MAXIMAL SINGULAR LOCI OF SCHUBERT VARIETIES IN $SL(n)/B$

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ABSTRACT. In this paper we give an explicit combinatorial description of the irreducible components of the singular locus of a Schubert variety X_w for an element $w \in \mathfrak{S}_n$. With our description, the computation of these components becomes computationally efficient ($O(n^6)$). The key to our result is a map $\phi_t : \mathcal{R}(y,w) \longrightarrow \mathcal{R}(x,w)$ where $x < y = xt \leq w$, t is a reflection, and $\mathcal{R}(x,w)$ is the set of reflections t' such that $x < xt' \leq w$. Our result depends on a characterization of the singular points of X_w in terms of the cardinality of $\mathcal{R}(x,w)$ due to Lakshmibai and Seshadri, [10].

1. Introduction

Starting with Chevalley in the 1950's [4], Schubert varieties in G/B and their singular loci have been carefully studied. Because a Schubert variety is the closure of a B-orbit, the irreducible components of the singular locus are themselves Schubert varieties. We denote the permutations indexing these components by $\mathrm{maxsing}(X_w)$ (see Definition 1).

There are many beautiful combinatorial criteria for determining smoothness at a point in X_w. These criteria can be used to find the elements of $\mathrm{maxsing}(X_w)$ by searching down through the Bruhat poset for the largest singular elements (see [1] for background). This method is computationally prohibitive in general because the size of the corresponding Weyl groups are so large. In this paper, we give the first explicit construction of the irreducible components for an arbitrary Schubert variety in $SL(n)/B$. This algorithm is very efficient, $O(n^6)$.[1]

In 1990, Lakshmibai and Sandhya [9] showed that the Schubert variety $X_w \subset SL(n)/B$ is smooth at every point if and only if the permutation matrix for w does not contain any 4×4 submatrix equal to 3412 or 4231. We use these two permutation patterns to produce the maximal permutations below w that correspond to points in the singular locus. This verifies the conjecture stated in [9]. (Gasharov, using a map similar to the one we introduce in Section 5, shows in [7] that the points constructed in [9] are singular. His result proves one direction of this conjecture.)

Our proof starts from an arbitrary maximal singular T-fixed point e_x in X_w and shows that w must contain a 4231 or 3412 pattern and x must contain a 2143 or 1324 pattern (respectively). The proof is based on

- the Lakshmibai-Seshadri basis for the tangent space of X_w at e_x [10] indexed by the Bruhat graph,
- a set of injective maps which imply Deodhar's inequality [5], and
- a two dimensional version of the tableaux criterion for Bruhat order on permutations.

In type A, smoothness is equivalent to rational smoothness ([5], [3]) so the maximal singular locus of X_w also determines the maximal permutations $x \leq w$ for which the corresponding Kazhdan-Lusztig polynomial is different from 1. We use the explicit form of $\mathrm{maxsing}(X_w)$ to compute all Kazhdan-Lusztig polynomials at maximal singular points; they are either $1 + q + \cdots + q^k$ or $1 + q^k$ depending on whether the corresponding bad pattern is 4231 or 3412 (respectively).

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[1] While in the process of preparing this submission, the authors have learned that Manivel [11] has recently independently proved a theorem equivalent to Theorem 1.

It turns out that elements of maxsing(X_w) are obtained by acting on w by certain cycles. These cycles, described in the following theorem, are best absorbed graphically in terms of the permutation matrices mat(x) and mat(w). Examples are shown in Figures 1 and 2.

We now give a precise definition of maxsing(X_w) and state our main theorem.

Definition 1. Let maxsing(X_w) be the maximal elements (under the Bruhat-Chevalley order) of the set $\{x \leq w : x \text{ is a singular point of } X_w\}$.

Theorem 1. *X_x is an irreducible component of the singular locus of X_w if and only if*

$$x = w \circ (\alpha_1, \ldots, \alpha_m, \beta_k, \ldots, \beta_1)$$

for disjoint sequences

$$1 \leq \alpha_1 < \cdots < \alpha_m \leq n, \text{ with } w(\alpha_1) > \cdots > w(\alpha_m), \text{ and}$$
$$1 \leq \beta_1 < \cdots < \beta_k \leq n, \text{ with } w(\beta_1) > \cdots > w(\beta_k),$$

the interiors of the shaded regions in Figures 1 and 2 do contain any other 1's in the permutation matrix of w, and one of the following cases holds:

(1) *4231 Case:*

$$k, m \geq 2 \text{ and}$$
$$\alpha_1 < \beta_1, \ldots, \beta_{k-1} < \alpha_2, \ldots, \alpha_m < \beta_k \text{ and}$$
$$w(\alpha_m) > w(\beta_1).$$

(2) *3412 Case:*

$$k, m \geq 2 \text{ and}$$
$$\beta_1, \ldots, \beta_{k-1} < \alpha_1 < \beta_k < \alpha_2, \ldots, \alpha_k \text{ and}$$
$$w(\alpha_{m-1}) > w(\beta_1) > w(\alpha_m) > w(\beta_2).$$

(3) *45312 Case:*

$$k = m = 2 \text{ and}$$
$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 \text{ and}$$
$$w(\alpha_1) > w(\beta_1) > w(\alpha_2) > w(\beta_2) \text{ and}$$
$$\text{entries of } \mathrm{mat}(w) \text{ in region } A \text{ of Figure 2.2 are in decreasing order.}$$

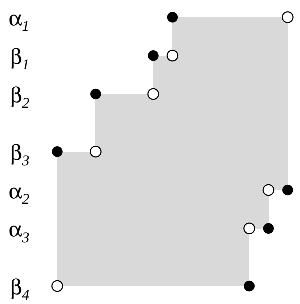
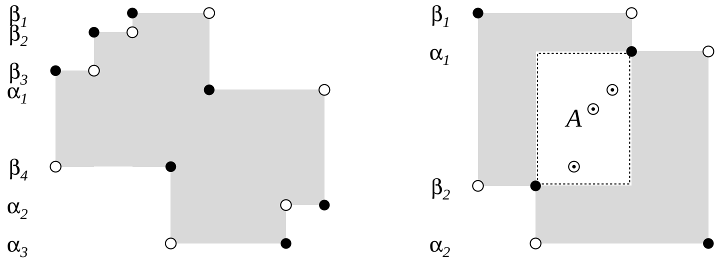


FIGURE 1. Example of Case 1 of Theorem 1. $\circ$'s denote 1's in mat(w), $\bullet$'s denote 1's in mat(x).



(1) (2)

FIGURE 2. Examples for Cases 2 and 3 of Theorem 1. For clarity in stating the theorem, region A in (2) is not shaded as it would be in the remainder of the paper. The $\odot$'s represent points where $\mathrm{mat}(x)$ and $\mathrm{mat}(w)$ both have entries.

To prove this theorem, we use the fact that (by definition) X_w is smooth at e_x if and only if the dimension of the Zariski tangent space at that point is equal to $l(w)$. Lakshmibai and Seshadri, [10], describe the dimension of this tangent space in terms of the root system. In particular, using the fact that $\#\{t \in \mathcal{T} : xt < x\} = l(x)$, we can paraphrase their result as:

Theorem 2. [10] *The Schubert variety X_w is smooth at e_x if and only if*

$$\#\mathcal{R}(x,w) := \#\{t \in \mathcal{T} : x < xt \leq w\} = l(w) - l(x). \tag{1.1}$$

Therefore, we have the following characterization of the permutations in $\mathrm{maxsing}(X_w)$:

Fact 1. $x \in \mathrm{maxsing}(X_w)$ if and only if
(1) $\#\mathcal{R}(x,w) > l(w) - l(x)$ and
(2) for all $t \in \mathcal{R}(x,w)$, we have $\#\mathcal{R}(xt,w) = l(w) - l(xt)$.

2. PRELIMINARIES

We begin by introducing our basic notation and terminology. Let $\mathfrak{S}_n$ denote the symmetric group on n letters. We will view elements of $\mathfrak{S}_n$ as permutations on $[1,\ldots,n]$. To this end, we identify s_i with the transposition $(i,i+1)$. Let $w(i)$ be the image of i under the permutation w. We have a one-line notation for a permutation w given by writing the image of $[1,\ldots,n]$ under the action of w: $[w(1),w(2),\ldots,w(n)]$. We will also often utilize the permutation matrix for w (denoted $\mathrm{mat}(w)$).

We use the standard presentation

$$\begin{aligned} \mathfrak{S}_n = \langle s_1,\ldots,s_{n-1} : s_i^2 = 1, \\ s_is_j = s_js_i \text{ for } |i-j| > 1, \text{ and} \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1}\rangle. \end{aligned} \tag{2.1}$$

Let $\mathcal{S} = \{s_i\}_{i\in[1,\ldots,n-1]}$ denote the generating set for $\mathfrak{S}_n$. Let $\mathcal{T}$ denote the set of all *transpositions* in $\mathfrak{S}_n$. The elements of $\mathcal{T}$ are all the conjugates of elements in $\mathcal{S}$:

$$\mathcal{T} = \{t_{j,k} = s_js_{j+1}\cdots s_{k-2}s_{k-1}s_{k-2}\cdots s_{j+1}s_j : 1 \leq j < k \leq n\}. \tag{2.2}$$

If we wish to refer to a transposition t that affects positions a and b, but the relative values of a and b are unknown, we will write $t_{\{a,b\}}$.

An *expression* is any product of generators s_i. The *length* $l(w)$ of an element $w \in \mathfrak{S}_n$ is the minimum r for which we have an expression $w = s_{i_1}\cdots s_{i_r}$. A *reduced expression* $w = s_{i_1}\cdots s_{i_r}$ is an expression for which $l(w) = r$. It is a standard fact that

$$l(w) = \#\{1 \leq i < j \leq n : w(i) > w(j)\}. \tag{2.3}$$

If $v, w \in \mathfrak{S}_n$, then $v \le w$ will signify that v is below w in the Bruhat (or Bruhat-Chevalley) order (see, e.g., [8]). The importance of this order comes from the correspondence

$$v \le w \Leftrightarrow X_v \subseteq X_w. \tag{2.4}$$

This order has a number of characterizations. One of the most common definitions is as the transitive closure of the relations $vt < v$ for $t \in \mathcal{T}$ if $l(vt) < l(t)$. However, shortly, we will present a description of this order that is more suitable for our discussions.

For any set $Z = \{z_1 < z_2 < \cdots < z_k\} \subseteq [1 \dots n]$, $x \in \mathfrak{S}_n$ define $\mathrm{fl}_Z(x)$ to be the "flattened" permutation on $[1 \dots k]$ whose elements are in the same relative order as $[x(z_1), x(z_2), \dots, x(z_k)]$.

It will be also useful to define the notion of *pattern avoidance*. We say that $w = [w(1), \dots, w(n)]$ *avoids* the pattern $v = [v(1), \dots, v(k)]$ for $k \le n$ if we cannot find $1 \le i_1 < \cdots < i_k \le n$ with $\mathrm{fl}_{\{i_1,\dots,i_k\}}(w) = v$.

3. Criterion for Bruhat order

Our main theorem is concerned not only with determining which points in a given Schubert variety are singular, but which are *maximally* singular (in the sense of Definition 1). To achieve this, we need to introduce some terminology that will let us relate $\mathcal{R}(x, w)$ and the Bruhat order. We now present the formulation of the Bruhat order which we will be using in this paper. We justify its suitability after giving the necessary definitions.

Definition 2. Let $x, w \in \mathfrak{S}_n$, $p, q \in \mathbb{Z}$. Define the *rank function* for w by $r_w(p, q) = \#\{i \le p : w(i) \ge q\}$. The *difference function* for the pair x, w is defined by $d_{x,w}(p, q) = r_w(p, q) - r_x(p, q)$.

Lemma 1. *We have $x \le w$ if and only if $d_{x,w}$ is everywhere non-negative.*

Corollary 3. *If $x \le y \le w$, then $d_{x,w} - d_{y,w}$ is everywhere non-negative.*

The function $d_{x,w}$ affords us a graphical view of the Bruhat order. Most importantly, it lets us see the set $\mathcal{R}(x, w)$. We will now introduce the graphical notation that is utilized in the proof of Theorem 1 as we feel it is interesting in its own right. A diagram displaying this notation is offered in Figure 3.

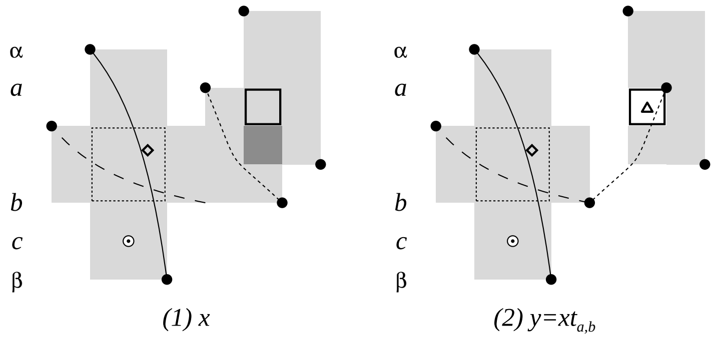


Figure 3. As mentioned in Definition 3, $\mathrm{pt}_x(c)$ denotes the position $(c, x(c))$. We see (among other facts) that $d_{x,w}(\diamond) \ge 1$, $d_{y,w}(\triangle) = d_{x,w}(\triangle) - 1 \ge 0$, $\mathrm{pt}_x(c) = \mathrm{pt}_w(c)$ and $t_{\alpha,\beta} \in \mathcal{R}(x, w)$.

First, we plot, as black disks, all or some of the positions containing 1's in the permutation matrix $\mathrm{mat}(x)$ of x. We will sometimes overlay $\mathrm{mat}(x)$ and $\mathrm{mat}(w)$. In these cases, 1's in $\mathrm{mat}(w)$ will be marked by open circles. Points that are simultaneously in both diagrams will consist of a black disk and a larger concentric circle. Let $[a, b] \times [c, d]$ denote the set of all points $(p, q) \in \mathbb{R}^2$ such that $a \le p \le b$ and $c \le q \le d$. The following notation will be handy:

Definition 3. For $t_{p,q} \in \mathcal{R}(x,w)$, set

$$\mathcal{A}_{p,q} := \mathcal{A}_{p,q}(x) = [p+1, q-1] \times [x(p)+1, x(q)-1], \tag{3.1}$$

$$\overline{\mathcal{A}_{p,q}} := \overline{\mathcal{A}_{p,q}(x)} = [p,q] \times [x(p), x(q)], \tag{3.2}$$

$$\mathrm{pt}_x(c) := (c, x(c)) \text{ for } c \in [1, \dots, n]. \tag{3.3}$$

Along with the points of $\mathrm{mat}(x)$, we can shade parts of our diagram in order to specify that $d_{x,w}$ satisfies a particular inequality on a given region. Light shading on a region signifies that $d_{x,w} \geq 1$ on that region. Dark shading signifies $d_{x,w} \geq 2$. No shading places no restrictions on the values $d_{x,w}$. A region with a black border is one where $d_{x,w}$ achieves the minimum possible value allowed by the shading on that region. Dotted borders are used to demarcate regions we wish to discuss in the text.

As mentioned above, the great utility of these diagrams arises from being able to visualize $\mathcal{R}(x,w)$ along with the information on the Bruhat order. To see how we do this, suppose we have some reflection $t_{a,b} \in \mathcal{R}(x,w)$ (which implies $x < xt_{a,b} \leq w$). Now compare the shading (with respect to w) in $\mathrm{mat}(x)$ and $\mathrm{mat}(xt_{a,b})$. We see (as in Figure 3), that in the region $\mathcal{A}_{a,b}(x)$, $d_{xt_{a,b},w} = d_{x,w} - 1$. Hence, by Lemma 1, we can state the following:

Fact 2. Let $t_{a,b} \in \mathcal{T}$ with $x < xt_{a,b}$. The transposition $t_{a,b}$ is in $\mathcal{R}(x,w)$ if and only if it corresponds to a region in $\mathrm{mat}(x)$ that is entirely shaded (i.e., $d_{x,w}|_{\mathcal{A}_{a,b}} \geq 1$). An example is given in Figure 4

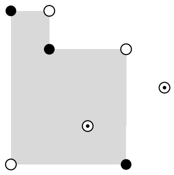


FIGURE 4. In the above configuration, $\mathcal{R}(x,w) = \{t_{1,2}, t_{2,4}, t_{2,5}, t_{4,5}\}$.

Note that the values of $d_{x,w}$ on the region $\overline{\mathcal{A}_{a,b}} \setminus \mathcal{A}_{a,b}$ are not considered in determining the membership of $t_{a,b}$ in $\mathcal{R}(x,w)$. As a final piece of notation, in order to highlight a reflection that we are particularly interested in, we can draw a corresponding arc in our diagram.

4. ELEMENTS OF $\mathcal{R}(x,w)$

As may be ascertained from Theorem 1, the criteria for x to be an element of $\mathrm{maxsing}(X_w)$ are local in nature. This implies that we may concentrate on only certain indices in our permutation w in order to determine $\mathrm{maxsing}(X_w)$. We now describe these indices explicitly.

Definition 4. Let

$$\Delta(x,w) = \{i, 1 \leq i \leq n : \exists\, j, 1 \leq j \leq n, t_{\{i,j\}} \in \mathcal{R}(x,w)\}. \tag{4.1}$$

If $\Delta(x,w) = \{d_1 < d_2 < \cdots < d_k\}$, set

$$\widetilde{x} = \mathrm{fl}_{\Delta(x,w)}(x) \text{ and} \tag{4.2}$$

$$\widetilde{w} = \mathrm{fl}_{\Delta(x,w)}(w). \tag{4.3}$$

Note that $\widetilde{x}$ and $\widetilde{w}$ are permutations in $\mathfrak{S}_k$.

We now give a sufficient condition for an index b to be in $\Delta(x,w)$.

Lemma 2. *Suppose $x < w$ and $x(b) \neq w(b)$ with $1 \leq b \leq n$.*

1. *If $w(b) < x(b)$ then $\exists\, c > b$ with $t_{b,c} \in \mathcal{R}(x,w)$ and $x(c) \neq w(c)$.*
2. *If $w(b) > x(b)$ then $\exists\, a < b$ with $t_{a,b} \in \mathcal{R}(x,w)$ and $x(a) \neq w(a)$.*

Lemma 2 tells us that if $x(i) \neq w(i)$ then $i \in \Delta(x,w)$. It turns out that the question of whether or not x is a maximal singular point (MSP) of X_w depends only on the pair $\widetilde{x}$,$\widetilde{w}$. This is borne out by the following three simple facts:

(1) $\widetilde{x} \leq \widetilde{w}$ if and only if $x \leq w$.
(2) $l(w) - l(x) = l(\widetilde{w}) - l(\widetilde{x})$.
(3) There exists a bijection $\mathcal{R}(\widetilde{x},\widetilde{w}) \xrightarrow{\sim} \mathcal{R}(x,w)$.

5. The map ϕ_t

We have already claimed in Fact 1 that maxsing(X_w) can be expressed in terms of $\mathcal{R}(x,w)$ for $x \leq w$. As we are interested in discovering the *maximal* singular points, we will need to relate $\mathcal{R}(x,w)$ to $\mathcal{R}(y,w)$ when x,y differ by an element of $\mathcal{T}$. So, for every triple $yt < y \leq w$ with $t \in \mathcal{T}$, we will define a map $\phi_t^{y,w} : \mathcal{R}(y,w) \longrightarrow \mathcal{T}$. The values of y,w are usually clear from context and we will often abbreviate $\phi_t^{y,w}$ as ϕ_t.

We do not give the definition of ϕ_t here, but we do note two properties of the map:

(1) If $tt' = t't$ then $\phi_t(t') = t'$.
(2) $\phi_{t_{a,b}}(t_{\{a,c\}})$ is either $t_{\{b,c\}}$ or $t_{\{a,c\}}$.

Remark 1. It is not sufficient to define $\phi_t(t_{b,c}) = t_{\{t(b),t(c)\}}$. For example, if $y = [3,1,2]$, then $t_{2,3} \in \mathcal{R}(y,w)$, but $t_{2,3} \notin \mathcal{R}(yt_{1,2},w)$.

For ϕ_t to be useful, we will need the following property.

Theorem 4. *The map $\phi_t(\mathcal{R}(y,w)) \hookrightarrow \mathcal{R}(yt,w) \setminus \{t\}$ is injective.*

The above theorem has as a simple corollary a special case of Deodhar's conjecture [5].

Corollary 5. $\#\mathcal{R}(y,w) \geq l(w) - l(y)$.

Various forms and generalizations of the preceding corollary have been proven by Dyer [6] (arbitrary Coxeter systems), Deodhar [5] (type A), Polo [12] (finite Weyl groups) and Carrell-Peterson [2] (crystallographic proofs).

The map ϕ_t can also be used to give a simple proof of the following well-known fact ([1, 8.2.10]).

Theorem 6. *If $x \leq w$ is an MSP, s, s' any simple reflections, then $sw < w$ and $ws' < w$ imply, respectively, that $sx < x$ and $xs' < x$.*

6. Structure of Proof of Theorem 1

The first step is to prove the following fact.

Lemma 3. *Let $x < w \in \mathfrak{S}_n$ be an MSP. If $t \in \mathcal{R}(x,w)$ then $l(xt) = l(x) + 1$.*

Recall that $\widetilde{x}$ and $\widetilde{w}$ are the restrictions of x and w to those positions that affect the composition of $\mathcal{R}(x,w)$. In order to determine the structure of maxsing(X_w), we then prove the following necessary conditions on $\widetilde{x}$ for any MSP x of X_w:

Lemma 4. *If $x < w$ is an MSP, then $\widetilde{x}$ is 231-, 312- and 1234-avoiding.*

Combining Lemma 4 and Lemma 3, we see that if $x \in$ maxsing(X_w), then

$$\widetilde{x} = [k, \ldots, 1, k+l, \ldots, k+1, k+l+m, \ldots k+l+1] \tag{6.1}$$

for some $k,l,m \geq 0$. If two out of three of k,l,m are 0, then $\widetilde{x}$ is strictly decreasing, so $x \leq w$ implies that $x = w$. But then x cannot possibly be an MSP. With some work, one can show that the cases of $k,l,m > 0$ and $k,l > 0, m = 0$ can both occur as elements of maxsing(X_w). With this in mind, we make the following definitions:

Definition 5.
For $k,m \geq 2$, define

$$x_{k,m} = [k,\ldots,1,k+m,\ldots,k+1], \tag{6.2}$$

$$w_{k,m} = [k+m,k,\ldots,2,k+m-1,\ldots,k+1]. \tag{6.3}$$

For $k,m \geq 1$ and $l \geq 2$, define

$$x_{k,l,m} = [k,\ldots,1,k+l,\ldots,k+1,k+l+m,\ldots,k+l+1], \tag{6.4}$$

$$\begin{aligned} w_{k,l,m} = [&k+l,k,\ldots,2,k+l+m,k+l-1,\ldots,k+2,\\ &1,k+l+m-1,\ldots,k+l+1,k+1]. \end{aligned} \tag{6.5}$$

We are then able to rephrase Theorem 1 in the following way:

Theorem 7. *x is an* MSP *of X_w if and only if*

(1) *$t \in \mathcal{R}(x,w)$ implies $l(xt) = l(x)+1$.*
(2) (a) *For some $k,m \geq 2$, we have $\widetilde{x} = x_{k,m}$ and $\widetilde{w} = w_{k,m}$*
or
(b) *For some $k,m \geq 1$, $l = 2$ or $k = m = 1, l \geq 2$, we have $\widetilde{x} = x_{k,l,m}$ and $\widetilde{w} = w_{k,l,m}$.*

To finish the proof of Theorem 1, one shows that the forms we have discovered for $\widetilde{x}$ in Lemma 4 are sufficient to show that these points correspond to MSP's in the appropriate Schubert variety. This task consists of two elementary steps which we omit:

(1) Show that the points x are singular points.
(2) Show that any cover of x that is still below w is a smooth point.

Remark 2. The cardinality of the set $\operatorname{maxsing}(X_w)$ for $w \in \mathfrak{S}_n$ may be $O(n^4)$. This is the case, for example, when $n = 2k$ and

$$w = [k+1,\ldots,2k,1,\ldots,k]. \tag{6.6}$$

Then $\#\operatorname{maxsing}(X_w) = \binom{k}{2}^2$.

7. Kazhdan-Lusztig polynomials at elements of $\operatorname{maxsing}(X_w)$

The set $\Delta(x,w)$ affords us the opportunity to reduce the computation of the Kazhdan-Lusztig polynomial associated to a certain pair of permutations to a pair of simpler permutations:

Theorem 8. $P_{x,w} = P_{\widetilde{x},\widetilde{w}}$.

This theorem enables us to easily calculate $P_{x,w}$ for $x \in \operatorname{maxsing}(X_w)$. There are three cases to consider, depending on the type of singularity. Note that as one moves down in the Bruhat order, the first non-trivial $P_{x,w}$ one encounters is of the form

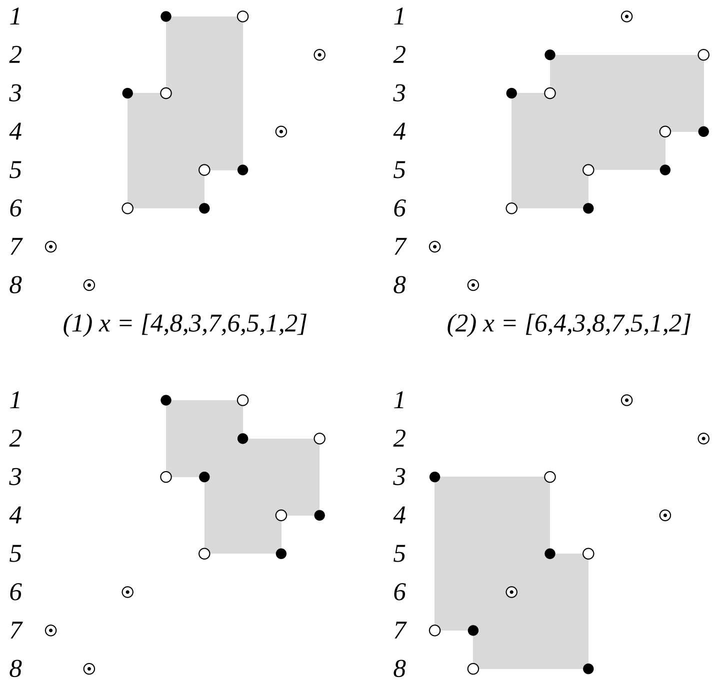
$$1+q^k \qquad \text{or} \qquad 1+q+q^2+\cdots+q^k.$$

Theorem 9. *The Kazhdan-Lusztig polynomials for $x \in \operatorname{maxsing}(X_w)$ are as follows.*

(1) *4231-type singularities:*
If $\widetilde{u} = x_{k,m}$ and $\widetilde{v} = w_{k,m}$ for $k,m \geq 2$, then
$$P_{u,v} = P_{\widetilde{u},\widetilde{v}} = 1+q+\cdots+q^{\min(k-1,m-1)}.$$
(2) *3412-type singularities:*
If $\widetilde{u} = x_{k,2,m}$ and $\widetilde{v} = w_{k,2,m}$ with $k,m \geq 1$, then
$$P_{u,v} = P_{\widetilde{u},\widetilde{v}} = 1+q.$$
(3) *45312-type singularities:*
If $\widetilde{u} = x_{1,l,1}$ and $\widetilde{v} = w_{1,l,1}$ with $l \geq 2$, then
$$P_{u,v} = P_{\widetilde{u},\widetilde{v}} = 1+q^{l-1}.$$

8. Examples calculating maxsing(X_w)

Example 1. Using Theorem 1, in Figure 5 we compute the singular locus maxsing(X_w) of X_w where $w = [6, 8, 4, 7, 5, 3, 1, 2]$.



(1) x = [4,8,3,7,6,5,1,2] (2) x = [6,4,3,8,7,5,1,2]

(3) x = [4,6,5,8,7,3,1,2] (4) x = [6,8,1,7,4,3,2,5]

Figure 5

Example 2. For $w = [6, 4, 5, 1, 2, 3]$,

$$\text{maxsing}(X_w) = \{[4,1,6,5,2,3],\ [4,2,6,1,5,3],\ [4,3,6,1,2,5],\\ [6,1,4,2,5,3],\ [6,1,4,3,2,5],\ [6,2,4,1,3,5]\}.$$

Example 3. Using a computer it is easy to calculate, for example, that for

$$w = [17, 6, 2, 15, 12, 11, 3, 8, 16, 7, 14, 5, 13, 9, 10, 1, 4], \tag{8.1}$$

$\#\,\text{maxsing}(X_w) = 29$.

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