mathcal{N}(\mathcal{G})$.*

As examples of building sets for face posets of polyhedral fans let us mention:

(1) the full set of faces, with the corresponding complex of nested sets being the order complex of $\mathcal{F}(\Sigma)$ (stellar subdivision in all cones results in the barycentric subdivision of the fan);
(2) the set of rays together with the non-simplicial faces of Σ;
(3) the set of irreducible elements in $\mathcal{F}(\Sigma)$: the set of rays together with all faces of Σ that are not products of some of their proper faces.

Remark 4.10. For a smooth toric variety X_Σ, the union of closed codimension 1 torus orbits is a local subspace arrangement, in particular, the codimension 1 orbits form a divisor with normal crossings. The intersection stratification of this local arrangement coincides with the torus orbit stratification of the toric variety. For any face τ in the defining fan Σ, the torus orbit $\mathcal{O}_\tau$ together with all orbits corresponding to rays in Σ form a geometric building set. Our proof in 4.1.4 applies in this context with $\mathcal{O}_\tau$ playing the role of G. We conclude that under blowup of X_Σ in the closed torus orbit $\mathcal{O}_\tau$, the incidence combinatorics of torus orbits changes exactly in the way described by a stellar subdivision of Σ in τ. This is the combinatorial part of the well-known fact that in the smooth case, the blowup of X_Σ in a torus orbit $\mathcal{O}_\tau$ corresponds to a regular stellar subdivision of the fan Σ in τ [12].

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