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ABSTRACT. We consider quotients of finitely generated Coxeter groups under the weak order. Björner and Wachs proved in [1] that every such quotient is a meet semi-lattice, and in the finite case is a lattice. Our result is that the quotient of an affine Weyl group by the corresponding finite Weyl group is a lattice, and that up to isomorphism, these are the only quotients of infinite Coxeter groups that are lattices. In this paper, we restrict our attention to the non-affine case; the affine case appears in [6]. We reduce to the hyperbolic case by an argument using induced subgraphs of Coxeter graphs. Within each quotient, we produce a set of elements with no common upper bound, generated by a Maple program. The number of cases is reduced because the sets satisfy the following conjecture: if a set of elements does not have an upper bound in a particular Coxeter group, then it does not have an upper bound in any Coxeter group whose graph can be obtained from the graph of the original group by increasing edge weights.

RÉSUMÉ. Nous considérons les quotients des groupes de Coxeter générés d'une façon finie sous l'ordre dit faible. Björner et Wachs ont établi dans [1] que tous ces quotients sont des demi treillis, et dans le cas fini sont des treillis. Notre résultat est que le quotient du groupe de Weyl affine par le groupe de Weyl fini correspondant est un treillis, et que les quotients isomorphes à ces quotients sont les seuls quotients de groupes de Coxeter infinis qui sont des treillis. Dans cette communication, nous limiterons notre attention au cas qui n'est pas affine; le cas affine est dans [6]. Nous réduisons au cas hyperbolique par un argument avec les sous-graphes induis. Dans chaque quotient, nous produisons un ensemble d'éléments sans supremum, engendré par un programme de Maple. Le nombre de cas est reduit parce que les ensembles satisfont à la conjecture suivante : si un ensemble d' éléments n'a pas de supremum dans un groupe de Coxeter particulier, alors il n'a pas de supremum dans n'importe quel groupe de Coxeter dont le graphe de Coxeter peut être obtenu du graphe du groupe original en augmentant les étiquettes des arêtes.

1. INTRODUCTION

Let (W, S) be a finitely generated *Coxeter system*, that is, let S be a finite set and let W be the group generated by S subject to the relations that each generator has order two and possibly additional relations specifying the orders of the products of some or all pairs of distinct generators. We call W a *Coxeter group*. For $w = s_1 \ldots s_k \in W$, with $s_i \in S$ for all i, and k minimal, we say $s_1 \ldots s_k$ is a *reduced expression* for w and $l(w) = k$ is the *length* of w. For $x, y \in W$, we write $x <_L y$ if there is a reduced expression for y which ends with a reduced expression for x; this partial ordering on W is called the (left) *weak order*.

For $J \subset S$, the subgroup of W generated by J is denoted W_J and is said to be a *parabolic subgroup*. Consider the corresponding (parabolic) *quotient* W/W_J. It is well-known [2] that

$$W^J := \{w \in W : l(ws) > l(w) \text{ for all } s \in J\}$$

is the set of *minimal coset representatives* for W/W_J, that is, each element of W^J is the unique element of minimal length within its coset. If $J = S - \{s\}$ for some $s \in S$, then the quotient W/W_J is known as the *maximal quotient* of W with respect to s, and we write $W^J = W^{\langle s \rangle}$. By popular abuse of language, we will refer to W^J as a quotient, and in particular when we make statements about quotients under the weak order, we are actually referring to the corresponding sets of minimal coset representatives as induced subposets $(W^J, <_L)$ of $(W, <_L)$.

We assume the reader is familiar with several definitions related to partially ordered sets as developed in Chapter 3 of [4].

The results in this paper are part of the author's doctoral dissertation.

The classification of the quotients which are distributive lattices under the weak order appears in [5] within the larger context of the following result:

Theorem 1 (Stembridge). *The following are equivalent:*

1. *W^J is a lattice under the Bruhat order.*
2. *W^J is a distributive lattice under the Bruhat order.*
3. *W^J is a distributive lattice under the weak order.*
4. *The weak order and Bruhat order coincide on W^J.*

The classification of these quotients in the finite case is also given in [3] within the context of the first condition and with the observation that the second condition is equivalent.

The quotients satisfying the conditions of the theorem are certain maximal quotients of finite Weyl groups known as minuscule quotients; the maximal quotients of the infinite dihedral group $I_2(\infty)$; and one maximal quotient of H_3.

In light of the equivalent conditions of Theorem 1, it is natural to ask when W^J is a lattice under the weak order. In this paper we will examine only the infinite non-affine case. In particular, we will show that no quotient of an infinite non-affine Coxeter group is a lattice. More generally, in [6], we prove that for infinite indecomposable $W, (W^J, <_L)$ is a lattice if and only if W is affine and W_J is isomorphic to the corresponding finite Weyl group; also for any $W, (W^J, <_L)$ is a lattice if and only if $W_I^{J\cap I}$ is a lattice for all $I \subset S$ such that the induced subgraph Γ_I is a connected component of Γ.

2. Reduction to the hyperbolic case

Recall that the finite Coxeter groups are precisely those for which the bilinear form matrix M is positive definite, and the affine Weyl groups are precisely those for which this matrix is positive semidefinite. The classifications of the finite and affine Coxeter groups are well-known and generally given by their Coxeter graphs [2]. We say that a matrix is of *positive type* if it is either positive definite or positive semidefinite; in case M is of positive type we will also say Γ and W are of positive type. Γ is said to be *hyperbolic* if it is not of positive type but all of its proper induced subgraphs are of positive type; similarly W is hyperbolic if it is not of positive type but all of its proper parabolic subgroups are of positive type.

Note that our definition of hyperbolic Coxeter group follows Humphreys [2] and does not include all discrete groups generated by reflections in hyperbolic space but only those with a simplex as fundamental domain.

We begin by reducing to the hyperbolic case.

Let (W, S) be a Coxeter system with positive roots Φ^+. Let Γ denote the Coxeter graph of W. For $w \in W$, let $\Phi^+(w)$ be the set of positive roots sent negative by w. We say that $\Theta = \{\alpha_1, \dots, \alpha_k\} \subset \Phi^+$ is a *forbidden configuration* if the positive linear span of Θ has infinite intersection with Φ^+. We say $\{w_1, \dots, w_k\} \subset W$ is a *forbidden configuration* if $\Phi^+(w_1) \cup \ldots \cup \Phi^+(w_k)$ is a forbidden configuration.

Our reduction is done in two parts, corresponding to the following two propositions.

Proposition 2. *Let $J \subset S$. Let j be a vertex of the induced subgraph Γ_J of Γ with vertex set J. If $F = \{w_1, w_2, \dots w_k\} \subset W_J^{\langle s_j \rangle}$ is a forbidden configuration with respect to W_J, then $F \subset W^{\langle s_j \rangle}$ is also a forbidden configuration with respect to W and therefore $W^{\langle s_j \rangle}$ is not a lattice.*

Proposition 3. *Suppose i and j are a pair of adjacent vertices of Γ with $j \in J$ and $i \notin J$. If $F = \{w_1, \dots, w_k\} \subset {W_J}^{\langle s_j \rangle}$ is a forbidden configuration with respect to W_J, then $W^{\langle s_i \rangle}$ contains a forbidden configuration and therefore is not a lattice.*

Corollary 4. *If the (connected) Coxeter graph Γ contains an induced subgraph Γ_J such that each $W_J^{\langle s_i \rangle}$ contains a forbidden configuration, then each $W^{\langle s_j \rangle}$ contains a forbidden configuration, and hence no $W^{\langle s_j \rangle}$ is a lattice.*

We demonstrate that no hyperbolic Coxeter group has any quotient for which W^J is a lattice by producing a forbidden configuration within each W^J. Consequently, for any group not of positive type, we can choose a minimal induced subgraph not of positive type, which will be hyperbolic by minimality, and then by Corollary 4 we will have that in our original group every W^J contains a forbidden configuration. So once we have shown that every W^J with W hyperbolic contains a forbidden configuration, it will follow that only groups of positive type have quotients which are lattices.

3. Description of the Maple program

Here we will describe the Maple program used to find forbidden configurations in W^J for maximal quotients of hyperbolic Coxeter groups. Our algorithm will make use of the following three facts.

Proposition 5. [2] *Let W be an arbitrary Coxeter group. For $w \in W, s \in \Delta$, we have $w^{-1}(\alpha_s) \in \Phi^-$ if and only if $l(sw) < l(w)$.*

Proposition 6. *If $\alpha, \beta \in \Phi^+$, and $\langle \alpha, \beta \rangle \leq -2$, then $\{\alpha, \beta\}$ is a forbidden configuration.*

Proposition 7. [2] *If $w = s_1 \cdots s_k$ is reduced, then*

$$\Phi^+(w) = \{\alpha_k, s_k\alpha_{k-1}, \ldots, s_k \cdots s_2\alpha_1\},$$

where α_i is the simple root corresponding to the generator s_i.

Note that as a corollary of 7, we have for all $x, y \in W, x <_L y$ if and only if $\Phi^+(x) \subset \Phi^+(y)$.

Purpose : The purpose of the program is to locate, within a given $W^J = W^{\langle s \rangle}$, a set of words having no common upper bound. In particular, we would like to find such a set which is in some sense minimal. First, we require that the longest word in our set is as short as possible; the program will generate a list of all such sets. Second, we will reduce to a list of such sets having minimal cardinality, and finally we choose the set which comes first in lexicographic order, where each word is represented by its lexicographically first reduced expression, and a set U of words is said to come before another set V if all words in $U - V$ come before the last word of $V - U$.

Input : We provide the bilinear form matrix $M, M_{ij} = \langle \alpha_i, \alpha_j \rangle$, to define the Coxeter group, and an index $i, 1 \leq i \leq n$, to specify the generator whose maximal quotient we wish to consider. We will call this generator the *distinguished generator*.

Output : The final output consists of a list of forbidden configurations. Each forbidden configuration is given both as a list of reduced expressions and as a list of roots; the roots are the last roots sent negative by the reduced expressions. If there are more than two roots, there will also be a matrix whose entries are the values of the bilinear form applied to pairs of the roots, which will be the bilinear form matrix of an infinite Coxeter system, demonstrating that there are infinitely many positive roots in the positive linear span of the roots.

Along the way, the program reports the number of elements of each length in W^J, up to and including the length of the longest word in the final output.

Algorithm : The algorithm maintains three lists: a list of elements of W^J with each element stored as its lexicographically first reduced expression, and the reduced expressions ordered first by length and then by lexicographic order; a list of roots sent negative by elements of W^J, with each root stored as its coordinates with respect to the simple roots, and the roots listed in the order in which a word sending them negative first appears in the list of words (note that a word cannot send two new roots negative since all shorter words appear before it in the list so in particular the word obtained from it by removing the first letter comes first and has already sent negative all but the last of the roots sent negative by the original word); and a list of maximal cliques of the graph whose vertices are in one-to-one correspondence with the roots sent negative, and in which two vertices are adjacent if and only if the value of the bilinear form applied to the corresponding roots is in $\{-2\cos(\pi/m) : m \in \mathbb{Z}^+\} \cup (-\infty, -2]$. Each clique is given by the indices in the root list of the roots it contains.

These lists are initialized as the unique element of length one in the quotient (whose unique reduced expression consists of the distinguished generator), the simple root corresponding to the distinguished generator, and the singleton clique consisting of the index 1.

The algorithm considers each length l of words in turn, performing the following series of steps in order to update the lists of words, roots, and cliques appropriately, and produce the desired output.

Consider each generator in turn in lexicographic order. We will examine in turn the expressions resulting from adding the current generator to each reduced expression of length $l-1$ in our existing list of words.

Step 1: Consider the expression formed by adding the current generator to the next reduced expression of length $l-1$ in the list of words. Determine whether or not this expression is a reduced expression for a word in W^J.

To accomplish this, calculate the image of the simple root corresponding to the generator we are attempting to add under the inverse of our original shorter word; by Proposition 5 this will be a negative root if the expression is not reduced and a positive root if the expression is reduced, and by Proposition 7, this will be a simple root if and only if the word is not in W^J. In particular, the sum of the coordinates of this root will be greater than 1 precisely when we have a reduced expression for a word in W^J. So we abandon this expression and skip to step 3 if the sum of the coordinates is less than or equal to 1; otherwise we continue investigating our expression.

Step 2: Determine whether this expression is the lexicographically first expression for the word it represents.

Consider the set of all reduced expressions for this word. Among them, the lexicographically first is among those which start with the smallest possible generator, and furthermore it is obtained by adding that generator to the beginning of the lexicographically first expression for the word of length one shorter. Because our list of words in W^J of length $l-1$ contains only the lexicographically first reduced expression for each word, it suffices to check that our reduced expression starts with the smallest possible generator.

Thus for each generator that comes before the generator under consideration, we must find out if our word has an expression starting with that generator, or equivalently if the inverse of our word has an expression ending with that generator. By Proposition 5, we can check this by calculating the images of the simple roots corresponding to the smaller generators under the inverse of our word; our expression will be the lexicographically first reduced expression if and only if all such images are positive. So we abandon this expression and skip to step 3 if any such image is negative.

Once we have identified an expression as the lexicographically first reduced expression for a word in W^J, we add this expression to our list of words in the quotient and proceed.

Step 3: Update the list of roots.

Calculate the last root sent negative by our reduced expression and determine whether or not it is a new root; if it is not, we skip to step 3, but if it is, we add it to our list of roots and proceed.

Step 4: Update the list of cliques.

Calculate the values of the bilinear form applied to the new root and each of the roots in our list in turn. If any of these values is less than or equal to -2, then by Proposition 6 we have found a pair of roots with infinitely many positive roots in their positive linear span, and we output these two roots and the corresponding words which first sent them negative.

For each clique in the existing list, we produce a new clique consisting of our new root and all adjacent old roots from the old clique, replacing the old clique with this new clique in the event that all roots from the old clique are adjacent to the new root and otherwise keeping the old clique. We check that our new clique has not already been produced; if it is new, we check to see if its bilinear form matrix is positive definite by testing the determinant (we index the rows and columns so our new root is last, and thus need only check the whole determinant since the remaining principal minors are known to have positive determinant because their corresponding roots have already been verified as a finite system). If we find that the matrix is not positive definite, we print out the matrix, and the corresponding roots and their corresponding words.

Step 5: Repeat steps 1 through 3 until we have exhausted the list of words in W^J of length $l-1$.
Step 6: Repeat steps 1 through 3 for each generator in turn.
Step 7: Output length information and increment length.

When we exhaust the set of generators, we output the old length and the total number of words of that length; if we have found a forbidden configuration, we terminate the program. Otherwise we increment l and repeat the whole process.

Remark 8: This process will terminate if W^J contains a forbidden configuration.

We eliminate all W^J of hyperbolic Coxeter groups as potential lattices. To do this, we provide explicit forbidden configurations for all of the $W^{\langle s\rangle}$; since any W^J contains a $W^{\langle s\rangle}$, this will establish that no W^J is a lattice.

For a reduced expression $x = s_1 \cdots s_k \in W$, we define the *last root sent negative* by $s_1 \cdots s_k$, denoted $\rho(s_1 \cdots s_k)$, or $\rho(x)$ when the choice of reduced expression is understood, to be the unique $\alpha \in \Phi^+$ such that $s_1 \cdots s_k(\alpha) \in \Phi^-$ but $s_2 \cdots s_k(\alpha) \in \Phi^+$, which by Proposition 7 is $s_k \cdots s_2\alpha_1$. In our forbidden configurations, we will list reduced expressions so that the set of last roots sent negative will be a forbidden configuration.

Here we outline a straightforward method that can be used to verify forbidden configurations. First, calculate the set of positive roots sent negative for each expression in the proposed forbidden configuration. Check that only the correct simple root is in this set, so the expression is indeed in the quotient under consideration. Use the cardinality of the root set to verify that each expression is reduced. For each pair of expressions in the proposed forbidden configuration, calculate the values of the bilinear form on the last roots sent negative by the two expressions. Finally, use these values to demonstrate that infinitely many positive roots must be sent negative by any expression that sends all of those roots negative.

It turns out that some of our forbidden configurations eliminate quotients for multiple hyperbolic Coxeter groups. The number of cases is reduced because the sets satisfy the following conjecture: if a set of elements does not have an upper bound in a particular Coxeter group, then it does not have an upper bound in any Coxeter group whose graph can be obtained from the graph of the original group by increasing edge weights.

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