POLYNOMIAL AND COMBINATORIAL IDENTITIES RELATED TO NEAR-TRIANGULAR MATRICES

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ABSTRACT. Combinatorial identities are obtained via detailed analysis of the growth for polynomial identities of near-triangular matrix algebras.
RÉSUMÉ. Identites combinatoire sont toucher sur base detaille etude identites polinomialle pur algébre presque-triangulaire matrices.

1. INTRODUCTION

A basis for the identities of associative algebra formed by upper triangular matrices over a field of zero characteristic was determined by Yu. Malcev in [9]. This result was extended for more general class of fields and rings by P. Siderov [13], U. Kaljulaid [7], S. Polin [11], and S. Vovsi [16]. It turns out that identities of associative algebra formed by upper triangular matrices of order m are generated by the product of m commutators. Algebras of upper triangular matrices with all zero elements on the main diagonal except one, at the fixed position, were studied in [5]. It was proved that ideals of identities for such algebras are generated by the polynomial $x_1 \cdots x_{i-1}[x_i, x_{i+1}]x_{i+2} \cdots x_{m+1}$, where i is the number of position of nonzero element at the main diagonal. It is of interest to study subalgebras of upper triangular matrix algebras, where coefficients are elements of a certain ring (for example, Grassman algebra). This algebras naturally appears in connection with associative and Lie algebra varieties defined by some extremal properties, see [15, 1]. In order to study such algebras it was proved in [12] that ideal of identities of algebra of matrices with coefficients from a certain PI-algebra R with at most one nonzero element at the main diagonal (on i-th position) is generated by polynomials $x_1 \cdots x_{i-1} Id(R) x_{i+1} \cdots x_m$, where $Id(R)$ is ideal of identities for algebra R. In [6] an ideal of identities of block-triangular matrix algebra, where each block is a full matrix algebra over a field of zero characteristic, was characterized. In Section 2 of the present paper we prove

Theorem 1.1. *Let F be an infinite field, $BT(m, F)$ be an algebra of block triangular matrices of order m over the field F, where some blocks are zero. Then the ideal of polynomial identities of algebra $BT(m, F)$ has the form:*
$Id(BT(m, F)) = U$, where U is a product of ideals of identities for blocks and the ideal corresponding to zero block is generated by a variable.

There are two corollaries from this result. The first one is the theorem by A. Giambruno and M. Zaicev [6, Theorem 2]. The second one is a characterization of an ideal of identities of the algebra of triangular matrices with an arbitrary arrangement of zero and non-zero elements at the main diagonal: this algebra corresponds to the case where blocks are of size one.

Corollary 1.2. *Let F be an infinite field. Then the ideal of identities of algebra $NT(m, F; i_1, \dots, i_k)$ has the form*

$$Id(NT(m, F; i_1, \dots, i_k)) = T(x_1 \cdots x_{i_1-1}[y_{i_1}, z_{i_1}]x_{i_1+1} \cdots [y_{i_k}, z_{i_k}]x_{i_k+1} \cdots x_m).$$

Section 3 is devoted to the study of a codimension sequence which measures the growth of identities. As of yet the knowledge about the codimensions is less than satisfactory. The codimensions

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are known only for a few algebras, see [2] for more details. We found an asymptotic behavior of codimension sequences for algebras $BT(m,F)$ and $NT(m,F;i_1,\dots,i_k)$.

In Section 4 we show that if an ideal of identities T asymptotically contains all proper identities (i.e., there exists k such that $(\bigcup_{i\geq k}\Gamma_i)\subseteq T$) then the corresponding algebra has a polynomial growth. The standard example of such algebra is an algebra $UT_0(k,F)$ formed by upper triangular matrices with equal elements on the main diagonal which is investigated in this section. Its codimension sequence is computed exactly. It is proved that all identities of the algebra $UT_0(k,F)$ are proper and asymptotically all proper identities holds in $UT_0(k,F)$. In Section 5 we derive a certain combinatorial identity as an applications of our results.

2. Basis of identities

Let F be an infinite field, R be an associative PI-algebra over F. Let us denote by $F\langle X\rangle = F\langle x_1,x_2,\dots\rangle$ a free associative algebra on the countable set of variables over the field F.

$F_p(R)=F_p(\operatorname{var}R)$ is called the relatively free algebra of rank p in the variety $\operatorname{var}R$.

Below we need the following construction by J. Lewin [8] of relatively free algebra such that its ideal of identities is the product of ideals of identities of two given algebras. We use his theorem in the formulation by V. Drensky [4, Theorem 3.9].

Let us denote by E_{ij}, $1\leq i,j\leq n$ the matrix units. For a countable set of independent variables $X=\{x_1,x_2,\dots\}$, let us denote by $\{u_i\}$, $\{v_i\}$, $\{w_i\}$ certain pair-wise non-intersecting subsets of X.

Theorem 2.1. *(Lewin) Let R_1 and R_2 be two PI-algebras with T-ideals respectively $Id(R_1)$ and $Id(R_2)$. Let $F_p(R_1)$ and $F_p(R_2)$ be generated, respectively, by y_i and z_i, $i=1,\dots,p$, and let $U_p=\sum_{i=1}^{p}F_p(R_1)u_iF_p(R_2)$ be a free $F_p(R_1)-F_p(R_2)$-bimodule. Then the subalgebra F_p is generated by*

$$x_i=y_iE_{11}+z_iE_{22}+u_iE_{12},\ i=1,\dots,m,$$

in the algebra of 2×2 block matrices

$$\begin{pmatrix} F_p(R_1) & U_p\\ 0 & F_p(R_2)\end{pmatrix}$$

is isomorphic to $F_p(R)$, where $Id(R)=Id(R_1)Id(R_2)$.

Our theorem can be formulated as follows

Theorem 2.2. *Let F be an infinite field, $BT(m,F)$ be an algebra formed by block-triangular matrices over F, where all blocks are matrices from either full matrix algebra or zero algebra. Then the ideal of identities of the algebra $BT(m,F)$ is the product of ideals of identities of matrix algebras corresponding to each block, where the ideal of identities of zero algebra is a T-ideal generated by a free variable.*

Proof. By induction on the number of blocks r of matrices from $BT(m,F)$ it is easy to see that polynomials of aforesaid type are identities in the algebra $BT(m,F)$. So, it remains to prove that all identities for $BT(m,F)$ are of the required form.

We prove by induction on r.

The base. Let $r=1$, i.e., there is only one block and the assertion of the theorem is trivial.

The step. Let $r>1$. Let us denote by A the algebra consisting of all blocks except the last one, by B we denote the algebra consisting of the last block. By the induction hypothesis it remains to check that the ideal of identities for the algebra $BT(m,F)$ is contained in the ideal $(Id(A)\cdot Id(B))$.

Let $BT(m,F)\subset M_m(F)$, $A\subset M_t(F)$, $B\subset M_s(F)$, $m=t+s$. Now one can say that the algebra B is a linear span of matrices E_{ij}, $t+1\leq i,j\leq t+s$, such that there exists a matrix from $BT(m,F)$ with non-zero element on (i,j)-th position, the algebra A is a linear span of matrices E_{kl}, $1\leq k,l\leq t$, such that there exists a matrix from $BT(m,F)$ with non-zero element on (k,l)-th position.

Let us denote $U = \{u^i_{jk}\}$, $V = \{v^i_{jk}\}$, $W = \{w^i_{jk}\}$, $i, j, k \geq 1$, are pair-wise non-intersecting subsets of the set $X = \{x_1, x_2, \dots\}$. For each i, $1 \leq i \leq p$, we consider the matrices $U^i = \sum\limits_{j,k} u^i_{jk} E_{jk}$, where $E_{jk} \in A$, $V^i = \sum\limits_{j,k} v^i_{jk} E_{t+j\, t+k}$, where $E_{t+j\, t+k} \in B$, $W^i = \sum\limits_{j=1}^{t} \sum\limits_{k=1}^{s} w^i_{jk} E_{j\, t+k}$. Then the family of matrices $\{U^i\}$ generates the relatively free algebra $\tilde{A}$ of rank p in the variety $\operatorname{var} A$ and the list of matrices $\{V^i\}$ generates the relatively free algebra $\tilde{B}$ of rank p in the variety $\operatorname{var} B$. By Lewin's theorem (Theorem 2.1) it remains to prove that matrices $W^1, \dots, W^p$ generate a $\tilde{A}$-$\tilde{B}$-bimodule which is a free submodule in the module ${}_{F_p(BT(m,F))} F_p(BT(m,F))_{F_p(BT(m,F))}$. We consider the set $\mathbf{a} = \{a_1, a_2, \dots\}$ of linearly independent elements of the algebra $\tilde{A}$ and the set $\mathbf{b^j} = \{b^j_1, b^j_2, \dots\}$, $j = 1, \dots, p$ of arbitrary non-zero elements of the algebra $\tilde{B}$. If either A or B or both of them are zero algebras (i.e., all blocks that form A or B or both of them, respectively) then we say that corresponding sets are empty. For the arbitrary set of variables $\{y_{ij}\}$ by the sum $\sum\limits_{i,j} a_i y_{ij}$ in the case of empty set $\mathbf{a}$ we mean the sum $\sum\limits_{j} y_{1j}$ and by the sum $\sum\limits_{i,j} y_{ij} b^k_i$ in the case of empty set $\mathbf{b^k}$ we mean $\sum\limits_{j} y_{1j}$. In order to prove that $\tilde{A}$-$\tilde{B}$-bimodule generated by matrices $W^1, \dots, W^p$ is free it suffice to check that $\sum\limits_{k} \sum\limits_{j=1}^{p} a_k W^j b^j_k \neq 0$. Since different elements W^j depend on non-intersecting sets of variables, it suffice to prove that

$$\sum_{k=1}^{n} a_k W^1 b^1_k \neq 0 \tag{1}$$

for each $n \geq 1$.

Let us assume that the condition (1) does not hold. We show that there exists such substitution of variables w^1_{ij} and v^1_{ij} in this case that the equality

$$\sum_{k=1}^{n} \lambda^k_{i_0 j_0} a_k E_{t j_0} = 0 \tag{2}$$

holds in the algebra $F\langle U\rangle/(Id(BT(m,F)) \cap F\langle U\rangle)$ and $\lambda^1_{i_0 j_0} \neq 0$.

Under the analysis of sum (1) the following situations can occur:

1. $\mathbf{b^1} = \emptyset$. Then the negation of the equality (1) has the form

$$\sum_{k=1}^{n} a_k W^1 = 0. \tag{3}$$

By setting $w^1_{t j_0} = 1$ for a certain j_0 and $w^1_{ij} = 0$ for $(i,j) \neq (t, j_0)$, one can conclude that W^1 is transformed to $E_{t j_0}$ and the equality (3) is transformed to the equality $\sum\limits_{k} a_k E_{t j_0} = 0$. So the equality (2) holds with the coefficients $\lambda^k_{i_0 j_0} = \lambda^k_{t j_0} = 1 \neq 0$.

2. $\mathbf{b^1} \neq \emptyset$. Then b^1_1 is a non-zero element of the relatively free algebra $\tilde{B}$. Hence, there exists a list of values for $\{v^i_{jk}\}$ such that b^1_1 has a non-zero value in B. This value is a linear span of elements E_{ij}, $t+1 \leq i, j \leq t+s$, and there exists (i_0, j_0) corresponding to the elements of B such that the coefficient under $E_{i_0 j_0}$ is non-zero. For this substitution of variables v^i_{jk} one has

$$\sum_{i,j=t+1}^{t+s} \left(\sum_{k} \lambda^k_{ij} a_k\right) W^1 E_{ij} = 0 \tag{4}$$

in the algebra $F\langle U,W\rangle/(Id(BT(m,F))\cap F\langle U,W\rangle)\subset F\langle X\rangle/(Id(BT(m,F))\cap F\langle X\rangle)$ and $\lambda^1_{i_0j_0}\neq 0$. Similarly with the case 1 one can conclude that there exists a substitution of variables $\{w^i_{jk}\}$ such that W^1 transforms to E_{ti_0}. Then it follows from (4) that $\sum\limits_j\sum\limits_k \lambda^k_{i_0j}a_kE_{tj}=0$.

For each fixed j the product a_kE_{tj} is a matrix with zero elements in the columns $1,\dots,j-1,j+1,\dots,t+s$. Thus, the last equality is equivalent to the following system of equalities: $\sum\limits_k \lambda^k_{i_0j}a_kE_{tj}=0$ for each j. In particular, for $j=j_0$ one has:

$$(\sum_k \lambda^k_{i_0j_0}a_k)E_{tj_0}=0.$$

So, in both cases the equality (2) holds in the algebra $F\langle U\rangle/(Id(BT(m,F))\cap F\langle U\rangle)$ and the coefficient $\lambda^1_{i_0j_0}\neq 0$.

Let us note that if the set $\mathbf{a}=\emptyset$ then the last equality has the form $\lambda^1_{i_0j_0}E_{tj_0}=0$, but $\lambda^1_{i_0j_0}E_{tj_0}\neq 0$ since the matrix E_{tj_0} is non-zero, $\lambda^1_{i_0j_0}\neq 0$. Hence, in this case we come to the contradiction with the assumption (that the condition (1) does not hold).

If the set $\mathbf{a}\neq\emptyset$ then the factor $f=\sum\lambda^k_{i_0j_0}a_k\in\tilde{A}$ under E_{tj_0} is a linear combination of linearly independent elements $a_1,\dots,a_n\in A$ and this combination is non-trivial since $\lambda^1_{i_0j_0}\neq 0$. Therefore, the element f is not an identity in A. By (2) one has that for an arbitrary substitution of variables u^k_{ij}, i.e., for an arbitrary homomorphism $\varphi:\tilde{A}\to A$,

$$\varphi(f)E_{tj_0}=0,$$

i.e., the matrix $\varphi(f)$ has zero t-th column. It follows that the last column of the matrix $\varphi(f)$ is zero since t is the size of matrices in A. According to [6], the set of values that associative polynomial with coefficients from F on m non-commuting variables can reaches on A is a Lie ideal in A. Hence the linear span $\langle\varphi(f)\rangle$ of all values of f in A is a Lie ideal. However all Lie ideals in A that have trivial projection on the last column are trivial. Therefore, we obtain the contradiction. Hence, (1) holds and the theorem is proved. □

We have two corollaries from this result. The first one is the theorem by A. Giambruno and M. Zaicev [6, Theorem 2]. The second one is a characterization of an ideal of identities of the algebra of triangular matrices with an arbitrary arrangement of zero and non-zero elements at the main diagonal: this algebra corresponds to the case where blocks are of size one.

Corollary 2.3. *Let F be an infinite field, $NT(m,F;i_1,\dots,i_k)$ be an algebra of triangular $m\times m$-matrices over F such that for a fixed subset $\{i_1,\dots,i_k\}\in\{1,\dots,m\}$ there are nonzero elements only at the positions $i_1,\dots,i_k$ on the main diagonal. Then the ideal of identities of $NT(m,F;i_1,\dots,i_k)$ has the form*

$$Id(NT(m,F;i_1,\dots,i_k))=$$
$$=T(x_1\cdots x_{i_1-1}[y_{i_1},z_{i_1}]x_{i_1+1}\cdots x_{i_k-1}[y_{i_k},z_{i_k}]x_{i_k+1}\cdots x_m),$$

here we have commutators at the positions $i_1,\dots,i_k$, that correspond to nonzero elements of the main diagonal, and variables at all positions corresponding to zero elements on the main diagonal, where $T(f_1,\dots,f_l)$ denotes T-ideal generated by the polynomials $f_1,\dots,f_l$.

3. The growth of identities

From now we assume that the base field F is of zero characteristic.

There are two basic questions one can ask about the polynomial identities of a given PI-algebra A: How many identities does the algebra have and what are they? The first question is answered by means of the codimension sequence $\{c_n(A)\}_{n=1}^\infty$ of the algebra A.

Definition 3.1. *The codimension sequence of algebra A is defined to be*

$$c_n=c_n(A)=c_n(T)=\dim(P_n/(P_n\cap T)),\quad n=1,2,\dots,$$

where $P_n = P_n(x_1, \ldots, x_n)$ denotes the F-vector subspace in $F\langle X\rangle$ of the multilinear polynomials in $x_1, \ldots, x_n$ and $T = Id(A)$ is the ideal of identities of the algebra A.

For the detailed and self-contained information on this subject see [1].

Applying our Theorem 2.2 together with the results from [2, Theorem 1.4] and [10, Theorem 3.1] we have

Theorem 3.2. *The asymptotic behavior of the codimension sequence growth of the algebra $BT(m,F)$ with l zero blocks and k non-zero blocks of order $d_1, \ldots, d_k$ correspondingly can be valued by formula*

$$c_n(BT(m,F)) \simeq a n^e t^n, \quad n \to \infty$$

where

$$t = d_1^2 + \ldots + d_k^2, \quad e = -\frac{1}{2}t + \frac{3}{2}k + l - 1, \quad a = d_1^2 \cdots d_k^2 \frac{a_1^{e_1} \cdots a_k^{e_k}}{(a_1 + \cdots + a_m)^e},$$

here

$$d_i = 0 \text{ if } i \neq 1, \ldots, k, \quad e_i = -\frac{1}{2}(d_i^2 - 1), \ i = 1, \ldots, m$$

$$a_i = (\frac{1}{\sqrt{2\pi}})^{d_i - 1} \cdot (\frac{1}{2})^{\frac{1}{2}(d_i^2 - 1)} \cdot 1! \cdot 2! \cdots (d_i - 1)! \cdot d_i^{\frac{1}{2}(d_i^2 + 4)}, \ i = 1, \ldots, m.$$

We wrote $NT(m,F;k) = NT(m,F;i_1,\ldots,i_k)$ if we do not need the exact positions of non-zero elements. The following two corollaries are straightforward

Corollary 3.3. *The growth of codimension sequence of the algebra $NT(m,F;k)$ is asymptotically behaved as $c_n(NT(m,F;k)) \simeq \frac{1}{k^{m-1}} \cdot n^{m-1} k^n$, $n \to \infty$.*

Corollary 3.4. [5, Theorem 2.4 (a)] *The growth of identities of the algebra $NT(m,F;1)$ is asymptotically behaved as $c_n(NT(m,F;1)) \simeq n^{m-1}$, $n \to \infty$.*

4. Algebras with polynomial growth of identities

Let F be a field of zero characteristics. We denote by Γ_n a linear space of all proper polynomials of degree n, see [14] for definitions.

Theorem 4.1. *If T-ideal T asymptotically contains all proper identities (i.e., there exists p such that $(\bigcup_{i \geq p} \Gamma_i) \subseteq T$), then the codimension sequence growth of T is polynomial.*

Proof. Let us denote l-th proper codimension of T by $\gamma_l(T)$. Then by conditions $\gamma_l(T) = 0$ if $l \geq p$ and $\gamma_l(T) \leq dim(\Gamma_l)$ if $l < p$. In the paper [3, Corollary 2.5] the following formula was proved:

$$c_n(T) = \sum_{l=0}^{n} \binom{n}{l} \gamma_l(T).$$

Speht, [14, Statement 14], has calculated the dimension of the space Γ_l formed by proper polynomials of a certain degree l:

$$dim(\Gamma_l) = l!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^l}{l!}\Big)$$

Thus if $n \geq p - 1$ one has:

$$c_n(T) \leq \sum_{l=0}^{p-1} \binom{n}{l} dim(\Gamma_l) = \sum_{l=0}^{p-1} \binom{n}{l} \cdot l! \cdot \sum_{i=0}^{l} \frac{(-1)^i}{i!} =$$

$$= \sum_{l=0}^{p-1} \frac{n!}{(n-l)! l!} \cdot l! \cdot \sum_{i=0}^{l} \frac{(-1)^i}{i!} = n! \sum_{l=0}^{p-1} \sum_{i=0}^{l} \frac{(-1)^i}{i!} \cdot \frac{1}{(n-l)!}$$

Therefore,

$$c_n(T) \leq \sum_{l=0}^{p-1} \sum_{i=0}^{l} \frac{(-1)^i}{i!} \cdot n \cdot (n-1) \cdots (n-l+1) \cong a \cdot n^{p-1}$$

if $n \to \infty$. □

Among the algebras with the structure of ideals of identities under consideration we have the following one. We denote by $UT_0(k,F)$ an algebra of upper triangular $k \times k$ matrices with the elements at the main diagonal equal to each other.

Algebra $UT_0(k,F)$ contains the identity element. Hence, (see. [14]), all its identities follow from proper ones. Let us show that in the algebra $UT_0(k,F)$ all proper identities asymptotically hold, i.e., there exists p such that $(\bigcup\limits_{i \geq p} \Gamma_i) \subseteq T$. Actually we prove that in this case $p = k$ is the order of matrices in algebra $UT_0(k,F)$.

Lemma 4.2. *There no identities of degree less than k in $UT_0(k,F)$.*

Proof. Let $h(x_1, \ldots, x_{k-1}) = \sum_{\sigma \in S_{k-1}} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(k-1)} \in T(UT_0(k,F))$. We assume that $h \neq 0$. Without loss of generality $\alpha_1 \neq 0$. Therefore,

$$h(e_{1,2}, \ldots, e_{k-1,k}) = \alpha_1 e_{1,k} \neq 0.$$

This is the contradiction. □

Theorem 4.3.

$$Id(UT_0(k,F)) = T(\bigcup_{n \geq k} \Gamma_n)$$

Proof. By Lemma 4.2 there are no identities of degree less than k.

$UT_0(k,F)$ contains a unity element. Therefore its T-ideal is uniquely determined by its proper part.

Then it is enough to prove that all proper polynomials of the degree greater than or equal to k are identities in $UT_0(k,F)$.

Let us consider an arbitrary proper polynomial $f(x_1, \ldots, x_{i_d})$. It is a linear combination of monomials of the form

$$g = [x_1, \ldots, x_{i_1}] \cdots [x_{i_{d-1}+1}, \ldots, x_{i_d}]$$

for some d and i_j, $1 \leq j \leq d$. Let us show that each of these monomials of degree greater than or equal to k are identities in $UT_0(k,F)$.

We note that $UT_0(k,F) = F \oplus J$, where $J = J(UT_0(k,F))$ is Jacobson radical of $UT_0(k,F)$, which consists of matrices with zero main diagonal. It is easy to see that $[F,J] \subseteq J^2$. Hence,

$$[UT_0(k,F), UT_0(k,F)] = [F \oplus J, F \oplus J] \subseteq [F,F] + [F,J] + [J,F] + [J,J] \subseteq J^2.$$

Similarly, $[J^i, UT_0(k,F)] \subseteq J^{i+1}$ for each i, $1 \leq i \leq k$ and $J^i J^j \subseteq J^{i+j}$. Then for each polynomial g and arbitrary matrices $P_1, \ldots, P_{i_d} \in UT_0(k,F)$ one has

$$g(P_1, \ldots, P_{i_d}) \in J^{i_1} \cdots J^{i_d - i_{d-1}} \subseteq J^{i_1 + i_2 - i_1 + \ldots + i_d - i_{d-1}} = J^{i_d} = \{0\},$$

if $i_d \geq k$. Therefore, $g(P_1, \ldots, P_{i_d}) = 0$ if $i_d \geq k$. This proves the theorem. □

Corollary 4.4.

$$c_n(UT_0(k,F)) = \begin{cases} n! \sum\limits_{l=0}^{k-1} \sum\limits_{i=0}^{l} \frac{(-1)^i}{i!} \cdot \frac{1}{(n-l)!} & \text{if} \quad n \geq k-1 \\ \\ n! \sum\limits_{l=0}^{n} \sum\limits_{i=0}^{l} \frac{(-1)^i}{i!} \cdot \frac{1}{(n-l)!} & \text{if} \quad n < k-1 \end{cases}$$

5. Combinatorial applications

According to Lemma 4.2 there are no polynomial identities of degree less than k in $UT_0(k,F)$. Hence, $c_n(UT_0(k,F)) = n!$ if $n < k$. Comparing this equality with the results of Corollary 4.4 one can obtain the following combinatorial identity:

Corollary 5.1.

$$\sum_{l=0}^{n}\sum_{i=0}^{l}\frac{(-1)^i}{i!}\cdot\frac{1}{(n-l)!} = 1, \quad \text{for all} \quad n \geq 1.$$

We also include here a direct combinatorial proof of this result.

Proof. Induction on n.

The base: $n=1$. The first sum consists from 3 summands:

1. $n=1$, $l=0$, $i=0$. Then $\frac{(-1)^0}{0!}\cdot\frac{1}{1!} = 1$.
2. $n=1$, $l=1$, $i=0$. Then $\frac{(-1)^0}{0!}\cdot\frac{1}{0!} = 1$.
3. $n=1$, $l=1$, $i=1$. Then $\frac{(-1)^1}{1!}\cdot\frac{1}{0!} = -1$.

That is $1+1-1=1$ and the equality holds.

The step.

Let us rewrite the required equality in the form

$$\sum_{l=0}^{n}\frac{1}{(n-l)!}\left(\sum_{i=0}^{l}\frac{(-1)^i}{i!}\right) = 1.$$

Let us consider

$$S_{n+1} = \sum_{l=0}^{n+1}\frac{1}{(n+1-l)!}\cdot\sum_{i=0}^{l}\frac{(-1)^i}{i!} =$$

$$\frac{1}{(n+1)!} + \sum_{l=1}^{n+1}\frac{1}{(n+1-l)!}\cdot\sum_{i=0}^{l}\frac{(-1)^i}{i!}.$$

Let us denote $t = l-1$. Then

$$S_{n+1} = \frac{1}{(n+1)!} + \sum_{t=0}^{n}\frac{1}{(n-t)!}\cdot\sum_{i=0}^{t+1}\frac{(-1)^i}{i!} =$$

$$= \frac{1}{(n+1)!} + \sum_{t=0}^{n}\frac{1}{(n-t)!}\cdot\sum_{i=0}^{t}\frac{(-1)^i}{i!} + \sum_{t=0}^{n}\frac{1}{(n-t)!}\cdot\frac{(-1)^{t+1}}{(t+1)!}.$$

By the induction hypothesis the first sum is equal to 1. Therefore,

$$S_{n+1} = \frac{1}{(n+1)!} + \sum_{t=0}^{n}\frac{1}{(n-t)!}\cdot\frac{(-1)^{t+1}}{(t+1)!} + 1.$$

Now we only need to prove that

$$-\frac{1}{(n+1)!} = \sum_{t=0}^{n}\frac{1}{(n-t)!}\cdot\frac{(-1)^{t+1}}{(t+1)!}.$$

Thus we have:

$$-1 = \sum_{t=0}^{n}\frac{(n+1)!}{(n-t)!(t+1)!}\cdot(-1)^{t+1};$$

$$1 = \sum_{t=0}^{n}\frac{(n+1)!}{(n-t)!(t+1)!}\cdot(-1)^{t};$$

$$1 = \sum_{t=0}^{n} \frac{(n+1)!}{((n+1)-(t+1))! \cdot (t+1)!} \cdot (-1)^t;$$

$$-1 = \sum_{t=0}^{n} C_{n+1}^{t+1} \cdot (-1)^{t+1};$$

$$C_{n+1}^{0} + \sum_{t=0}^{n} C_{n+1}^{t+1} \cdot (-1)^{t+1} = 0.$$

Let us denote $p = t + 1$. Then we have

$$\sum_{p=0}^{n+1} C_{n+1}^{p} \cdot (-1)^p = (1-1)^{n+1} = 0.$$

This concludes the proof. □

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