TORIC IDEALS OF ROOT SYSTEMS

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ABSTRACT. The present paper is based on a joint work with Takayuki Hibi. Let $\Phi \subset \mathbb{Z}^n$ be one of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$ and write Φ^+ for the set of positive roots of Φ. Let, in addition, $\widetilde{\Phi}^+$ be the set Φ^+ together with the origin of $\mathbb{R}^n$. Let $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ denote the Laurent polynomial ring $K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ over a field K and let $\mathcal{R}_K[\Phi^+] = K[\{x_\mathbf{a}\,;\, \mathbf{a} \in \Phi^+\}]$ denote the polynomial ring over K. The toric ideal I_{Φ^+} $(\subset \mathcal{R}_K[\Phi^+])$ of Φ^+ is the kernel of the homomorphism $\pi : \mathcal{R}_K[\Phi^+] \to K[\mathbf{t}, \mathbf{t}^{-1}, s]$ defined by setting $\pi(x_\mathbf{a}) = \mathbf{t}^\mathbf{a} s$ for all $\mathbf{a} \in \Phi^+$, where $\mathbf{t}^\mathbf{a} = t_1^{a_1} \cdots t_n^{a_n}$ if $\mathbf{a} = (a_1, \ldots, a_n)$. We can define $I_{\widetilde{\Phi}^+}$ by the same way as above. In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov discovered a quadratic squarefree initial ideal of the toric ideal $I_{\widetilde{\mathbf{A}}_{n-1}^+}$ of $\widetilde{\mathbf{A}}_{n-1}^+$. The purpose of the present paper is to discuss (a) the existence of a reverse lexicographic squarefree quadratic initial ideal of the toric ideal of each of $\widetilde{\mathbf{B}}_n^+$, $\widetilde{\mathbf{C}}_n^+$, $\widetilde{\mathbf{D}}_n^+$ and $\widetilde{\mathbf{BC}}_n^+$; (b) which subconfiguration $\widetilde{\mathcal{A}} = \mathcal{A} \cup \{(0, 0, \ldots, 0)\}$ of $\widetilde{\mathbf{BC}}_n^+$ possesses a (regular) unimodular triangulation (i.e., which $I_{\widetilde{\mathcal{A}}}$ posseses a squarefree initial ideal); (c) which subconfiguration $\mathcal{A}$ of $\mathbf{BC}_n^+$ possesses a unimodular triangulation or a unimodular covering.

1. INTRODUCTION

A *configuration* in $\mathbb{R}^n$ is a finite set $\mathcal{A} \subset \mathbb{Z}^n$. Let $\mathrm{conv}(\mathcal{A})$ denote the convex hull of $\mathcal{A}$ in $\mathbb{R}^n$ and write $\sharp(\mathcal{A})$ for the cardinality of $\mathcal{A}$ (as a finite set). A subset $F \subset \mathcal{A}$ is said to be a *simplex belonging to* $\mathcal{A}$ if $\mathrm{conv}(F)$ is a simplex in $\mathbb{R}^n$ of dimension $\sharp(F) - 1$. A *triangulation* of $\mathcal{A}$ is a collection Δ of simplices belonging to $\mathcal{A}$ such that (i) if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$; (ii) $\mathrm{conv}(F) \cap \mathrm{conv}(F') = \mathrm{conv}(F \cap F')$ for all $F, F' \in \Delta$, and (iii) $\mathrm{conv}(\mathcal{A}) = \bigcup_{F \in \Delta} \mathrm{conv}(F)$. Such a triangulation Δ of $\mathcal{A}$ is called *unimodular* if the normalized volume [13, p. 36] of $\mathrm{conv}(F)$ is equal to 1 for each $F \in \Delta$ with $\dim \mathrm{conv}(F) = \dim \mathrm{conv}(\mathcal{A})$. A *unimodular covering* of $\mathcal{A}$ is a collection Δ of simplices belonging to $\mathcal{A}$ such that (i) for each $F \in \Delta$, $\dim \mathrm{conv}(F) = \dim \mathrm{conv}(\mathcal{A})$ and the normalized volume of $\mathrm{conv}(F)$ is equal to 1, and (ii) $\mathrm{conv}(\mathcal{A}) = \bigcup_{F \in \Delta} \mathrm{conv}(F)$.

Let $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ denote the Laurent polynomial ring over a field K. We associate each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with the monomial $\mathbf{t}^\alpha s = t_1^{\alpha_1} \cdots t_n^{\alpha_n} s \in K[\mathbf{t}, \mathbf{t}^{-1}, s]$ and write $K[\mathcal{A}]$ for the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by all monomials $\mathbf{t}^\alpha s$ with $\alpha \in \mathcal{A}$. Let $K[\mathbf{x}] = K[\{x_\alpha\,;\, \alpha \in \mathcal{A}\}]$ denote the polynomial ring in $\sharp(\mathcal{A})$ variables over K and $I_\mathcal{A} \subset K[\mathbf{x}]$ the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[\mathcal{A}]$ defined by setting $\pi(x_\alpha) = \mathbf{t}^\alpha s$ for all $\alpha \in \mathcal{A}$. The ideal $I_\mathcal{A}$ is called the *toric ideal* of the configuration $\mathcal{A}$.

Let $<$ be a monomial order ([1, p. 53], [14, p. 9]) on $K[\mathbf{x}]$ and $in_<(I_\mathcal{A}) \subset K[\mathbf{x}]$ the initial ideal ([1, p. 73], [14, p. 10]) of $I_\mathcal{A}$ with respect to $<$. Let $\sqrt{in_<(I_\mathcal{A})}$ denote the radical ideal of $in_<(I_\mathcal{A})$ and $\Delta_<(\mathcal{A}) = \{F \subset \mathcal{A}\,;\, \prod_{\alpha \in F} x_\alpha \notin \sqrt{in_<(I_\mathcal{A})}\}$. It then follows that $\Delta_<(\mathcal{A})$ is a triangulation of $\mathcal{A}$, called the *regular triangulation* of $\mathcal{A}$ with respect to the monomial order $<$. It is known [13, Corollary 8.9] that $\Delta_<(\mathcal{A})$ is unimodular if and only if $in_<(I_\mathcal{A})$ is squarefree, i.e., $in_<(I_\mathcal{A}) = \sqrt{in_<(I_\mathcal{A})}$.

Recently, the following six properties on a configulation $\mathcal{A}$ have been investigated by many papers on commutative algebra and combinatorics:

(i) $\mathcal{A}$ is unimodular, i.e., all triangulations of $\mathcal{A}$ are unimodular;

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(ii) $\mathcal{A}$ is compressed, i.e., the regular triangulation with respect to any reverse lexicographic monomial order is unimodular;
(iii) $\mathcal{A}$ possesses a regular unimodular triangulation;
(iv) $\mathcal{A}$ possesses a unimodular triangulation;
(v) $\mathcal{A}$ possesses a unimodular covering;
(vi) $\mathcal{A}$ is normal, i.e., the semigroup ring $K[\mathcal{A}]$ is normal.
The hierarchy (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) is easy to prove; while the converse of each of the five implications is false.

Fix $n \geq 2$. Let $\mathbf{e}_i$ denote the i-th unit coordinate vector of $\mathbb{R}^n$. We write $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$, $\mathbf{D}_n^+$ and $\mathbf{BC}_n^+$ for the set of positive roots of root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$, respectively ([4, pp. 64 – 65]):

$$
\begin{aligned}
&\mathbf{A}_{n-1}^+ = \{\mathbf{e}_i - \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\};\\
&\mathbf{B}_n^+ = \{\mathbf{e}_i \, ; \, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\};\\
&\mathbf{C}_n^+ = \{2\mathbf{e}_i \, ; \, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\};\\
&\mathbf{D}_n^+ = \{\mathbf{e}_i + \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\};\\
&\mathbf{BC}_n^+ = \mathbf{B}_n^+ \cup \mathbf{C}_n^+.
\end{aligned}
$$

Let, in addition, $\widetilde{\mathbf{\Phi}}^+ = \mathbf{\Phi}^+ \cup \{(0,0,\ldots,0)\}$, where $\mathbf{\Phi} = \mathbf{A}_{n-1}, \mathbf{B}_n, \mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{BC}_n$ and where $(0,0,\ldots,0)$ is the origin of $\mathbb{R}^n$. An explicit regular unimodular triangulation of the configuration $\widetilde{\mathbf{A}}_{n-1}^+$ is constructed in [3, Theorem 6.3]. Moreover, for *any* subconfiguration $\mathcal{A}$ of $\mathbf{A}_{n-1}^+$, the configuration $\widetilde{\mathcal{A}} = \mathcal{A} \cup (0,0,\ldots,0)$ possesses a regular unimodular triangulation ([10, Example 2.4 (a)]). Stanley [11, Exercise 6.31 (b), p. 234] computed the Ehrhart polynomial of the convex polytope $\mathrm{conv}(\widetilde{\mathbf{A}}_{n-1}^+)$. Recently, Fong [2] constructs certain triangulations of the configurations $\widetilde{\mathbf{B}}_n^+$ $(= \mathrm{conv}(\widetilde{\mathbf{D}}_n^+) \cap \mathbb{Z}^n)$ and $\mathrm{conv}(\widetilde{\mathbf{C}}_n^+) \cap \mathbb{Z}^n$ $(= \widetilde{\mathbf{BC}}_n^+)$, and computes the Ehrhart polynomials of $\mathrm{conv}(\widetilde{\mathbf{B}}_n^+)$ and $\mathrm{conv}(\widetilde{\mathbf{C}}_n^+)$. The triangulations studied in [2] are, however, non-unimodular and it seems to be reasonable to ask if the configurations $\widetilde{\mathbf{B}}_n^+$, $\widetilde{\mathbf{C}}_n^+$, $\widetilde{\mathbf{D}}_n^+$ and $\widetilde{\mathbf{BC}}_n^+$ possess unimodular triangulations.

Our goal is to discuss
(a) the existence of quadratic initial ideals of $I_{\widetilde{\mathbf{\Phi}}^+}$;
(b) which subconfiguration $\widetilde{\mathcal{A}} = \mathcal{A} \cup \{(0,0,\ldots,0)\}$ of $\widetilde{\mathbf{BC}}_n^+$ possesses a unimodular triangulation;
(c) which subconfiguration $\mathcal{A}$ of $\mathbf{BC}_n^+$ possesses a unimodular triangulation or a unimodular covering.
All subconfigurations of $\{2\mathbf{e}_i \, ; \, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j \, ; \, 1 \leq i < j \leq n\}$ having unimodular coverings are completely classified ([6]). See also [12].

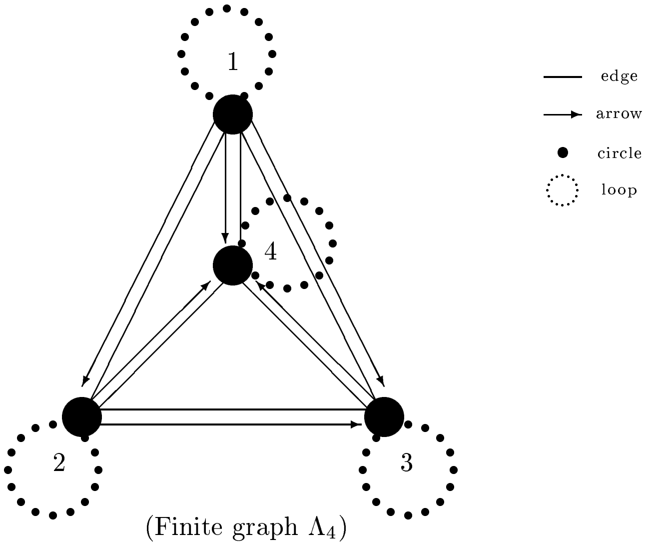
2. SEMIGROUP RINGS ASSOCIATED WITH ROOT SYSTEMS

Fix $n \geq 2$. Let $[\,n\,] = \{1, 2, \ldots, n\}$ be the vertex set and write Λ_n for the finite graph on $[\,n\,]$ consisting of the edges $\{i,j\}$, $1 \leq i \neq j \leq n$, the arrows (i,j), $1 \leq i < j \leq n$, the circles γ_i, $1 \leq i \leq n$, and the loops δ_i, $1 \leq i \leq n$. Let $E(\Lambda_n)$ denote the set of edges, arrows, circles and loops of Λ_n.

A *subgraph* of Λ_n is a pair $\Sigma = (V(\Sigma), E(\Sigma))$ of $(\emptyset \neq)\ V(\Sigma) \subset [\,n\,]$ and $E(\Sigma) \subset E(\Lambda_n)$ such that
(i) if either $\{i,j\} \in E(\Sigma)$ or $(i,j) \in E(\Sigma)$, then $i, j \in V(\Sigma)$;
(ii) if either $\gamma_i \in E(\Sigma)$ or $\delta_i \in E(\Sigma)$, then $i \in V(\Sigma)$;
(iii) each $i \in V(\Sigma)$ is a vertex of some $\xi \in E(\Sigma)$.
Fix a subgraph G of Λ_n. Let $\rho : E(\Lambda_n) \to \mathbb{Z}^n$ denote the map defined by setting $\rho(\{i,j\}) = \mathbf{e}_i + \mathbf{e}_j$, $\rho((i,j)) = \mathbf{e}_i - \mathbf{e}_j$, $\rho(\gamma_i) = \mathbf{e}_i$ and $\rho(\delta_i) = 2\mathbf{e}_i$. The map $\rho : E(\Lambda_n) \to \mathbb{Z}^n$ enables us to associate G with the configurations

$$
\begin{aligned}
\mathcal{A}(G) &= \{\rho(\xi) \, ; \, \xi \in E(G)\};\\
\widetilde{\mathcal{A}}(G) &= \mathcal{A}(G) \cup \{(0,0,\ldots,0)\}
\end{aligned}
$$

in $\mathbb{R}^n$. Here $(0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$.



(Finite graph Λ_4)

Let K be a field. The subalgebra $K[\mathcal{A}(G)]$ of $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ is generated by the monomials $t_i t_j s$ with $\{i, j\} \in E(G)$, $t_i t_j^{-1} s$ with $(i, j) \in E(G)$, $t_i s$ with $\gamma_i \in E(G)$ and $t_i^2 s$ with $\delta_i \in E(G)$. In addition, $K[\widetilde{\mathcal{A}}(G)]$ is generated by the above monomials together with s, i.e., $K[\widetilde{\mathcal{A}}(G)] = (K[\mathcal{A}(G)])[s]$.

Let $\mathcal{R}_K[\Lambda_n]$ and $\widetilde{\mathcal{R}}_K[\Lambda_n]$ denote the polynomial rings

$$\begin{aligned}
\mathcal{R}_K[\Lambda_n] &= K[\{y_i\}_{1 \le i \le n} \cup \{z_i\}_{1 \le i \le n} \cup \{e_{i,j}\}_{1 \le i < j \le n} \cup \{f_{i,j}\}_{1 \le i < j \le n}];\\
\widetilde{\mathcal{R}}_K[\Lambda_n] &= K[\{x\} \cup \{y_i\}_{1 \le i \le n} \cup \{z_i\}_{1 \le i \le n} \cup \{e_{i,j}\}_{1 \le i < j \le n} \cup \{f_{i,j}\}_{1 \le i < j \le n}]
\end{aligned}$$

over K, and set

$$\begin{aligned}
\mathcal{R}_K[G] &= K[\{y_i\}_{\gamma_i \in E(G)} \cup \{z_i\}_{\delta_i \in E(G)} \cup \{e_{i,j}\}_{\{i,j\} \in E(G)} \cup \{f_{i,j}\}_{(i,j) \in E(G)}];\\
\widetilde{\mathcal{R}}_K[G] &= K[\{x\} \cup \{y_i\}_{\gamma_i \in E(G)} \cup \{z_i\}_{\delta_i \in E(G)} \cup \{e_{i,j}\}_{\{i,j\} \in E(G)} \cup \{f_{i,j}\}_{(i,j) \in E(G)}].
\end{aligned}$$

Write $\pi : \widetilde{\mathcal{R}}_K[\Lambda_n] \to K[\mathbf{t}, \mathbf{t}^{-1}, s]$ for the homomorphism defined by setting $\pi(x) = s$, $\pi(y_i) = t_i s$, $\pi(z_i) = t_i^2 s$, $\pi(e_{i,j}) = t_i t_j s$ and $\pi(f_{i,j}) = t_i t_j^{-1} s$. If $\operatorname{Ker} \pi$ denote the kernel of π, then the toric ideals $I_{\mathcal{A}(G)}$ of $\mathcal{A}(G)$ and $I_{\widetilde{\mathcal{A}}(G)}$ of $\widetilde{\mathcal{A}}(G)$ is

$$\begin{aligned}
I_{\mathcal{A}(G)} &= \operatorname{Ker} \pi \cap \mathcal{R}_K[G];\\
I_{\widetilde{\mathcal{A}}(G)} &= \operatorname{Ker} \pi \cap \widetilde{\mathcal{R}}_K[G].
\end{aligned}$$

The research object of the present paper is the configurations both $\mathcal{A}(G)$ and $\widetilde{\mathcal{A}}(G)$ $(\subset \widetilde{\mathcal{A}}(\Lambda_n))$ associated with root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. To simplify the notation, we write A_{n-1}, B_n, C_n, D_n and BC_n for the subgraphs of Λ_n with

$$\begin{aligned}
&E(A_{n-1}) = \{(i, j)\,;\, 1 \le i < j \le n\};\\
&E(B_n) = \{\gamma_i\,;\, 1 \le i \le n\} \cup \{\{i, j\}\,;\, 1 \le i < j \le n\} \cup \{(i, j)\,;\, 1 \le i < j \le n\};\\
&E(C_n) = \{\delta_i\,;\, 1 \le i \le n\} \cup \{\{i, j\}\,;\, 1 \le i < j \le n\} \cup \{(i, j)\,;\, 1 \le i < j \le n\};\\
&E(D_n) = \{\{i, j\}\,;\, 1 \le i < j \le n\} \cup \{(i, j)\,;\, 1 \le i < j \le n\};\\
&E(BC_n) = E(B_n) \cup E(C_n),
\end{aligned}$$

respectively. Note that $BC_n = \Lambda_n$.

First, we are interested in the questions when the toric ideal of an affine semigroup ring is generated by quadratic binomials as well as when the toric ideal of an affine semigroup ring possesses a quadratic initial ideal. Consult, e.g., [8] and [7]. First of all, we study the unimodular triangulation of $\widetilde{\mathbf{A}}_{n-1}^{+}$ constructed in [3]. Let $<_{lex}$ denote the lexicographic monomial order on the polynomial ring

$$\widetilde{\mathcal{R}}_K[A_{n-1}] = K[\{x\} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

over K induced by the ordering of the variables

$$f_{1,2} > f_{1,3} > \cdots > f_{1,n} > f_{2,3} > \cdots > f_{n-1,n} > x,$$

and let $<_{rev}$ denote the reverse lexicographic monomial order on $\widetilde{\mathcal{R}}_K[A_{n-1}]$ induced by the ordering of the variables

$$x < f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < \cdots < f_{n-1,n}.$$

Then each of the initial ideals $in_{<_{lex}}(I_{\widetilde{\mathbf{A}}_{n-1}^{+}})$ and $in_{<_{rev}}(I_{\widetilde{\mathbf{A}}_{n-1}^{+}})$ of the toric ideal $I_{\widetilde{\mathbf{A}}_{n-1}^{+}}$ is generated by the squarefree quadratic monomials $f_{i,k}f_{j,\ell}$ with $1 \leq i < j < k < \ell \leq n$ and $f_{i,j}f_{j,k}$ with $1 \leq i < j < k \leq n$.

We have the same result for other root systems:

Theorem 1. *Fix $n \geq 2$. Let $G \in \{B_n, C_n, D_n, BC_n\}$. Then, there exists a reverse lexicographic order $<$ such that the initial ideal $in_<(I_{\widetilde{\mathcal{A}}(G)})$ of the toric ideal $I_{\widetilde{\mathcal{A}}(G)}$ with respect to $<$ is generated by squarefree quadratic monomials.*

Second, we study the problem which subconfiguration $\widetilde{\mathcal{A}} = \mathcal{A} \cup \{(0,0,\ldots,0)\}$ of $\widetilde{\mathbf{BC}}_n^{+}$ possesses a unimodular triangulation. For a subgraph G of A_{n-1}, the following is known: Since the matrix associated with A_{n-1} is totally unimodular and since $(0,0,\ldots,0) \in \widetilde{\mathcal{A}}(G)$, by virtue of [10, Example 2.4 (a)] the initial ideal $in_<(I_{\widetilde{\mathcal{A}}(G)})$ is squarefree for *any* reverse lexicographic monomial order $<$ on the polynomial ring $\widetilde{\mathcal{R}}_K[G]$ with $x < f_{i,j}$ for all $(i,j) \in E(G)$. We fix the reverse lexicographic monomial order $<_{\Lambda_n}$ on the polynomial ring $\widetilde{\mathcal{R}}_K[\Lambda_n]$ in n^2+n+1 variables over a field K induced by the ordering of the variables

$$\begin{aligned} y_1 < y_2 &< \cdots < y_n < x < f_{1,2} < f_{1,3} < \cdots < f_{1,n} < f_{2,3} < \cdots < f_{n-1,n} \\ &< e_{1,2} < e_{1,3} < \cdots < e_{1,n} < e_{2,3} < \cdots < e_{n-1,n} < z_1 < z_2 < \cdots < z_n. \end{aligned}$$

If G is a subgraph of Λ_n, then we write $<_G$ for the reverse lexicographic monomial order on $\widetilde{\mathcal{R}}_K[G]$ obtained by $<_{\Lambda_n}$ with the elimination of variables; in other words, for monomials u and v of $\widetilde{\mathcal{R}}_K[G]$, $u <_G v$ if and only if $u <_{\Lambda_n} v$.

We can show the following for the subconfiguration of BC_n:

Theorem 2. *Fix $n \geq 2$. Let G be a subgraph of Λ_n satisfying the following conditions:*

(2.1) *All edges of Λ_n belong to G;*

(2.2) *If $1 \leq i < j < k \leq n$ and if the arrows (i,j) and (j,k) belong to G, then the arrow (i,k) belongs to G;*

(2.3) *Either all circles of Λ_n belong to G or no circle of Λ_n belongs to G.*

Then the regular triangulation $\Delta_{<_G}(\widetilde{\mathcal{A}}(G))$ of the configuration $\widetilde{\mathcal{A}}(G)$ with respect to the reverse lexicographic monomial order $<_G$ is unimodular, i.e., the initial ideal $in_{<_G}(I_{\widetilde{\mathcal{A}}(G)})$ is generated by squarefree monomials.

The situation for $\mathcal{A}(G)$ is, however, completely different from that of $\widetilde{\mathcal{A}}(G)$ and, in general, $\mathcal{A}(G)$ is not normal. For example, if $n=5$ and G is a subgraph of A_4 with

$$E(G) = \{(1,2),(2,3),(3,4),(4,5),(1,5),(1,4)\},$$

then $I_{\mathcal{A}(G)} = (e_{1,2}e_{1,5}^2e_{2,3}e_{3,4} - e_{1,4}^3e_{4,5}^2)$ and $\mathcal{A}(G)$ is non-normal.

Nevertheless, we have the following for $G \in \{A_{n-1}, B_n, BC_n\}$:

Proposition 3. *The configuration $\mathcal{A}(A_{n-1})$ associated with the root system $\mathbf{A}_{n-1}$ possesses a regular unimodular triangulation. More precisely, if $<$ is the reverse lexicographic monomial order on $\mathcal{R}_K[A_{n-1}]$ induced by the ordering of the variables*

$$f_{1,2} < f_{1,3} < \cdots < f_{1,n} < f_{2,3} < \cdots < f_{n-1,n},$$

then the initial ideal $in_<(I_{\mathcal{A}(A_{n-1})})$ of the toric ideal $I_{\mathcal{A}(A_{n-1})}$ with respect to $<$ is squarefree.

Theorem 4. *Fix $n \geq 2$. Let $G \in \{B_n, BC_n\}$. Then, $\mathcal{A}(G)$ possesses a regular unimodular triangulation.*

We now discuss the existence of unimodular coverings of subconfigurations of $\mathbf{A}_{n-1}^+$. For a subgraph G of Λ_n with $E(G) \subset \{\{i,j\}\,;\,1 \leq i < j \leq n\}$, a combinatorial characterization for the toric ideal of the configuration $\mathcal{A}(G)$ to be generated by quadratic binomials is obtained in [8, Theorem 1.2]. It is known [7] that if G is, in addition, bipartite, then the toric ideal of the configuration $\mathcal{A}(G)$ possesses an initial ideal generated by quadratic monomials if and only if every cycle Γ of even length ≥ 6 appearing in G possesses at least one "chord," i.e., an edge $\xi = \{v,w\} \in E(G)$ with $v \in V(\Gamma)$, $w \in V(\Gamma)$ and $\xi \notin E(\Gamma)$.

All normal subconfigurations of $\{2\mathbf{e}_i\,;\,1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j\,;\,1 \leq i < j \leq n\}$ are completely classified ([6], [12]). More precisely, [6, Corollary 2.3] says that, for a connected subgraph G of Λ_n with $E(G) \subset \{\{i,j\}\,;\,1 \leq i < j \leq n\} \cup \{\delta_i\,;\,1 \leq i \leq n\}$, the following conditions are equivalnet:

(i) $\mathcal{A}(G)$ is normal;

(ii) $\mathcal{A}(G)$ possesses a unimodular covering;

(iii) If each of Γ_1 and Γ_2 is either a loop or an odd cycles of length ≥ 3 appearing in G and if Γ_1 and Γ_2 possess no common vertex, then there is a "bridge" between Γ_1 and Γ_2, i.e., an edge $\{v_1, v_2\} \in E(G)$ with $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$.

Question 5. *Find a combinatorial characterization of subgraphs G of A_{n-1} such that the configuration $\mathcal{A}(G)$ possesses a unimodular covering.*

We now turn to the discussion of the existence of a unimodular covering of a subconfiguration $\mathcal{A} \subset \mathbf{A}_{n-1}^+$, where $n \geq 2$, satisfying the condition (2.2). The fundamental technique to prove the following theorem is already developed in [6].

Theorem 6. *Fix $n \geq 2$. Let G be a subgraph of A_{n-1} satisfying the condition (2.2) appearing in Theorem 2. Then, $\mathcal{A}(G)$ possesses a unimodular covering.*

References

[1] D. Cox, J. Little and D. O'Shea, "Ideals, Varieties and Algorithms," Springer–Verlag, Berlin, Heidelberg, New York, 1992.

[2] W. Fong, Triangulations and Ehrhart polynomials of polytopes associated with root systems, Dissertation, M. I. T., June, 2000.

[3] I. M. Gelfand, M. I. Graev and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, *in* "Arnold–Gelfand Mathematics Seminars, Geometry and Singularity Theory" (V. I. Arnold, I. M. Gelfand, M. Smirnov and V. S. Retakh, Eds.), Birkhäuser, Boston, 1997, pp. 205 – 221.

[4] J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Second Printing, Revised, Springer–Verlag, Berlin, Heidelberg, New York, 1972.

[5] H. Ohsugi, J. Herzog and T. Hibi, Combinatorial pure subrings, *Osaka J. Math.* **37** (2000), 745 – 757.

[6] H. Ohsugi and T. Hibi, Normal polytopes arising from finite graphs, *J. Algebra* **207** (1998), 409 – 426.

[7] H. Ohsugi and T. Hibi, Koszul bipartite graphs, *Advances in Applied Math.* **22** (1999), 25 – 28.

[8] H. Ohsugi and T. Hibi, Toric ideals generated by quadratic binomials, *J. Algebra* **218** (1999), 509 – 527.

[9] H. Ohsugi and T. Hibi, Compressed polytopes, initial ideals and complete multipartite graphs, *Illinois J. Math.* **44** (2000), 391 – 406.

[10] R. P. Stanley, Decompositions of rational convex polytopes, *Ann. Discrete Math.* **6** (1980), 333 – 342.

[11] R. P. Stanley, "Enumerative Combinatorics, Volume II," Cambridge University Press, Cambridge, New York, Sydney, 1999.

[12] A. Simis, W. V. Vasconcelos and R. H. Villarreal, The integral closure of subrings associated to graphs, *J. Algebra* **199** (1998), 281 – 289.
[13] B. Sturmfels, "Gröbner Bases and Convex Polytopes," Amer. Math. Soc., Providence, RI, 1995.
[14] W. V. Vasconcelos, "Computational Methods in Commutative Algebra and Algebraic Geometry," Springer–Verlag, Berlin, Heidelberg, New York, 1998.

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